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Equilibrium States of the Ising Model in the Two-Phase Region

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We prove that at low enough temperature all translationally invariant equilibrium states for the Ising ferromagnet are a superposition of only two extremal states, i.e., the positively and negatively magnetized pure phases. In particular this proves, at low temperature and in two dimensions, the identity of the spontaneous magnetization and the Onsager's value $M_0 = [1-(sh\beta)^{-4}]^{1/8}$.

I. INTRODUCTION

Consider an Ising ferromagnet enclosed in a box Λ in a square lattice. Assume that if $\underline{\sigma}$ is a spin configuration, then its energy is given by

$$H_0(\underline{\sigma}) = -\frac{1}{2}\sum_{\langle ij\rangle}\sigma_i\sigma_j\ , \tag{1}$$

where $\sum_{\langle ij\rangle}$ denotes, as usual, the sum over the nearest-neighbors pairs in Λ. We have put the strength of the interaction $J=-\frac{1}{2}$ for simplicity, and the external magnetic field $h=0$, since we are interested in the two-phase region.

Suppose also that fixed spins $\underline{\tau}$ are placed on the lattice sites adjacent to the boundary of Λ and define

$$H_{\underline{\tau}}(\underline{\sigma}) = -\frac{1}{2}\sum_{\langle ij\rangle}\sigma_i\sigma_j - \frac{1}{2}\sum_i \sigma_i\ \tau_i - \frac{1}{2}\sum_{\langle ij\rangle}\tau_i\tau_j\ , \tag{2}$$

where the second sum runs over the couples of spins (σ_i, τ_i) adjacent to the boundary; the last term in (2) is $\underline{\sigma}$ independent and has been added only for convenience.

If $p_\Lambda(\underline{\tau})$ is a probability distribution over the set $\underline{\tau}$, we define the probability of a spin configuration $\underline{\sigma}$ as

$$P(\underline{\sigma}) = \sum_{\underline{\tau}}\Big(e^{-\beta H_{\underline{\tau}}(\underline{\sigma})}\Big/\sum_{\underline{\sigma}'} e^{-\beta H_{\underline{\tau}}(\underline{\sigma}')}\Big)\, p_\Lambda(\underline{\tau})\ . \tag{3}$$

It will be more convenient to introduce instead of $P(\underline{\sigma})$ the set of correlation functions

$$\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle_{p_\Lambda} = \textstyle\sum_{\underline{\sigma}}\sigma_{x_1}\cdots\sigma_{x_n}P(\underline{\sigma})\ ,$$

$$n=1,\,2,\,\ldots;\quad x_1,x_2,\ldots \text{ in } \Lambda\ . \tag{4}$$

We shall be interested in the set of functions $\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle$ which can be written [for a suitable choice of $p_\Lambda(\underline{\tau})$] as

$$\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle = \lim_{\Lambda\to\infty}\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle_{p_\Lambda}\ ,\quad n=1,\,2,\ldots \tag{5}$$

for all $x_1, \ldots, x_n$ in the lattice, and furthermore are such that

$$\langle\sigma_{x_1+a}\cdots\sigma_{x_n+a}\rangle \equiv \langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle \text{ for all } a\ . \tag{6}$$

A set or correlation functions verifying (5) and (6) will be called an equilibrium state at temperature β^{-1}.

At β large enough it is known that there are at least two different equilibrium states, which will be denoted as $\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^+$ and $\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^-$. These states can be obtained by choosing in (5) the distribution $p_\Lambda(\underline{\tau})$ to be, respectively,

$$p^{\pm}_\Lambda(\tau) = \prod_i \delta_{\tau_i,\pm 1}\ , \tag{7}$$

i.e., by fixing the boundary spins to be all up or all down. These states have very special physical properties,[1] which explain why they are usually called the up-magnetized and the down-magnetized pure phases; for instance,

$$\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^+ = (-1)^n\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^-\ ,$$

$$\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^+ = \lim_{h\to 0^+}\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle_h\ ,$$

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle^- = \lim_{h \to 0^-} \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_h ,$$

$$\langle \sigma_x \rangle_\pm = m^* , \quad m^* = \left. \frac{\partial f(\beta, h)}{\partial h} \right|_{h = 0^+} \tag{8}$$

where $f(\beta, h)$ denotes the free energy in external field $h = 0$, and $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_h$ the corresponding equilibrium state which is unique and analytic in β, h when $h \neq 0$.

