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Ising Model with a Scaling Interaction

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A spin-infinity Ising model is presented in which there is a hierarchy of spin groupings, for which the interaction potential satisfies exactly the scaling hypothesis that group-group interactions involve only the group mean spin. The calculation of the critical exponents for this model is reduced to the solution of an eigenvalue problem which appears to be numerical tractable. The value of the critical exponent η appears as a parameter of the model and formulas for the critical exponents δ, γ, ν are presented. Assuming $\eta=0$ for three dimensions, we obtain $\delta=5$, $\gamma=1.30$, and $\nu=0.65$.

I. INTRODUCTION AND SUMMARY

Since Widom[1] introduced his homogeneity arguments and Kadanoff[2] introduced the scaling idea, there has been considerable interest as to whether they are valid for the d-dimensional, spin-s, nearest-neighbor Ising models near their ferromagnetic critical points ($d \geq 2$). The two-dimensional Ising model results agree very well with these ideas, but the question is open for $d \geq 3$.[3] The fundamental idea (in the scaling picture) is that, near the critical point where the spin-spin correlations are very long ranged, it does not matter whether we consider the fundamental interaction to be that of a single spin with its neighbor or that of a group of spins with a neighboring group of spins, since all those spins in the group would usually be aligned anyway. The consequence of this line of reasoning has been to derive a series of equations which relate any critical index of divergence (or convergence) of the various thermodynamic variables at the critical point to the behavior of, at most, two fundamental ones and the dimensionality of the system. These results then allow values to be deduced for all the critical indices once any two independent ones are known. The trouble was, until recently, that scaling arguments provided no way to obtain numerical values for any of these. Recently, Wilson[4] has shown how to compute (by means of a series of scaling-idea type of approximations) the value of one independent index (he assumes that the scattering intensity index $\eta=0$) for an Ising-like continuous-spin model. The value he obtained of $\gamma=1.22$ for the three-dimensional case is close to the accepted[3] value of $\gamma=1.25$ for the Ising model.

In this paper we clarify, refine, and extend the work of Wilson.[4] We introduce a ferromagnetic spin-infinity Ising model for which Wilson's key approximations are exact. The structure of the spin-spin interaction is such that there exists a hierarchy of groupings of the spins, such that at each level in the hierarchy the spin-spin interactions can be broken into the intragroup spin interactions and an interaction between the mean spin of the group with the mean spins of the other groups. The lattice structure of the groups is the same as that of the original spin lattice. As one reaches levels of the hierarchy in which each group consists of a large number of spins, the behavior of the mean spin, except for amplitude, should cease (in the critical region) to depend on precisely which layer of the hierarchy one is on and depend only on the hierarchy structure. Thus, there exists an intrinsic scaling behavior for our family of models. We will show for our model how to obtain in a numerically feasible way the various critical indices.

In Sec. II we show how we were heuristically led to define our model in one dimension and show that it is indeed a ferromagnetic Ising model. We describe some of the general results that apply to models of this type.

In Sec. III of this paper we show how the scaling ideas can be applied to reduce the properties of the model to the discussion of a recursively defined family of functions of a single real variable. We also derive an expression for the field-free magnetic susceptibility in this model.

In Sec. IV we extend our model to higher dimension ($d \geq 2$). Again, a recursively defined family of functions of a single real variable results. The defining recursion formula now involves $(2^d - 1)$ integrations rather than the single integration required for $d=1$ and required in Wilson's approximate solution for his model. In addition, a slightly different model is introduced which requires only a single integration.

In Sec. V we investigate the behavior of our recursively defined family of functions. We argue that for $T > T_c$ the mean spin will be normally distributed and we calculate the order of magnitude of its variance. At the critical point this variance

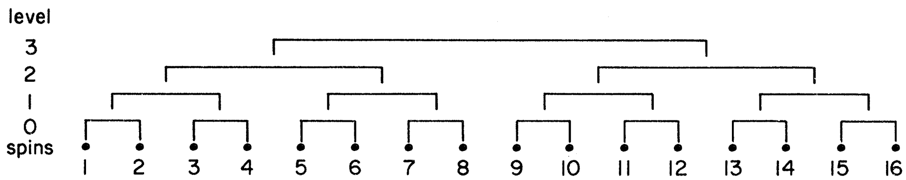

FIG. 1. The spin grouping hierarchy used for our one-dimensional model.

becomes infinite, relative to its previous scale of size. We then derive an approximate linearization of the recursion relation, and solve for the behavior of the free energy at the critical point. We find that for our model the scaling relation[3]

$$\delta = (d+2-\eta)/(d-2+\eta) \tag{1.1}$$

must hold, where δ is the magnetization index along the critical isotherm. We next investigate the region immediately above T_C by perturbation theory in $(T-T_C)$ and derive an eigenvalue equation whose dominate eigenvalue is denoted by 2^{ζ}. In terms of this eigenvalue we obtain

$$\gamma = (2-\eta)/\zeta \ , \tag{1.2}$$

where γ is the zero-field susceptibility index. From (1.2) it easily follows[3] that the correlation-length index is

$$\nu = 1/\zeta \ . \tag{1.3}$$

Finally, the numerical calculations of Wilson[4,5] in his approximation are reported, which correspond fairly well to the ordinary Ising model results.

II. CONSTRUCTION OF ISING MODEL WITH SCALING PROPERTY

In this section we will construct a family of Ising models which possesses a rigorous scaling property. The ideas involved are very similar to those of Wilson.[4] Our approach is to construct a model where Wilson's type of procedures are exact, whereas in his approach Wilson used his procedures as approximations to a nearest-neighbor Ising model. Our approach will permit us to draw conclusions about a particular family of Ising models and the relevance of these models to nature and to other models can then be discussed separately from the procedures of solution. For ease of presentation, we will first discuss one-dimensional models and generalize to arbitrary dimension in a later section, even though we are well aware that we would not expect a phase transition in one dimension and that many of our procedures are somewhat artificial when applied to one-dimensional systems.

The first step in our approach is to introduce a linear array of Ising spins $-1 \le \nu_j \le 1$, $j = 1, 2, 3, \ldots, 2^L$. These are the spin infinity or classical spins. The reason for making the total number a power of 2 will become apparent later, as will the use of continuously distributed rather than discrete spins. A change of spin variable is desirable now so that one group will involve only two adjacent spins at a time, the next group will involve two adjacent sets of two spins, and so on. Figure 1 illustrates the kind of grouping contemplated. Explicitly we define

$$\left.\begin{aligned} s_{m,0} &= (1/\sqrt{2})(\nu_{2m-1} - \nu_{2m}) \\ \hat{s}_{m,0} &= (1/\sqrt{2})(\nu_{2m-1} + \nu_{2m}) \end{aligned}\right\} \quad m = 1, \ldots, 2^{L-1} \ . \tag{2.1}$$

This transformation is easily seen to be an orthonormal one. Each of the new variables involves two, and only two, adjacent spins. We may define recursively successively higher and higher levels of such variables. We therefore define

$$\left.\begin{aligned} s_{m,l+1} &= (1/\sqrt{2})(\hat{s}_{2m-1,l} - \hat{s}_{2m,l}) \\ \hat{s}_{m,l+1} &= (1/\sqrt{2})(\hat{s}_{2m-1,l} + \hat{s}_{2m,l}) \end{aligned}\right\} \quad m = 1, \ldots, 2^{L-2-l} \tag{2.2}$$

for $l = 0, 1, \ldots L-2$. Each of these transformations of variables is again orthonormal. The final set of variables which we choose is

$$\begin{aligned} &s_{m,l}, \ m = 1, \ \ldots \ 2^{L-l-1}, \quad l = 0, \ldots, L-1 \ , \\ &\hat{s}_{1,L-1} = 2^{-0.5L} \sum_{j'} \nu_{j'} \ , \end{aligned} \tag{2.3}$$

a total of 2^L new variables. In order to understand the behavior of these variables, it is convenient to reexpress them in terms of the more familiar momentum transformed spin variables. Thus we introduce $(N = 2^L)$

$$\begin{aligned} &\mu_q = N^{-1/2} \sum_{j=1}^{N} e^{2\pi i q j} \nu_j \ , \\ &q = 0, \ \frac{1}{N}, \ \frac{2}{N}, \ \ldots, \ \frac{N-1}{N} \end{aligned} \tag{2.4}$$

or

$$\nu_j = N^{-1/2} \textstyle\sum_q e^{-2\pi i q j} \mu_q \tag{2.5}$$

by the standard procedure of Fourier inversion. Upon substituting (2.5) into (2.1), we obtain

$$\begin{aligned} s_{m,0} &= (2N)^{-1/2} \textstyle\sum_q e^{-4\pi i q m} \mu_q (e^{2\pi i q} - 1) \ , \\ \hat{s}_{m,0} &= (2N)^{-1/2} \textstyle\sum_q e^{-4\pi i q m} \mu_q (e^{2\pi i q} + 1) \ . \end{aligned} \tag{2.6}$$