We shall prove that, if β is large enough, any translationally invariant equilibrium state $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle$ [i.e., any set of spin correlation functions verifying (5) and (6)], is such that, for some $0 \leq \alpha \leq 1$,

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle = \alpha \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle^+ + (1 - \alpha) \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle^- . \tag{9}$$

It has been shown that the set of states given by (5) and (6) exhausts all translationally invariant equilibrium states, even if one allows for other reasonable definitions of equilibrium states[2]; hence (9) provides a satisfactory answer to the question of how many pure phases can coexist in the Ising model at low temperature. As discussed in Ref. 3, formula (9) also implies that at least at low temperature and in two dimensions

$$m_0 = [1 - (sh\beta)^{-4}]^{1/8} = \left. \frac{\partial f(\beta, h)}{\partial h} \right|_{h = 0^+} = m^* , \tag{10}$$

i.e., that the Onsager's value for the spontaneous magnetization is the right limit of the derivative of the free energy. The validity, at large β, of formula (10) has been recently also proved by Martin-Löf.[4] The results of this paper provide a slight improvement of the region of β in which (10) has, so far, been verified.

II. FORMULATION OF PROBLEM AND MAIN PROBLEMS

We shall restrict ourselves to the two-dimensional Ising model. However, unless explicitly stated, the results and techniques easily extend to all dimensions.

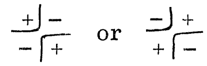
Let Λ be a square box and let $\underline{\tau}$ be a fixed set of values for the spins adjacent to the boundary of Λ. If $\underline{\sigma}$ is a spin configuration in Λ we assign to $\underline{\sigma}$ the relative weight $e^{-\beta H_{\underline{\tau}}(\underline{\sigma})}$

The configuration $\underline{\sigma} \cup \underline{\tau}$ can also be described in a different way: For each lattice bond having opposite spins at its extremes we draw a unit segment cutting the bond, perpendicularly, at its midpoint.

We thereby associate to $\underline{\sigma} \cup \underline{\tau}$ a set of lines which separate regions with spins + from regions with spins −. For convenience we shall deform slightly these lines when one of the following two circumstances happens:

$$\begin{array}{c|c} + & - \\ \hline - & + \end{array} \quad \text{or} \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array} ,$$

and we shall draw instead



After this modification the drawn lines split into a certain number of disjoint self-avoiding contours.[5] There will be k contours $\lambda_1, \ldots, \lambda_k$ which begin and end on the boundary ($k = 0, 1, \ldots$), and h contours $\gamma_1, \ldots, \gamma_h$ which do not touch the boundary and are therefore closed ($h = 0, 1, \ldots$). We denote by $|\gamma|$, $|\lambda|$ the length of a contour. The lines $\lambda_1, \ldots, \lambda_k$ and $\gamma_1, \ldots, \gamma_h$ separate regions of opposite spins.

From the definition of the contours it immediately follows that

$$H_{\underline{\tau}}(\underline{\sigma}) = -\tfrac{1}{2} \text{ (number of bonds with spins at their extremes)} + \textstyle\sum_i |\lambda_i| + \sum_j |\gamma_j| ; \tag{11}$$

hence, if we define

$$Z_{\underline{\tau}}(\Lambda) = \textstyle\sum_{\underline{\sigma}} \exp[-\beta(\sum_i |\lambda_i| + \sum_j |\gamma_j|)] , \tag{12}$$

where $(\lambda_1, \ldots, \lambda_k, \gamma_1, \ldots, \gamma_h)$ is the set of contours associated with a general configuration $\underline{\sigma}$ when, on the boundary of Λ, the spins are fixed to be $\underline{\tau}$, we find that [see (3) and (4)] for $x_1, \ldots, x_n$ in Λ

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{p_\Lambda} = \textstyle\sum_{\underline{\tau}} p_\Lambda(\underline{\tau}) \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\underline{\tau}} , \tag{13}$$

where

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\underline{\tau}} = \sum_{\underline{\sigma}} \sigma_{x_1} \cdots \sigma_{x_n} \frac{\exp[-\beta(\sum_i |\lambda_i| + \sum_j |\gamma_j|)]}{Z_{\underline{\tau}}(\Lambda)} . \tag{14}$$

It is convenient to introduce the translationally averaged correlation functions

$$A \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{p_\Lambda} = \frac{1}{|\Lambda|} \sum_a \langle \sigma_{x_1 + a} \cdots \sigma_{x_n + a} \rangle_{p_\Lambda} , \tag{15}$$

where the sum runs over the a's such that $x_1 + a, \ldots, x_n + a$ are all in Λ. The reason why it is convenient to define (15) lies in the following fact[1,2]:

Proposition 1. If $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle$ is a set of translationally invariant correlation functions as in (5) and (6), then one can find a suitable sequence of distributions $p_\Lambda(\underline{\tau})$ such that, for all $x_1, \ldots, x_n$,

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle = \lim_{\Lambda \to \infty} \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{p_\Lambda} . \tag{16}$$

This result, together with the remark that

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{p_\Lambda} = \sum_{\underline{\tau}} p_\Lambda(\underline{\tau}) \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\underline{\tau}} , \tag{17}$$

will imply the main result of this paper expressed by (9) if the following theorem holds.

Theorem 1. If β is large enough, one can find a family of numbers $\alpha_{\Lambda,\underline{\tau}}$ such that $0 \le \alpha_{\Lambda,\underline{\tau}} \le 1$ and such that

$$A\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle_{\underline{\tau}} = \alpha_{\Lambda,\underline{\tau}}\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^{+} + (1-\alpha_{\Lambda,\underline{\tau}})\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^{-} + \psi(\underline{\tau},\Lambda,x_1,\ldots,x_n)\,, \tag{18}$$

and for all $\underline{\tau}$

$$|\psi(\underline{\tau},\Lambda,x_1,\ldots,x_n)| \le C(\beta,x_1,\ldots,x_n,\Lambda)\,, \tag{19}$$

where $C(\beta, x_1, \ldots, x_n, \Lambda)$ is a translationally invariant function tending to zero as $\Lambda\to\infty$.

A clear physical picture of what is behind the above formal discussion can be obtained by reading the texts of the two lemmas of Sec. III and then Sec. IV. One essentially can say that, at large β (say, $\beta > \ln 3$) if the system is large enough, the "long" contours $\lambda_1, \ldots, \lambda_k$ are very far from all but a negligible fraction of translates of $x_1, \ldots, x_n$; therefore, if $\theta_1, \ldots, \theta_{k+1}$ are the disjoint regions into which Λ is divided by $\lambda_1, \ldots, \lambda_k$, then a translate $x_1+a, \ldots, x_n+a$ of $x_1, \ldots, x_n$ is always in the "middle" of some θ_i, and therefore $\langle\sigma_{x_1+a}\cdots\sigma_{x_n+a}\rangle \cong \langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^{+}$, or $\langle\sigma_{x_1+a}\cdots\sigma_{x_n+a}\rangle \cong \langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^{-}$, according to whether the spins adjacent (from the inside) to the boundary of θ_i are $+1$ or -1.