From this starting point we can derive by use of (2.2) that

$$s_{m,l}=(2^{l+1}N)^{-1/2}\sum_q \exp(-2^{l+2}\pi iqm)\mu_q[\exp(2^{l+1}\pi iq)-1]\times\prod_{\lambda=0}^{l-1}[1+\exp(2^{\lambda+1}\pi iq)]\,,$$

$$\hat{s}_{m,l}=(2^{l+1}N)^{-1/2}\sum_q \exp(-2^{l+2}\pi iqm)\,\mu_q\times\prod_{\lambda=0}^{l}[1+\exp(2^{l+1}\pi iq)]\,. \quad (2.7)$$

The norm of each level (indexed by l) of variables may be computed as

$$\sum_{m=1}^{2^{L-1-l}} s_{m,l}^2=\frac{1}{2^{L+l+1}}\sum_q\sum_{q'}\sum_m \exp[-2^{l+2}\pi i(q+q')m]\mu_q\,\mu_{q'}\times\prod_{\lambda=0}^{l-1}\{[1+\exp(2^{\lambda+1}\pi iq)][1+\exp(2^{\lambda+1}\pi iq')]\}[\exp(2^{l+1}\pi iq)-1][\exp(2^{l+1}\pi iq')-1]\,, \quad (2.8)$$

or summing over m we get

$$\sum_{m=1}^{2^{L-1-l}} s_{m,l}^2=\frac{1}{2^{2l+2}}\sum_q\sum_{q'}\delta(l,q,q')\,\mu_q\,\mu_{q'}\times\prod_{\lambda=0}^{l-1}\{[1+\exp(2^{\lambda+1}\pi iq)][1+\exp(2^{\lambda+1}\pi iq')]\}[\exp(2^{l+1}\pi iq)-1][\exp(2^{l+1}\pi iq')-1]\,, \quad (2.9)$$

where

$$\delta(l,q,q')=1 \ \text{ if } 2^{l+1}(q+q')=\text{integer}$$
$$=0 \ \text{ otherwise.} \quad (2.10)$$

For the $l=0$ level, there are two types of nonvanishing δ's. They are

$$q'=1-q\,,\quad q'=\tfrac{1}{2}-q\,,$$

which yields (as μ_q is periodic with period 1)

$$\textstyle\sum_m s_{m,0}^2=\sum_q[\mu_q\,\mu_{-q}\sin^2\pi q-\frac{1}{2}\,\mathrm{Im}(\mu_q\,\mu_{0.5-q})\sin 2\pi q]\,. \quad (2.11)$$

In the hope that they are representative, we will study only the diagonal $(q+q'=1)$ terms. From (2.9) these are, for $l\geq 1$,

$$\sum_{m=1}^{2^{L-1-l}} s_{m,l}^2\simeq\sum_q\Big[\prod_{\lambda=0}^{l-1}\cos^2(2^\lambda\pi q)\Big]\sin^2(2^l\pi q)\mu_q\,\mu_{-q} \quad (2.12)$$

or, for very small q

$$\textstyle\sum_m s_{m,l}^2\simeq 2^{2l}\pi^2\sum_q q^2\,\mu_q\,\mu_{-q}\,. \quad (2.13)$$

Now we also know that

$$\textstyle\sum_j(\nu_j-\nu_{j+1})^2=\sum_q\sin^2(\pi q)\,\mu_q\,\mu_{-q}\approx\pi^2\sum_q q^2\,\mu_q\,\mu_{-q}\,. \quad (2.14)$$

Hence, in some sense, at least for small q, (that is, with regard to the long-range behavior) the sum on the left-hand side of (2.13) and (2.14) are very similar. The sum on the left-hand side of (2.14) differs from the usual Ising energy only slightly as

$$\textstyle -J\sum_j\nu_j\,\nu_{j+1}\equiv\frac{1}{2}J\sum_j(\nu_j-\nu_{j+1})^2-J\sum_j\nu_j^2\,. \quad (2.15)$$

From (2.12) it is evident that the variables in the lth level mainly concern momenta with q of the order 2^{-l}. From their analysis of the spin-spin correlations near the critical point, Fisher and Burford[6] conclude (among many other things) that

$$\langle\mu_q\,\mu_{-q}\rangle\approx\hat{D}/(qa)^{2-\eta}\quad(q\to 0)\,, \quad (2.16)$$

with a possible nonzero exponent η and a finite amplitude $\hat{D}$. Putting this consideration of the magnitude of the μ_q near the critical point (if any) together with (2.13) and taking account of the cutoff in the sum due to higher terms for $q\approx 2^{-l}$, we are led to consider a model Hamiltonian of the form

$$\mathcal{H}=J\sum_{l=0}^{L-1}2^{-l(2-\eta)}\sum_{m=1}^{2^{L-1-l}}(s_{m,l})^2-\frac{1}{2}J\left(\frac{1-2^{-L}}{1-2^{\eta-3}}\right)\sum_j\nu_j^2\,. \quad (2.17)$$

The considerations which have led us to this form have been somewhat heuristic; however, we will be able to derive exact consequences of (2.17).

A picture of the structure of (2.17) can be obtained from Fig. 1. The spins, grouped by an interaction line on level zero, are coupled by the interaction $\frac{1}{2}(\nu_5-\nu_6)^2$, for example. Those spins in level-zero groups, which are joined together by an interaction line at level one, interact as $\frac{1}{4}(\nu_5+\nu_6-\nu_7-\nu_8)^2$, with various strengths, and so on through the higher levels.

It is easy to see that every $\nu_j\,\nu_k$ interaction is ferromagnetic. First we note that before a certain level (value of l) is reached, $\nu_j\,\nu_k$ will not appear at all. The first time it appears, ν_i will be in one $\hat{s}_{2m-1,l-1}$ and ν_k in the next $\hat{s}_{2m,l-1}$. Thus, in the first level of appearance l the interaction will be (for $J>0$) of ferromagnetic sign. In every subsequent level, ν_j and ν_k will appear only through $\hat{s}_{m,l}$ and so have the antiferromagnetic sign. Thus we may, by summing a geometric progression, compute the spin-spin interaction in terms of the level of first appearance as

$$\frac{-J\nu_j\nu_k}{2^{l(3-\eta)}}\left(\frac{2^{3-\eta}-2}{2^{3-\eta}-1}\right), \tag{2.18}$$

where we have taken the limit $L\to\infty$ in writing this expression. As long as $\eta<2$ the total $\nu_j\nu_k$ interaction is ferromagnetic. It is also to be noted that the ν_j^2 terms have been arranged to cancel exactly. Due to the quadratic nature of the interactions, although our Hamiltonian is not translationally invariant, it has at least spin symmetry, i.e., if $\nu_j \to -\nu_j$ for all j, $\mathcal{H}$ is unchanged.