III. TWO COMBINATORIAL LEMMAS

Let Λ be a square containing L^2 points and fix the spins $\underline{\tau}$ around Λ; consider, for each spin configuration $\underline{\sigma}\cup\underline{\tau}$, the open contours $\lambda_1, \ldots, \lambda_k$. The following lemma says that $\sum_i|\lambda_i|$ cannot be too large if we assign to a spin configuration $\underline{\sigma}$ the weight [see (12)]

$$\exp[-\beta(\sum_i|\lambda_i| + \sum_j|\gamma_j|)]\,. \tag{20}$$

Lemma 1. Suppose the spin configurations are given the probability (20). If $\beta > \ln 3$,

$$p(\tau, L^{4/3}) = P(\sum_i|\lambda_i| \ge L^{4/3}) \le \epsilon(L)\,, \tag{21}$$

where $\epsilon(L)$ is a function independent of the choice of $\underline{\tau}$ and tending to zero when $L\to\infty$.

Proof: We have

$$p(\underline{\tau}, L^{4/3}) = \sum_{\lambda_1,\ldots,\lambda_k,\sum_i|\lambda_i|\ge L^{4/3}}\ \sum_{\gamma_1,\ldots,\gamma_n}\frac{\exp[-\beta(\sum_i|\lambda_i|+\sum_j|\gamma_j|)]}{Z_{\underline{\tau}}(\Lambda)}\,. \tag{22}$$

Suppose we define for a region θ of the lattice

$$\zeta(\theta) = \sum_{\gamma_1,\ldots,\gamma_n\ \theta} e^{-\beta\sum_i|\gamma_i|}\,, \tag{23}$$

where the sum runs over the set of disjoints sets of closed self-avoiding[5] contours $(\gamma_1, \ldots, \gamma_n)$, $n = 0, 1, \ldots,$ contained in θ. In terms of (23) we can write

$$p(\underline{\tau}, L^{4/3}) = \sum_{\lambda_1,\ldots,\lambda_k,\sum_i|\lambda_i|\ge L^{4/3}} e^{-\beta|\lambda_i|}\,\frac{\zeta(\theta_1)\cdots\zeta(\theta_{k+1})}{Z_{\underline{\tau}}(\Lambda)}\,, \tag{24}$$

where $\theta_1, \ldots, \theta_{k+1}$ are the $k+1$ regions into which $\lambda_1, \ldots, \lambda_k$ divide Λ.

Clearly, we have

$$\zeta(\theta_1)\cdots\zeta(\theta_{k+1}) \le \zeta(\Lambda)\,. \tag{25}$$

Denoting now by Λ_1 the square concentric to Λ and with side equal to $L-2$, we then have

$$Z_{\underline{\tau}}(\Lambda) \ge e^{-6\beta L}\,\zeta(\Lambda_1)\,. \tag{26}$$

In fact, this inequality is obtained by restricting the sum defining $Z_{\underline{\tau}}(\Lambda)$ to a few terms, namely, the ones in which the contours $\lambda_1, \ldots, \lambda_k$ have a very special form, while the contours $\gamma_1, \ldots, \gamma_n$ are put in Λ_1; the chosen $\lambda_1, \ldots, \lambda_k$ are constructed as follows: Draw a set of contours $\bar\lambda_1, \ldots, \bar\lambda_k$ which isolate the + spins of $\underline{\tau}$ and run parallel to the boundary of Λ and next to it. Then $\sum_i|\bar\lambda_i| \le 6L$, and therefore (26) follows.

We shall need also the following inequality:

$$\zeta(\Lambda) \le \exp\left(-\frac{(3e^{-\beta})^4}{1-3e^{-\beta}}\,8L\right)\zeta(\Lambda_1)\,; \tag{27}$$

in fact,

$$\zeta(\Lambda) = \sum_{\gamma_1,\ldots,\gamma_n\subset\Lambda} e^{-\beta\sum_i|\gamma_i|} \le \sum_{\gamma_1,\ldots,\gamma_n\subset\Lambda} e^{-\beta\sum_i|\gamma_i|}\,,$$
$$\sum_{\gamma'_1,\ldots,\gamma'_s\not\subset\Lambda_1} e^{-\beta\sum_i|\gamma'_i|} = \zeta(\Lambda_1)\sum_{\gamma'_1,\ldots,\gamma'_s\not\subset\Lambda_1} e^{-\beta\sum_i|\gamma'_i|}\,, \tag{28}$$

where $\gamma'_i, \ldots, \gamma'_s\not\subset\Lambda_1$ means that none of the $\gamma'_1, \ldots, \gamma'_s$ is in Λ_1. We have

$$\sum_{\gamma'_1,\ldots,\gamma'_s\not\subset\Lambda_1} e^{-\beta|\gamma'_i|} \le \sum_{r=1}^{\infty}\frac{1}{r!}\left(\sum_{\gamma'\subset\Lambda_1} e^{-\beta|\gamma'|}\right)^r \le \sum_{r=0}^{\infty}\frac{1}{r!}\left(8L\sum_{\gamma'\ni 0} e^{-\beta|\gamma'|}\right) = \exp\left(8L\sum_{\gamma'\ni 0} e^{-\beta|\gamma'|}\right)\,, \tag{29}$$

where $\sum_{\gamma'\ni 0}$ means the sum over all the contours passing through a point 0 (we are simply saying that, if $\gamma'\not\subset\Lambda_1$, it has to pass through some of the $8L$ points outside Λ_1 and inside Λ). Now it is easy to show that, if $\beta > \ln 3$,

$$\sum_{\gamma'\ni 0} e^{-\beta|\gamma'|} \le \sum_{n=4}^{\infty} 3^n e^{-\beta n} = \frac{(3e^{-\beta})^4}{1-3e^{-\beta}}\,, \tag{30}$$

because the number of contours starting from a given point and having perimeter n is at most 3^n.

Formula (27) then follows.

From (24)–(27) and $\beta > \ln 3$ it follows that

$$p(\underline{\tau}, L^{4/3}) \leq \left(\sum_{\lambda_1, \cdots, \lambda_k, \sum_i |\lambda_i| \geq L^{4/3}} e^{-\beta \sum_i |\lambda_i|} \right) \times \exp\left(\frac{8+6\beta}{1-3e^{-\beta}}\right) L \,. \quad (31)$$

To estimate the sum in the bracket we again observe that the number of contours starting on a given point and having perimeter l is not larger than 3^l, and furthermore we observe that, if $\underline{\tau}$ is fixed, the number k of contours $\lambda_1, \ldots, \lambda_k$ is also assigned and there are at most

$$\binom{2k}{k}$$

ways of choosing k beginning points between the $2k$ that are possible; hence, since $2k \leq 4L$,

$$\sum_{\lambda_1, \cdots, \lambda_k, \sum_i |\lambda_i| \geq L^{4/3}} \prod_{i=1}^{k} e^{-\beta |\lambda_i|}$$

$$\leq \binom{2k}{k} \sum_{l_1, \cdots, l_k, \sum_i l_i \geq L^{4/3}} 3^{l_1} e^{-\beta l_1} \cdots 3^{l_k} e^{-\beta l_k}$$

$$\leq \binom{2k}{k} \binom{L^{4/3}}{k} (3e^{-\beta})^{L^{4/3}} \sum_{l_1', \cdots, l_k' \geq 0} 3^{l_1'} e^{-\beta l_1'} \cdots 3^{L_k'} e^{-\beta L_k'}$$

$$\leq \left(\frac{4}{1-3e^{-\beta}}\right)^{2L} \max_{k \leq 2L} \binom{L^{4/3}}{k} (3e^{-\beta})^{L^{4/3}} \,, \quad (32)$$

we have

$$p(\underline{\tau}, L^{4/3}) \leq \left(\frac{4}{1-3e^{-\beta}}\right)^{2L} \left(\exp \frac{8+6\beta}{1-3e^{-\beta}}\right)^{L} \times \max_{k \leq 2L} \binom{L^{4/3}}{k} (3e^{-\beta})^{L^{4/3}} = \epsilon(L) \,, \quad (33)$$

and Lemma 1 follows, by noting that for large L,

$$\binom{L^{4/3}}{k} \leq \binom{L^{4/3}}{2L} \,,$$

which has a bound proportional to $e^{\text{const}\, L \ln L}$.