An important consequence of the fact that all the interactions are ferromagnetic is that the Lee-Yang theorem,[7] as extended by Asano,[8] by Griffiths,[9] and by Suzuki and Fisher,[10] holds for this model. Various important consequences follow from the theorem.[11,12] Consequently, for any positive temperature the free energy as a function of magnetic field is analytic, provided $H\neq 0$. It appears plausible that that the result of Gallavotti, Miracle-Sole, and Robinson,[13] which shows that the free energy is analytic for $H=0$, provided T is large enough, can be extended to our model. For although our potential is not translationally invariant, it can be uniformly bounded by one that is, and the spin-infinity Ising coupling is physically more difficult to order and hence less likely to produce a singularity in the free energy than is the spin $-\frac{1}{2}$ case which they treated. Thus, on general grounds we expect our model to behave (both the one-dimensional model and the higher-dimensional models described in Sec. IV) as any other more or less short-ranged interaction ferromagnetic Ising model as regards the appearance of a critical point and the general appearance of the phase diagram.

According to the criterion of Thompson[14] (stated only for spin $-\frac{1}{2}$ Ising models), there would not be expected to be any long-range order for any positive T as long as $\eta\leq 1$. This result follows since two spins separated by $|j-k|$ cannot enter with an l smaller than $\log_2|j-k|$ and thus, by (2.18), the interaction must fall off like $|j-k|^{\eta-3}$. We would regard the spin-infinity Ising model as no more likely to develop order than the spin $-\frac{1}{2}$ model. The probable lack of long-range order for our one-dimensional model is consistent with the known similar lack in the arbitrary-spin, nearest-neighbor, one-dimensional Ising model.

We note that in higher dimensions, d, discussed in Sec. IV, Eq. (2.18) will generalize to

$$\frac{-2^{1-d}J\nu_j\nu_k}{2^{l(2+d-\eta)}}\left(\frac{2^{2+d-\eta}-2}{2^{2+d-\eta}-1}\right), \tag{2.19}$$

and hence the interaction will decay as $r^{\eta-2-d}$. Ferromagnetic interactions are maintained for $\eta<d+1$. For any dimension we will require that $\eta<2$, however, to yield a finite total interaction strength.

III. REDUCTION OF PARTITION FUNCTION TO FUNCTIONAL EQUATION

In this section we consider the standard partition function for the Hamiltonian of Eq. (2.17). It is given by

$$Z=\int\cdots\int_{|\nu_j|\leq 1}\exp\left(-\beta J\sum_{l=0}^{L-1}2^{-l(2-\eta)}\sum_{m=1}^{2^{L-1-l}}(s_{m,l})^2 + \frac{1}{2}\beta J\,\frac{1-2^{L(\eta-3)}}{1-2^{\eta-3}}\sum_j \nu_j^2\right)\prod_{j=1}^{2^L}d\nu_j\,. \tag{3.1}$$

It is convenient to replace the finite integral in (3.1) with an infinite one. Thus let

$$\int_{-1}^{+1}d\nu_j=\int_{-\infty}^{+\infty}d\nu_j\,U(1-\nu_j^2)$$

$$=\int_{-\infty}^{+\infty}d\nu_j\exp\left[-\frac{1}{2}K\left(\frac{1-2^{L(\eta-3)}}{1-2^{\eta-3}}\right)\nu_j^2-\frac{1}{2}P(\nu_j)\right], \tag{3.2}$$

where $U(x)$ is the unit step function and $K=\beta J$. Equation (3.2) defines $P(x)$ by equating the integrands. With this notation, Eq. (3.1) becomes

$$Z=\int_{-\infty}^{+\infty}\cdots\int\exp\left(-K\sum_{l=0}^{L-1}2^{-l(2-\eta)}\sum_{m=1}^{2^{L-1-l}}(s_{m,l})^2 -\frac{1}{2}\sum_{j=1}^{2^L}P(\nu_j)\right)\prod_{j=1}^{2^L}d\nu_j\,. \tag{3.3}$$

Let us now solve Eq. (2.1) to reexpress ν_j in terms of the variables $s_{m,0}$ and $\hat{s}_{m,0}$. We obtain

$$Z=\int_{-\infty}^{+\infty}\cdots\int^{2}\prod_{m=1}^{L-1}(ds_{m,0}\,d\hat{s}_{m,0})\exp\left\{-K\sum_{l=1}^{L-1}2^{-l(2-\eta)}\sum_{l=1}^{2^{L-1-l}}(s_{m,l})^2 -K\sum_{m-1}^{2^{L}-1}(s_{m,0})^2-\frac{1}{2}\sum_{m=1}^{2^{L}-1}\left[P\left(\frac{1}{\sqrt{2}}(\hat{s}_{m,0}+s_{m,0})\right)+P\left(\frac{1}{\sqrt{2}}(\hat{s}_{m,0}-s_{m,0})\right)\right]\right\}. \tag{3.4}$$

If we define

$$I_0(x)=\int_{-\infty}^{+\infty}dy\exp\left[-Ky^2-\tfrac{1}{2}Q_0(x+y)-\tfrac{1}{2}Q_0(x-y)\right], \tag{3.5}$$

where

$$Q_0(2^{1/2}x)=P(x) \tag{3.6}$$

and define

$$Q_1(x) = -2\ \ln[I_0(2^{0.5-0.5\eta}x)I_0(0)]\ , \qquad (3.7)$$

then

$$Z = (2^{2-\eta})^{2^{L-2}}[I_0(0)]^{2^{L-1}}$$

$$\times \int_{-\infty}^{+\infty}\cdots\int \exp\left[-K\sum_{l=0}^{L-2} 2^{-l(2-\eta)} \sum_{m=1}^{2^{L-2-l}} \frac{(s_{m,l+1})^2}{2^{2-\eta}}\right.$$

$$\left. - \sum_{m=1}^{2^{L-1}} \frac{1}{2} Q_1\left(\frac{2^{1/2}\hat{s}_{m,0}}{2^{1-0.5\eta}}\right)\right] \prod_{m=1}^{2^{L-1}} \left(\frac{d\hat{s}_{m,0}}{2^{1-0.5\eta}}\right). \qquad (3.8)$$

It will now be observed that Eq. (3.8), aside from the factor in front of the integral sign, the replacement of ν_j by $(\hat{s}_{m,0}/2^{1-0.5\eta})$, and the replacement of L by $L-1$, is identical to (3.3) because of the recursive nature of the definition of our variables by Eq. (2.2). Consequently, if we define recursively the functions

$$I_l(x) = \int_{-\infty}^{+\infty} dy\ \exp[-Ky^2 - \tfrac{1}{2}Q_l(x+y) - \tfrac{1}{2}Q_l(x-y)] \qquad (3.9)$$

and

$$Q_{l+1}(x) = -\ 2\ \ln[I_l(2^{0.5-0.5\eta}x)I_l(0)], \qquad (3.10)$$

where $Q_0(x)$ is given by (3.6), then

$$Z = \prod_{l=0}^{L-1} \{(2^{2-\eta})^{2^{L-2-l}} [I_l(0)]^{2^{L-1-l}}\}$$

$$\times \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}Q_L\left(\frac{2^{1/2}\hat{s}_{1,L-1}}{2^{(2-\eta)L/2}}\right)\right]\frac{d\hat{s}_{1,L-1}}{2^{(2-\eta)L/2}} \qquad (3.11)$$

or the free energy per spin will be

$$F/N = -kT\ \ln\frac{Z}{N}$$

$$= -kT\left(1-\frac{1}{N}\right)(2-\eta)\ \ln 2 - kT\sum_{l=0}^{L-1} 2^{-1-l}\ \ln[I_l(0)]$$

$$-\frac{kT}{N}\ \ln \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}Q_l\left(\frac{2^{1/2}s_{1,L-1}}{2^{(2-\eta)L/2}}\right)\right]\frac{d\hat{s}_{1,L-1}}{2^{(2-\eta)L/2}}\Bigg\}\ . \qquad (3.12)$$

It is to be noted that these results, except for the last integral, and the introduction of η are identical to those of Wilson[4] for one dimension.