We next prove another simple lemma.

Let θ be an arbitrary region and let $x_1, \ldots, x_n$ be n points; suppose we fix the spins adjacent to the boundary to be all $+1$ (or all -1) and assign, using the notation of Sec. I, the relative probability

$$\exp(-\beta H_{\underline{\tau}}(\underline{\sigma})) \,, \quad (34)$$

and denote by $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\theta, \pm}$ the average of $\sigma_{x_1} \cdots \sigma_{x_n}$ under the probability distribution generated by (34). Then the following lemma holds.

Lemma 2. If D is the distance of $x_1, \ldots, x_n \in \theta$ from the boundary of θ, we have

$$|\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\theta, +} - \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle^{+}| \leq f(x_1, \ldots, x_n, D) \,, \quad (35)$$

where $f(x_1, \ldots, x_n, D)$ is a translationally invariant function tending to zero as $D \to \infty$. A similar result holds for $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\theta, -}$.

Proof: In fact, the second Griffiths inequality[6] implies that $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\theta, +}$ decreases when θ increases. Call Q_D a square centered at the baricenter of $x_1, \ldots, x_n$ and with side $\sqrt{D}$, then

$$0 \leq \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\theta, +} - \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{+}$$

$$\leq \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{Q_D, +} - \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle^{+}$$

$$= f(x_1, \ldots, x_n, D) \,. \quad (36)$$

The function $f(x_1, \ldots, x_n, D)$ is translationally invariant and decreases to zero as $D \to \infty$ (again by the second Griffiths inequality).

IV. PROOF OF THEOREM 1

Let $\underline{\tau}$ be a fixed boundary condition and let $\theta_1, \ldots, \theta_{k+1}$ be the $k+1$ regions into which contours $\lambda_1, \ldots, \lambda_k$ associated with a given configuration $\underline{\sigma}$ divide the region Λ. We call θ_i a "positive" region if the spins adjacent to the boundary of θ_i from the inside are $+1$, and a "negative" region if they are -1.

We can clearly compute $\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\underline{\tau}}$ as

$$\langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\underline{\tau}} = \sum_{\lambda_1, \ldots, \lambda_k, \sum_i |\lambda_i| \geq L^{4/3}} \sum_{\underline{\sigma}}{}^{*} \sigma_{x_1} \cdots \sigma_{x_n} \times \frac{\exp[-\beta(\sum_i |\lambda_i| + \sum_j |\gamma_j|)]}{Z_{\underline{\tau}}(\Lambda)} + \epsilon(x_1, \ldots, x_n, \underline{\tau}) \,, \quad (37)$$

where the first sum runs over the sets of open contours associated to some spin configuration and the second sum runs over the spin configurations $\underline{\sigma}$ which have $\lambda_1, \ldots, \lambda_k$ as associated open contours; the function $\epsilon(x_1, \ldots, x_n, \underline{\tau})$ is such that (see Lemma 1)

$$|\epsilon(x_1, \ldots, x_n, \tau)| \leq \epsilon(L) \,. \quad (38)$$

Let $A_{\lambda_1, \ldots, \lambda_k}$ be the set of points at a distance not exceeding $\frac{1}{2}(L^{1/3})$ from the set of contours $\lambda_1, \ldots, \lambda_k$. The number of points in $A_{\lambda_1, \ldots, \lambda_k}$ is not larger than $L^{5/3}$ if $\sum_i |\lambda_i| \leq L^{4/3}$.