If we add a magnetic field interaction to the Hamiltonian (2.17) of the form

$$-mH\sum_{j=1}^{N}\nu_j = -mHN^{1/2}\hat{s}_{1,L-1} \qquad (3.13)$$

then the integral in (3.11) and (3.12) is modified to be

$$\int_{-\infty}^{+\infty}\left[\exp + mH\beta N^{1/2}\ \hat{s}_{1,L-1} - \frac{1}{2}Q_L\left(\frac{2^{1/2}\hat{s}_{1,L-1}}{2^{(2-\eta)L/2}}\right)\right]\frac{d\hat{s}_{1,L-1}}{2^{(2-\eta)L/2}}. \qquad (3.14)$$

One can deduce by the usual statistical mechanical formula that for $T > T_c$ and $H = 0$ the magnetic susceptibility is

$$\chi = m^2\beta N\ \frac{\int_{-\infty}^{+\infty} (\hat{s}_{1,L-1})^2 \exp[-\frac{1}{2}Q_L(2^{1/2}\hat{s}_{1,L-1}/2^{(2-\eta)L/2})]\ d\hat{s}_{1,L-1}}{\int_{-\infty}^{+\infty} \exp[-\frac{1}{2}Q_L(2^{1/2}\hat{s}_{1,L-1}/2^{(2-\eta)L/2})]\ d\hat{s}_{1,L-1}}\ . \qquad (3.15)$$

For a finite χ $(T > T_c)$ the function Q_L must be such, despite the factor in its argument which tends to infinity, as to produce a distribution of finite width for $(\hat{s}_{1,L-1})$. At the critical point (if any), the requirement that the $Q_L(x)$ tend to a limit would lead to the conclusion that

$$\chi \approx O(2^{L(2-\eta)})\ . \qquad (3.16)$$

Since the minimum q for a system of this size is $O(2^{-L})$, Eq. (3.16) corresponds correctly to the assertion that η is, in fact, the scattering intensity critical index of Eq. (2.16), which was its intended role.

IV. EXTENSION TO HIGHER DIMENSIONS

There are many ways to extend the results of Secs. II and III to higher dimension. In this section we will present only the simplest ones, and that on only the hypercubic lattices (plane square, simple cubic, etc.). We could have based our analysis of the one-dimensional model on triplication (see Fig. 2) instead of duplication as we did. One would expect that as larger fundamental groups of spins are taken the results approach those of the nearest-neighbor Ising model more closely. Our method for the construction of a model is to take a nearest-neighbor Ising model on a regular lattice, and to group the spins into cells of two or more spins in such a way that the cells have the same lattice structure as did the original lattice. The interactions between spins in the same cell are retained and those between spins of different cells

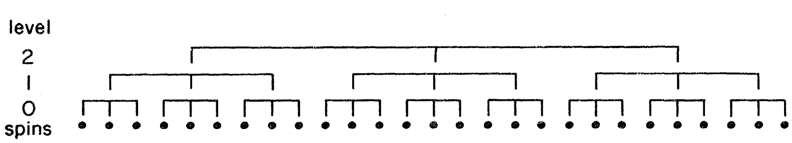

FIG. 2. An alternate hierarchy grouping for a one-dimensional model.

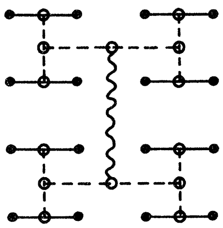

FIG. 3. A grouping of a two-dimensional lattice of spins which is structurally equivalent to Fig. 1.

are replaced by an interaction between the mean cell spins, just as in the one-dimensional case. This procedure is the repeated grouping of cells into higher level cells.

We believe that it is necessary for each fundamental cell to retain the full dimensionality of the original lattice in order for at least the rotational symmetry of the original lattice to be retained. For example, one might consider breaking up the plane-square lattice as in Fig. 3, however, this model can be easily shown to be identical to that of Fig. 1. We will discuss this alternate breakup further at the end of this section.

To illustrate our extension to higher dimension, we will break up the plane-square lattice as shown in Fig. 4. The first step is to introduce an orthonormal change of variable which diagonalizes the intracell interactions. In the two-dimensional plane-square case, we need to diagonalize the quadratic form:

$$E_c = (\nu_{1,1}-\nu_{1,2})^2 + (\nu_{1,2}-\nu_{2,2})^2 + (\nu_{2,1}-\nu_{2,2})^2 + (\nu_{1,1}-\nu_{2,1})^2 \, . \quad (4.1)$$

The one transformation which does this is (d is the dimension, 2)

$$\begin{aligned} u &= 2^{-0.5d}(\nu_{1,1}+\nu_{1,2}+\nu_{2,1}+\nu_{2,2}) \, , \\ v_1 &= 2^{-0.5d}(\nu_{1,1}-\nu_{1,2}+\nu_{2,1}-\nu_{2,2}) \, , \\ v_2 &= 2^{-0.5d}(\nu_{1,1}+\nu_{1,2}-\nu_{2,1}-\nu_{2,2}) \, , \\ v_3 &= 2^{-0.5d}(\nu_{1,1}-\nu_{1,2}-\nu_{2,1}+\nu_{2,2}) \, , \end{aligned} \quad (4.2)$$

where there are 2^d spins per fundamental cell. Reexpressing (4.1) by (4.2) we have

$$E_c = 2v_1^2 + 2v_2^2 + 4v_3^2 \, . \quad (4.3)$$

We may now define (total system size is now taken as $N = 2^{dL}$)

$$\begin{aligned} s_{\alpha,\vec m,0} &= \sum_{\text{cell } \vec m} c_{\alpha\vec\beta}\, \nu_{2\vec m-\vec\beta} \quad (2^{d(L-1)} \text{ vector values of } \vec m), \\ \hat s_{\vec m,0} &= 2^{-d/2} \sum_{\text{cell } \vec m} \nu_{2\vec m-\vec\beta} \, , \end{aligned} \quad (4.4)$$

where $\vec\beta$ is a vector index which runs over the fundamental cell. The index $\alpha = 1, \ldots, 2^d - 1$ indexes the variables which enter into the intracell energy. For the breakup (Fig. 4) for the plane-square lattice, the $c_{\alpha\vec\beta}$ are given by the v_i equations of (4.2). The intracell interaction energy will be given by

$$E_{\vec m} = \sum\nolimits_\alpha \epsilon_\alpha (s_{\alpha,\vec m,0})^2 \, , \quad (4.5)$$

where the ϵ_α are given by comparison with (4.3) for the plane-square-lattice case. Following (2.2) we now introduce recursively the orthonormal transformations

$$\begin{aligned} s_{\alpha,\vec m,l+1} &= \sum_{\text{cell } \vec m} c_{\alpha\vec\beta}\, \hat s_{2\vec m-\vec\beta,l} \, , \\ \hat s_{\vec m,l+1} &= 2^{-d/2} \sum_{\text{cell } \vec m} \hat s_{2\vec m-\vec\beta,l} \, , \end{aligned} \quad (4.6)$$

for $l = 0, 1, \ldots, L-2$, $\alpha = 1, \ldots, 2^d - 1$, and there are $2^{d(L-2-l)}$ vector values of $\vec m$. As before, the final set of variables will be

$$\begin{aligned} & s_{\alpha,\vec m l} \quad (2^{d(L-2-l)} \text{ vector values of } \vec m,\ 2^{d-1} \text{ values of } \alpha,\ l = 0, 1, \ldots, L-1) \\ & \hat s_{1,L-1} = 2^{-0.5dL} \sum\nolimits_j \nu_j \, . \end{aligned} \quad (4.7)$$

In a manner similar to (2.17) we introduce the Hamiltonian

$$\mathcal{H} = J \sum_{l=0}^{L-1} 2^{-l(2-\eta)} \sum_{\vec m} \sum_{\alpha} \epsilon_\alpha (s_{\alpha,\vec m,l})^2 - \frac{d}{2} J \left(\frac{1-2^{L(\eta-d-2)}}{1-2^{\eta-2-d}} \right) \sum_j \nu_j^2 \, . \quad (4.8)$$

The partition function will again be

$$Z = \int \cdots \int \left(\prod d\nu_j \right) e^{-\beta\mathcal{H}} \, . \quad (4.9)$$

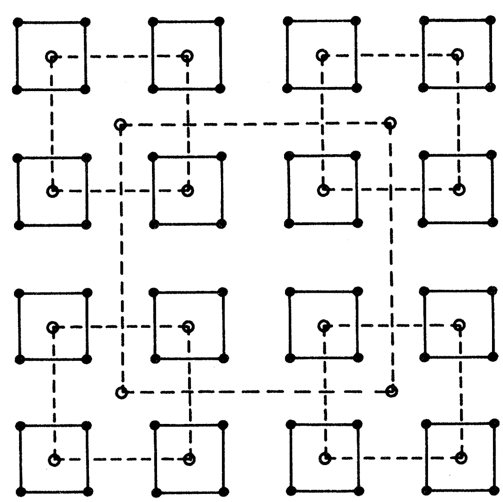

FIG. 4. The spin grouping hierarchy used for our two-dimensional model.

If we introduce the change of variable

$$\nu_{2\vec{m}-\vec{\beta}} = 2^{-d/2} \sum_\alpha d_{\vec{\beta}\alpha} s_{\alpha,\vec{m},0} , \tag{4.10}$$

where the sum on α is extended to $\alpha=0$ and $s_{0,\vec{m},0} \equiv \hat{s}_{\vec{m},0}$. Then we may write (4.9), using the notation of Sec. III, as

$$Z = \int_{-\infty}^{+\infty} \cdots \int \prod_{\vec{m}} (d\hat{s}_{\vec{m},0} \prod_\alpha ds_{\alpha,\vec{m},0}) \exp[-K \sum_{l=1}^{L-1} 2^{-l(2-\eta)} \sum_{\vec{m}} \sum_\alpha \epsilon_\alpha (s_{\alpha,\vec{m},l})^2 - K \sum_{\vec{m}} \sum_\alpha \epsilon_\alpha (s_{\alpha,\vec{m},0})^2 - 2^{-d} \sum_{\vec{m}} \sum_{\vec{\beta}} 2^{d-1} P(\sum_\alpha 2^{-d/2} d_{\vec{\beta}\alpha} s_{\alpha,\vec{m},0})] , \tag{4.11}$$

where P is defined in an analogous manner to (3.2) to absorb the ν_j^2 terms. We may now integrate separately over the variables associated with each cell. This possibility leads us to again introduce

$$I_0(x_0) = \int_{-\infty}^{+\infty} \cdots \int (\prod_{\alpha=1}^{2^d-1} dx_\alpha) \times \exp[-K \sum_{\alpha=1}^{2^d-1} \epsilon_\alpha x_\alpha^2 - 2^{-d} \sum_{\vec{\beta}} Q_0 (\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta}\alpha} x_\alpha)] , \tag{4.12}$$

where

$$Q_0(2^{d/2} x) = 2^{d-1} P(x) , \tag{4.13}$$

and defining

$$Q_1(x) = -2^d \ln[I_0(2^{1-0.5d-0.5\eta} x)/I_0(0)] , \tag{4.14}$$

then

$$Z = (2^{1-\eta/2} I_0(0))^{2^{d(L-1)}} \int_{-\infty}^{+\infty} \cdots \int \prod_{\vec{m}} \left(\frac{d\hat{s}_{\vec{m},0}}{2^{(2-\eta)/2}} \right) \times \exp\left[-K \sum_{l=0}^{L-2} 2^{-l(2-\eta)} \sum_{\vec{m}} \sum_\alpha \frac{\epsilon_\alpha (s_{\alpha,\vec{m},l+1})^2}{2^{2-\eta}} - 2^{-d} \sum_{\vec{m}} Q_1 \left(\frac{2^{d/2} \hat{s}_{\vec{m},0}}{2^{1-\eta/2}} \right) \right] , \tag{4.15}$$

which is again form invariant. Thus, as before, we can define a sequence of functions recursively and evaluate the partition function in terms of them. We therefore introduce, in analogy to (3.9) and (3.10),

$$I_l(x_0) = \int_{-\infty}^{+\infty} \cdots \int (\prod_\alpha^{2^d-1}) dx_\alpha \times \exp[-K \sum_{\alpha=1}^{2^d-1} \epsilon_\alpha x_\alpha^2 - 2^{-d} \sum_{\vec{\beta}} Q_l (\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta}\alpha} x_\alpha)] \tag{4.16}$$

and

$$Q_{l+1}(x) = -2^d \ln[I_l(2^{1-0.5d-0.5\eta} x)/I_l(0)] , \tag{4.17}$$

where $Q_0(x)$ is given by (4.13). (It is to be noted that on the plane-square lattice, and indeed on all the hypercubical lattices the $d_{\vec{\beta}\alpha} = \pm 1$.) The partition function is now given by

$$Z = \prod_{l=0}^{L-1} \{ (2^{1-\eta/2} I_l(0))^{2^{d(L-1-l)}} \} \times \int_{-\infty}^{+\infty} \exp\left[-2^{-d} Q_L \left(\frac{2^{d/2} \hat{s}_{1,L-1}}{2^{(2-\eta)L/2}} \right) \right] \frac{d\hat{s}_{1,L-1}}{2^{(2-\eta)/2}} . \tag{4.18}$$

The susceptibility for $T > T_c$, $H=0$ is given by

$$\chi = m^2 \beta N \frac{\int_{-\infty}^{+\infty} (\hat{s}_{1,L-1})^2 \exp[-2^{-d} Q_L (2^{d/2} \hat{s}_{1,L-1}/2^{(2-\eta)L/2})] d\hat{s}_{1,L-1}}{\int_{-\infty}^{+\infty} \exp[-2^{-d} Q_L (2^{d/2} \hat{s}_{1,L-1}/2^{(2-\eta)L/2})] d\hat{s}_{1,L-1}} . \tag{4.19}$$

We included in Tables I and II are the parameters $d_{\vec{\beta},\alpha}$ for the plane-square and the simple-cubic lattices.

The factors in these tables allow one to write out explicitly Eq. (4.16) for two and three dimensions. It will be further observed that Table II can be gotten from Table I by writing each horizontal row twice and doubling its length by writing it a second time on the same row first with the same signs and next with the signs reversed; a similar construction will extend to all the hypercubical lattices.

In the breakup of the plane-square lattice of Fig. 3 the dimensions are not treated in a symmetrical manner, since one direction is selected for the original spins and another for the first level grouping. This breakup procedure can be extended in an obvious way to any number of dimensions. Now there are d levels required to involve all the spins treated in one unit cell before. Thus for this breakup we introduce the Hamiltonian

$$\mathcal{H} = J \sum_{\mu=0}^{Ld-1} 2^{-\mu(2-\eta)/d} \sum_{m=1}^{2^{Ld-1-\mu}} (s_{m,\mu})^2 - \frac{1}{2} J \left(\frac{1 - 2^{L(\eta-d-2)}}{1 - 2^{(\eta-d-2)/d}} \right) \sum_j \nu_j^2 . \tag{4.20}$$

Since the structure of this breakdown is, for any

TABLE I. $d_{\vec{\beta},\alpha}$ for the plane-square lattice.