If $x_1, \ldots, x_n \notin A_{\lambda_1, \ldots, \lambda_k}$ then $x_1, \ldots, x_n$ must be in θ_i for some i, and the sum $\sum^*$ appearing in (37) can be written as

$$\sum_{\underline{\sigma}}{}^{*} \sigma_{x_1} \cdots \sigma_{x_n} \frac{\exp[-\beta(\sum_i |\lambda_i| + \sum_j |\gamma_j|)]}{Z_{\underline{\tau}}(\Lambda)} = p_{\underline{\tau}}(\lambda_1, \ldots, \lambda_k) \langle \sigma_{x_1} \cdots \sigma_{x_n} \rangle_{\theta_i, \pm} \,, \quad (39)$$

where the sign has to be chosen to be the same as the one of the region θ_i, and $p_{\underline{\tau}}(\lambda_1, \ldots, \lambda_k)$ is the probability of the spin configurations containing $\lambda_1, \ldots,$

λ_k as open contours (at fixed boundary condition $\underline{\tau}$). This formula simply follows from the fact that if λ_1, ..., λ_k are fixed, then the probability distributions of the spins inside the regions θ_i are independent and generated by the weight (34).

Let $N^+(\lambda_1, \ldots, \lambda_k)$ be the number of points in the positive θ_i's; then put

$$\alpha_{\Lambda, \underline{\tau}} = \sum_{\underline{\sigma}} p_{\underline{\tau}}(\lambda_1, \ldots, \lambda_k) \frac{N^+(\lambda_1, \ldots, \lambda_k)}{L^2} . \tag{40}$$

We find, using (37)–(39) and definition (15), that

$$\begin{aligned} |A\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle_{\underline{\tau}} - \alpha_{\Lambda,\tau}\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^+ \\ -(1-\alpha_{\Lambda,\tau})\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^-| \\ \le \epsilon(L) + 2f(x_1, \ldots, x_n, L^{1/3}) \\ + C(L^{5/3}/L^2) , \end{aligned} \tag{41}$$

where the first term comes from the error term in (37), the second comes from the replacement of $\langle\sigma_{x_1+a}\cdots\sigma_{x_n+a}\rangle_{\theta_i,\pm}$, with $\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^\pm$ for all the a's such that $(x_1+a, \ldots, x_n+a)$ has no points in $A_{\lambda_1,\ldots,\lambda_k}$, and from the use of Lemma 2 to estimate the involved error; finally C is $\underline{\tau}$ independent and depends only on $\max_{ij}|x_i - x_j|$, and comes from the contribution of the a's such that $(x_1+a, \ldots, x_n+a)$ has points in common with $A_{\lambda_1,\ldots,\lambda_k}$.

Formula (41) proves Theorem 1.

V. FINAL REMARKS

If we examine critically the calculations performed in the paper in the two-dimensional case we realize that a simple improvement can be achieved by replacing the estimate 3^l on the number of contours of length l by μ^l, where μ is any number larger than the connective constant μ_0.[5] This allows us to replace 3 in (27) and (32) by μ_0, and the results discussed in this paper hold for

$$\beta > \ln\mu_0 ; \tag{42}$$

as is known, $\ln\mu_0$ is an estimate from below of the critical β_0 to within 9%.[7] Notice that, in order to get the proof of the statement that there are only two extremal translationally invariant equilibrium states, we have used neither the Minlos-Sinai equations nor the fact that $\sum_{\gamma\ni 0} e^{-\beta|\gamma|}$ is small (we used only that $\sum_{\gamma\ni 0} e^{-\beta|\gamma|}$ converges fast enough); this is the reason why our results hold so close to the critical temperature.

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[1]D. Ruelle, J. Math Phys. (to be published). The expressions (8) hold for all β. Assuming that β is large enough, they are a consequence of the results of Minlos and Sinai, which give also a more detailed picture of the states $\langle\sigma_{x_1}\cdots\sigma_{x_n}\rangle^\pm$. See R. A. Minlos and Y. G. Sinai, Mat. Sb. 73, 375 (1967) [Math. U. S. S. R. 2, 335 (1968)].

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[5]Notice that in this definition of self-avoiding contours we allow the contour to meet itself at some points, as in the pictures. In particular, the connective constant $\tilde{\mu}_0$ associated with such contours is not *a priori* the same as the usual connective constant. However, at the expense of technical complications, one could work with really self-avoiding contours and get formula (43) with $\ln\mu_0$ replacing $\ln\tilde{\mu}_0$.

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