$\vec{\beta}\backslash\alpha$	0	1	2	3
0, 0	1	1	1	1
0, 1	1	1	−1	−1
1, 0	1	−1	1	−1
1, 1	1	−1	−1	1
ϵ_α	0	2	2	4

number of dimensions, equivalent to the structure of the Hamiltonian studied in Sec. III, the results can be transcribed to our present case. Thus, we define

$$U(1-\nu_j^2) = \exp\left[-\frac{1}{2}K\left(\frac{1-2^{L(\eta-d-2)}}{1-2^{(\eta-d-2)/d}}\right)\nu_j^2 - \frac{1}{2}P(\nu_j)\right] \quad (4.21)$$

and

$$Q_0(2^{1/2}x) = P(x) \quad . \quad (4.22)$$

We obtain the recursion formulas

$$I_\mu(x) = \int_{-\infty}^{+\infty} dy \exp[-Ky^2 - \tfrac{1}{2}Q_\mu(x+y) - \tfrac{1}{2}Q_\mu(x-y)] \quad (4.23)$$

and

$$Q_{\mu+1}(x) = -2\ln[I_\mu(2^{(1-0.5\eta-0.5d)/d}x)/I_\mu(0)] \quad . \quad (4.24)$$

The partition function is given by

$$Z = \prod_{\mu=0}^{Ld-1}[2^{(1-0.5\eta)/d}I_\mu(0)]2^{Ld-1-\mu} \times \int_{-\infty}^{+\infty}\exp\left[-\frac{1}{2}Q_{Ld}\left(\frac{2^{1/2}\hat{s}_{1,Ld-1}}{2^{(2-\eta)L/2}}\right)\right]\frac{d\hat{s}_{1,Ld-1}}{2^{(2-\eta)L/2}} \quad . \quad (4.25)$$

The results of this breakup procedure, by virtue of the resultant structure, equivalent to a one-dimensional system with a long-range force, reduce the calculation to essentially the equations of Sec. III. There is an immense computational advantage in that only one integration per iteration is required rather 2^d-1 as in the other breakup.

It is instructive at this point to note the similarities and differences to the work of Wilson.[4] The structure of his recursion relations is extremely similar. The only difference arises in his formula corresponding to (4.24). Wilson's result is (4.23) coupled with (4.17). Basically, his recursion relation attempts to advance d levels at one step. Our results show that, at least in the context of our models, a better procedure would be either to use a more couplex recursion relation, such as (4.16) with (4.17), or to take d steps of the type (4.23) and (4.24). In the solution of his equations, Wilson found values of $\nu = 0.61$ and $\gamma = 1.22$ ($\eta = 0$) for the three-dimensional case, which are not very different from the commonly accepted values.

V. SOME PROPERTIES OF THE SOLUTION TO THE FUNCTIONAL EQUATION

In the previous sections we have defined a family of models and expressed their solutions in terms of a sequence of functional iterations. In this section we will deduce some of the properties from these equations. We will use Eqs. (4.16) and (4.17) as the basis for this section. The consequences of (4.23) and (4.24) are quite analogous.

First let us consider the high-temperature region where K is very small. Let us compute from (4.16)

$$F_l(s) = \int_{-\infty}^{+\infty} e^{-sx_0} I_l(x_0)dx_0 \quad . \quad (5.1)$$

If we convert to the variables

$$v_{\vec{\beta}} = \sum_{\alpha=0} d_{\vec{\beta},\alpha}x_\alpha \quad , \quad (5.2)$$

then we have

$$F_l(s) = \int_{-\infty}^{+\infty}\cdots\int\left(\prod_{\vec{\beta}} 2^{-d/2}dv_{\vec{\beta}}\right) \times \exp\{-2^{-d}\sum_{\vec{\beta}}[sv_{\vec{\beta}} + Q_l(v_{\vec{\beta}})] - 2^{-d}K\sum_{\text{bonds}}(v_{\vec{\beta}} - v_{\vec{\gamma}})^2\} \quad . \quad (5.3)$$

From (4.17), we also have

$$F_l(2^{0.5d+0.5\eta-1}s) = \int_{-\infty}^{+\infty} dx \exp(-s\,2^{0.5d+0.5\eta-1}x)I_l(x) = 2^{1-0.5d-0.5\eta}\int_{-\infty}^{+\infty} d\hat{x}\, e^{-s\hat{x}} I_l(2^{1-0.5d-0.5\eta}x) = 2^{1-0.5d-0.5\eta}I_l(0)\int_{-\infty}^{+\infty} d\hat{x} \exp[-s\hat{x} - 2^{-d}Q_{l+1}(\hat{x})] \quad . \quad (5.4)$$

TABLE II. $d_{\vec{\beta},\alpha}$ for the simple-cubic lattice.

$\vec{\beta}\backslash\alpha$	0	1	2	3	4	5	6	7
0, 0, 0	1	1	1	1	1	1	1	1
0, 1, 0	1	1	1	1	−1	−1	−1	−1
1, 0, 0	1	1	−1	−1	1	1	−1	−1
1, 1, 0	1	1	−1	−1	−1	−1	1	1
0, 0, 1	1	−1	1	−1	1	−1	1	−1
0, 1, 1	1	−1	1	−1	−1	1	−1	1
1, 0, 1	1	−1	−1	1	1	−1	−1	1
1, 1, 1	1	−1	−1	1	−1	1	1	−1
ϵ_α	0	2	2	4	2	4	4	6

Thus we obtain, for the Laplace transform of the step $l \to l+1$,

$$\int_{-\infty}^{+\infty} d\hat{x} \exp[-s\hat{x} - 2^{-d} Q_{l+1}(\hat{x})]$$

$$= 2^{0.5d+0.5\eta-1} I_l(0)^{-1} \int_{-\infty}^{+\infty} \cdots \int (\prod_{\vec{\beta}} 2^{-d/2} dv_{\vec{\beta}})$$

$$\times \exp\{-\sum_{\vec{\beta}} [2^{0.5\eta-1-0.5d} s v_{\vec{\beta}} + 2^{-d} Q_l(v_{\vec{\beta}})]$$

$$- 2^{-d} K \sum_{\text{bonds}} (v_{\vec{\beta}} - v_{\vec{\gamma}})^2\} . \quad (5.5)$$

We observe from (5.5) that

$$f_{l+1}(\hat{x}) = \exp[-2^{-d} Q_{l+1}(\hat{x})] \quad (5.6)$$

corresponds to a distribution which is the sum of 2^d independent identical distributions, except for the K-dependent part, and a change of scale on $v_{\vec{\beta}}$. Since $f_0(x)$ is a distribution possessing all moments (and mean zero), it follows from (5.5) that so also do all the higher $f_l(x)$. For $K=0$ we have exactly

$$\int_{-\infty}^{+\infty} d\hat{x} \exp[-s\hat{x} - 2d^{-d} Q_{l+1}(\hat{x})]$$

$$= \prod_{k=0}^{l} \left[\left(\frac{2^{0.5d+0.5\eta-1}}{I_l(0)} \right) 2^{(l-k)d} \{ 2^{-d/2} \int_{-\infty}^{+\infty} d\hat{x} \right.$$

$$\left. \times \exp[-2^{(\eta-2-d)l/2} s\hat{x} - 2^{-d} Q_0(\hat{x})] \}^{2^{dl}-1} \right] . \quad (5.7)$$

In the limit when l is very large, we can expand

$$\{\cdots\}^{2^{dl}-1}$$

$$= \exp[-(2^{dl}-1)(s^2 \kappa_2 2^{(\eta-2-d)l} + s^4 \kappa_4 2^{(\eta-2-d)l} + \cdots)], \quad (5.8)$$

where κ_2, κ_4, ... are constants of order unity obtained from the $F_0(s)$ function. To obtain the scale of the resultant distribution we substitute $s = 2^{0.5(\eta-2)l} \hat{s}$. This leads us to conclude at once that the transform of (5.7) tends to a pure Gaussian in $\hat{s}$, in the limit of large l and $K=0$. This form corresponds to a Gaussian distribution for $f_l(x)$ and the conclusion is independent of the details of the original distribution. (This argument is a variant of the central limit theorem of probability theory.)

Let us now consider the inclusion of the K-dependent terms. For the spin-∞ Ising model the original distribution $f_0(x)$ is bounded in the sense that it vanishes for x greater than some fixed value. Thus if we expand (5.5) in powers of K (for $l=0$) and integrate we get an entire function in K. If we then move this series into the exponent we will get K-dependent terms in the κ_i of (5.8) (again for $l=1$). As the width of the sum of two (or more) finite distributions is again finite, we may repeat this process with full confidence to any l we choose. It should be noted, however, that the limiting function of K as $l \to \infty$ need not be entire.

As a result of this argument, we conclude, for K small enough,

$$f_l(x) \propto \exp\{-2^{(2-\eta)l} x^2 / [2\kappa_2(K, l)] + \cdots\} . \quad (5.9)$$

As long as $\infty > \kappa_2(K) > 0$ we deduce from (4.19) that the magnetic susceptibility is

$$\chi = m^2 \beta N \kappa_2(N, L)/2^d . \quad (5.10)$$

All the calculation difficulties are now concealed in the calculation of $\kappa_2(K, L)$. At the critical point we expect $\kappa_2(K, L) \gg 1$.

In order to obtain an approximate idea concerning the solution of (4.16) and (4.17), we will replace them by a linearized form which, while not a really good approximation, should preserve most of the essential features and is much more tractable. To this end let us expand $Q_l(x)$ to second order about x_0 in (4.16). We then have

$$I_l(x_0) \approx \int_{-\infty}^{+\infty} \cdots \int (\prod_{\alpha=1}^{2^d-1} dx_\alpha)$$

$$\times \exp[-K \sum_{\alpha=1}^{2^d-1} \epsilon_\alpha x_\alpha^2 - Q_l(x_0) - Q_l''(x_0) \sum_{\alpha=1}^{2^d-1} x_\alpha^2] , \quad (5.11)$$

where use has been made of the orthogonality of the $(d_{\vec{\beta},\alpha})$. We may now do the integrals in (5.11) analytically as it is a product of independent Gaussians. The result is

$$I_l(x_0) \approx \prod_{\alpha=1}^{2^d-1} \left(\frac{\pi}{K\epsilon_\alpha + Q_l''(x_0)} \right)^{1/2} e^{-Q_l(x_0)} . \quad (5.12)$$

Thus, by (4.17) we have

$$Q_{l+1}(x) \approx 2^d Q_l(2^{1-0.5d-0.5\eta} x) 2^{d-1}$$

$$\times \sum_{\alpha=1}^{2^d-1} \ln \left(\frac{K\epsilon_\alpha + Q_l''(2^{1-0.5d-0.5\eta} x)}{K\epsilon_\alpha + Q_l''(0)} \right), \quad (5.13)$$

where the condition $Q_l(0)=0$ is maintained. To complete the linearization, we expand the ln terms in (5.13) and obtain

$$Q_{l+1}(x) \approx 2^d \, Q_l(2^{1-0.5d-0.5\eta} x)$$

$$+ \Gamma_d [Q_l''(2^{1-0.5d-0.5\eta}) - Q_l''(0)], \quad (5.14)$$

where we define

$$\Gamma_d = 2^{d-1} \sum_{\alpha=1}^{2^d-1} (K\epsilon_\alpha)^{-1} . \quad (5.15)$$

For $d=2$ and 3, Eq. (5.14) becomes

$$Q_{l+1}(x) = 4Q_l(2^{-\eta/2} x) + 2.5[Q_l''(2^{-\eta/2} x) - Q_l''(0)]/K , \quad (5.16)$$

$$Q_{l+1}(x) = 8Q_l(x/2^{0.5+0.5\eta}) + \frac{29[Q_l''(x/2^{0.5+0.5\eta}) - Q_l''(0)]}{(3K)},$$

respectively. Here we have used $\Gamma_2 = 2.5/K$ and

$\Gamma_3 = 29/(3K)$.

As we remarked at the end of Sec. III, our identification of η as a critical point scattering intensity exponent depends on the Q_l tending to a finite limit. We shall seek such a solution. Suppose that

$$Q_l(x) = \sum_{-\infty}^{+\infty} a_{l,k} x^{2k+\lambda} , \tag{5.17}$$

where we select powers differing by 2 because those are the ones linked by (5.15). Then we must have

$$a_{l+1,k} = a_{l,k}\, 2^{d+(2k+\lambda)(1-0.5d-0.5\eta)} + \Gamma_d a_{l,k+1}(2k+\lambda+2)(2k+\lambda+1)\, 2^{(2k+\lambda)(1-0.5d-0.5\eta)} . \tag{5.18}$$

As we seek the limit $a_{l+1,k} = a_{l,k}$, we can derive

$$a_{k+1} = \frac{1-2^{d+(2k+\lambda)(1-0.5d-0.5\eta)}}{(2k+\lambda+2)(2k+\lambda+1)\Gamma_d\, 2^{(2k+\lambda)(1-0.5d-0.5\eta)}}\, a_k . \tag{5.19}$$

Now for $(1-0.5d-0.5\eta) < 0$, which we anticipate to be the interesting case, the a_k must diverge as $k\to\infty$ like $2^{k^2(d+\eta-2)}$ with constant sign, unless they are identically zero. The divergence implies a series sum of infinity. This cancellation will occur for (define $k=0$ for that one)

$$\lambda = 2d/(d-2+\eta) . \tag{5.20}$$

With this choice, if λ is an integer, then a polynomial solution results; otherwise, an infinite sequence of negative powers of x with finite coefficients diverging like $(2k)!$ and alternating in sign results. According to Carleman's theorem[15] there will be, at most, one function in the cut $(-\infty < x^{-1} < 0)$ plane defined by this formal series solution. The important property of this solution is that for large x it behaves like x^λ, where λ is given by (5.20). Also, it is in the large-x region that we expect approximation (5.14) to be best since there, $Q'' \ll Q$.

It might be wondered, since $Q_l(x) = A_l x^2$ is a solution of (4.16) and (4.17), whether at the critical point when the coefficients of $2^{(2-\eta)l} x^2$ in (5.10) goes to zero, if the result might not be simply $A_l x^2$ with $A_l \ll 2^{(2-\eta)l}$, instead of the solution we have just given. It will turn out that, for large x at least, this type of solution is impossible since, following the procedures described below, we would then find an infinite magnetization per spin, which is impossible for our spins of finite magnitude.

In terms of this solution which we expect to be characteristic of the critical point, we may now investigate the behavior of the magnetization as a function of the magnetic field on the critical isotherm. First, we have seen that the limiting function Q_c depends only on the temperature and not on the magnetic field, as the field only interacts via $\hat{s}_{1,L-1}$. Thus we have, from differentiation of (3.12) as modified by (3.14),

$$M = m(N)^{1/2}\, \frac{\int_{-\infty}^{+\infty} \hat{s}_{1,L-1} \exp[mH\beta(N)^{1/2}\hat{s}_{1,L-1} - 2^{-d} Q_L(2^{d/2}\hat{s}_{1,L-1}/2^{(2-\eta)L/2})]\,d\hat{s}_{1,L-1}}{\int_{-\infty}^{+\infty} \exp[mH\beta(N)^{1/2}\hat{s}_{1,L-1} - 2^{-d} Q_L(2^{d/2}\hat{s}_{1,L-1}/2^{(2-\eta)L/2})]\,d\hat{s}_{1,L-1}} , \tag{5.21}$$

where M is the magnetization. If we transform to the integration variable $y = (N)^{-1/2}\hat{s}_{1,L-1}$, then, provided $(1-0.5d-0.5\eta) < 0$, the region of large argument of Q_L becomes important, and then, using our solution of approximation (5.14) and the saddle point method we obtain

$$M/N \propto H^{1/(\lambda-1)} . \tag{5.22}$$

Thus, for our model, we have derived the relation for the magnetization index δ along the critical isotherm

$$\delta = (d+2-\eta)/(d-2+\eta) , \tag{5.23}$$

provided the denominator is positive, which is the usual result of scaling. The substitution of our solution of approximation (5.14) into (4.19) leads, as expected, to (3.16), confirming the exponent η.

It can be shown that the reason Wilson's model[4] changes to a gaussian model for $d > 4$ is because he chose $P(x) = ax^2 + x^4$. If he had chosen, say $P(x) = ax^2 + |x|^3$, then the change would have come at $d > 6$. These results follow from application of (5.14) for large x. The essential difference seems to be that while it is true that the shaping influence of the recursion relations spreads out from $x=0$, it does not spread fast enough to overwhelm the effect of the initial $P_0(x)$ at those values of x which are important for any fixed $H \neq 0$.

Next, we wish to investigate the behavior near, but slightly above, T_c in order to determine the rate of divergence of the susceptibility. In this analysis we follow the methods of Wilson.[4] We assume we are near T_c and seek to linearize the recursion relations (4.16) and (4.17) in this region. First, in (4.16) we expand the integral to first order in $(K-K_c)$ which yields

$$I_l(x_0) = \int_{-\infty}^{+\infty} \cdots \int \Big(\prod_{\alpha=1}^{2^d-1}\Big) dx_\alpha$$

$$\times \exp\left[-K_c \sum_{\alpha=1}^{2^d-1} \epsilon_\alpha x_\alpha^2 - 2^{-d} \sum_{\vec{\beta}} Q_l\left(\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta},\alpha} x_\alpha\right)\right]$$

$$\times\left[1-(K-K_c)\sum_{\alpha=1}^{2^d-1} \epsilon_\alpha x_\alpha^2 + O((K-K_c)^2)\right] . \qquad (5.24)$$

If we also expand the solution as

$$Q_l(x) = Q_c(x) + (K-K_c)\,\mathcal{Q}_l(x), \qquad (5.25)$$

then substitution into (5.24) and (4.17) yields, to first order in $(K-K_c)$,

$$Q_{l+1}(x) = Q_c(x) + (K-K_c)\,\mathcal{Q}_{l+1}(x)$$

$$= Q_c(x) - (K-K_c)\left\{\sum_{\alpha=1}^{2^d-1} \epsilon_\alpha\left(\langle x_\alpha^2\rangle_c - \langle x_\alpha^2\rangle_0\right)\right.$$

$$\left.+ 2^{-d}\sum_{\vec{\beta}}\left[\left\langle \mathcal{Q}_l\left(\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta},\alpha} x_\alpha\right)\right\rangle_c - \left\langle \mathcal{Q}_l\left(\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta},\alpha} x_\alpha\right)\right\rangle_0\right]\right\}$$

$$+ O((K-K_c)^2), \qquad (5.26)$$

where $\langle\ \rangle_c$ means the expected value where Q_c is used for Q_l in the exponent of (5.24) with $K-K_c=0$, and evaluated at $x_0 = 2^{1-0.5d-0.5\eta}x$. The brackets $\langle\ \rangle_0$ means the same thing except $x_0=0$. If we define

$$q_l(x) = \mathcal{Q}_l(x) - \Omega(x) , \qquad (5.27)$$

where we define

$$\Omega(x) = \sum_{\alpha=1}^{2^d-1} \epsilon_\alpha\left(\langle x_\alpha^2\rangle_c - \langle x_\alpha^2\rangle_0\right)$$

$$+ 2^{-d}\sum_{\beta}\left[\left\langle \Omega\left(\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta},\alpha} x_\alpha\right)\right\rangle_c - \left\langle \Omega\left(\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta},\alpha} x_\alpha\right)\right\rangle_0\right], \qquad (5.28)$$

then

$$q_{l+1}(x) = 2^{-d}\sum_{\vec{\beta}}\left[\left\langle q_l\left(\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta},\alpha} x_\alpha\right)\right\rangle_c\right.$$

$$\left.- \left\langle q_l\left(\sum_{\alpha=0}^{2^d-1} d_{\vec{\beta},\alpha} x_\alpha\right)\right\rangle_0\right] + O(K-K_c) \qquad (5.29)$$

or

$$q_{l+1} = T(q_l) \qquad (5.30)$$

gives, to leading order in $(K-K_c)$, a *linear* recursion relation in the q_l's. Now the behavior of such operators is well known. We expect there to be an eigensolution corresponding to a unique maximum eigenvalue 2^ζ. As the iteration proceeds, this term will dominate and we will have

$$q_l(x) \approx 2^{l\zeta} q_c(x) . \qquad (5.31)$$

Now as the difference between q_l and $\mathcal{Q}_l$ does not grow in l, we can neglect it compared to the dominate term (5.31). Thus, substituting back into (5.25), we obtain, to first order in $(K-K_c)$,

$$Q_l(x) \approx Q_c(x) + (K-K_c)\,2^{l\zeta} q_c(x) . \qquad (5.32)$$

However, we can also derive from (5.6) and (5.9) that

$$Q_l(x) \approx 2^{d-1}\,2^{(2-\eta)l} x^2/\kappa_2(l, K_c+(K-K_c)) . \qquad (5.33)$$

Thus, for very large l where

$$\lim_{l\to\infty} \frac{1}{\kappa^2(l, K_c+(K-K_c))} = g(K-K_c) \qquad (5.34)$$

we must have the relation [from (5.32)–(5.34)]

$$2^{(2-\eta)} g(K-K_c) = g((K-K_c)\,2^\zeta) . \qquad (5.35)$$

Repeated application of (5.35) leads to the conclusion that (in the absence of pathological behavior of g)

$$g(x) \approx A\,x^{(2-\eta)/\zeta} . \qquad (5.36)$$

Now by (5.34) and (5.10) it is concluded that

$$\gamma = (2-\eta)/\zeta , \qquad (5.37)$$

where γ is the usual index of divergence of the high-temperature magnetic susceptibility. It follows then (Fisher[3]) that the correlation-length divergence index is given by

$$\nu = \gamma/(2-\eta) = 1/\zeta . \qquad (5.38)$$

To summarize this section, for our model, a value of η is assumed. From the recursion relations (4.16) and (4.17) [or (4.23) and (4.24) in case of the alternate breakup] the Q_l's are determined as a function of K and, in particular, $\kappa_2(l,K)$. At the value of K for which $\kappa_2(l,K)$ diverges to $+\infty$, we solve for the critical function $Q_c(x)$. From $Q_c(x)$ via (5.29) we construct a linear operator. The maximum eigenvalue of this operator is denoted by 2^ζ, where ζ is a function of η and the dimension. (In the case of the alternate breakup, an eigenvalue of $2^{\zeta/d}$ is obtained.) From ζ, η, and the dimension d we can compute γ, ν, and δ by (5.23), (5.37), and (5.38). We see no reason to suppose that the remainder of the scaling relations do not hold so that the specific-heat index α, and the magnetization index β, and the gap index Δ would be given by

$$\alpha = 2 - d/\zeta ,$$
$$\beta = \tfrac{1}{2}(d-2+\eta)/\zeta , \qquad (5.39)$$
$$\Delta = \tfrac{1}{2}(d+2-\eta)/\zeta .$$

We will not discuss the numerical aspects in this paper. All the calculations that have been actually carried out have been done by Wilson[5] and the procedures used by him are described in his paper.[4] He has approximated the recursion procedure by using (4.23) with (4.17), thereby taking d steps at once. He has obtained the following two solutions:

$$d=2\,,\quad \eta=\tfrac{1}{4}\,,\quad \nu=0.9033\,,\quad \gamma=1.5808\,,$$
$$d=3\,,\quad \eta=0\,,\quad \nu=0.61\,,\quad \gamma=1.22\,, \tag{5.40}$$

which are in rough agreement with the values obtained for the nearest-neighbor Ising model by other methods[3] ($d=2$, $\nu=1.0$, $\gamma=1.75$; $d=3$, $\nu=0.643$, $\gamma=1.25$). He has also approximated [Eq. (3.2)] $P(x)$ by ax^2+x^4.

Wilson[5] has also computed the following solution for the breakup of Fig. 3, again using $P(x)$ approximated bv ax^2+x^4:

$$d=3,\quad \eta=0,\quad \nu=0.6496,\quad \gamma=1.2991,$$
$$\delta=5,\quad \beta=0.3248,\quad \alpha=0.0513\,. \tag{5.41}$$

Note added in proof. It has come to our attention that the breakup (4.23) and (4.24) is equivalent to a special case of Dyson's[16] hierarchical model defined by

$$\mathcal{H}=\sum_l 2^l\, b_l \sum_m (\hat{s}_{m,2})^2\,, \tag{5.42}$$

where we must choose

$$b_l = J(2^{3-\eta}-2)\,2^{-(3-\eta)(l+1)},\quad 0<l<L-1$$
$$b_{L-1}=J\,2^{(3-\eta)(L-1)}\,. \tag{5.43}$$

Thus, Dyson[16] has proved that long-range order exists, for T low enough.

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