an antiferromagnetic surface layer described by Meiklejohn[12] is a special pseudohelicoidal structure where Pincus's calculation is still valid. Afterwards, Soohoo[13] and Wigen[14] confirmed and developed these surface spin pinning mechanisms in agreement with some experimental results.

However, the pseudohelicoidal structure which generalizes the previous simple cases differs from a real helicoidal one because a few layers only are perturbed and thus the structure is not periodical. The shorter the range of the pseudohelicoidal structure is, the stronger the pinning.

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PHYSICAL REVIEW B VOLUME 5, NUMBER 1 1 JANUARY 1972

General Theorems on Ferromagnetism and Ferromagnetic Spin Waves*

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(Received 9 June 1970; revised manuscript received 1 March 1971)

Ideal ferromagnetism in perfect crystals (and/or in free space), where spin-orbit interactions may be neglected, is investigated at zero temperature under the following conditions: The fermion system considered here should have the inversion symmetry of the space coordinates and the thermodynamic limit. Its ground state is nondegenerate for a fixed eigenvalue of S_z. In other respects the ferromagnets considered are quite general and may cover all possible types of ferromagnetism: insulators, metals, and free fermions. Dynamical spin-spin correlation functions are studied. Sum rules for them are developed so as to exclude the contributions from Stoner excitations. Spin waves are considered by means of these sum rules. In the case of complete ferromagnetism (all electron spins being aligned in one direction), it is shown rigorously that no consistent result can be obtained; the excitation energies of magnons cannot be finite in the form of Dq^2, but are vanishing. This suggests that the complete ferromagnetism, if it could exist, must violate one of the above conditions.

I. INTRODUCTION

Ferromagnetism and ferromagnetic spin waves have been discussed for many years.[1,2] The discussions have mostly concerned specific models, such as ideal Heisenberg ferromagnets and itinerant-electron models. If the spin-orbit interaction is neglected and hence the spontaneous magnetization of a ferromagnet may take any direction, then a well-defined acoustic spin-wave mode with frequency spectrum $\omega = Dq^2$ is always obtained for small values of the wave number q. This fact, which can be easily inferred from the results derived for the specific models, has also been discussed by some authors[3] from the point of view of the Goldstone theorem[4] relating the acoustic mode with symmetry breaking down. However, such discussions have again been confined to the models.

A simple and undoubtedly clear derivation of the magnon mode was given for the ideal Heisenberg ferromagnet,[1] which is, however, an oversimplification of real ferromagnets. Actually, the problems concerning the nonorthogonality and variety of ionic configurations of atoms in a solid must inevitably be kept in mind whenever we go beyond the simple-minded pictures in which we neglect the nonorthogonality and assume fixed atomic orbital configurations. These problems will destroy all the advantage of the Heisenberg model in its mathematical simplicity even in the case of ferromagnetic insulators.

In the spin-wave theory[2] of metallic ferromagnets, which was initiated by the famous work of Herring and Kittel, we have been dealing with some models which are again approximate pictures. Yet we have not been successful in rigorously deriving ferromagnetism and ferromagnetic spin waves in those models; some approximations have always been introduced in deriving the spin waves. From a physical point of view, all these approximate theories are significant in the sense that they can describe some characteristic features of metallic ferromagnetism and its spin waves. However, for people who prefer mathematical rigor to qualitative arguments, these theories would be unsatisfactory. They could claim, first of all, that the very existence of ferromagnetism has not yet been proved in those models usually assumed or approximately shown to be ferromagnetic. The only models in which ferromagnetism has been rigorously shown to exist are (1) one-dimensional fermion gas with hard-core interactions (Lieb and Mattis[5]), and (2) the so-called Hubbard model in the limit of the very narrow band just half-filled, with a single hole inserted (Nagaoka[6]). We will not consider here any model in which ferromagnetism arises by artificial means (for instance, by including the ferromagnetic spin-spin interactions in the original Hamiltonian).

The Lieb-Mattis theorem states generally that $E(S+1) \geqq E(S)$ in any one-dimensional nonrelativistic system where $E(S)$ is the lowest energy for the states possessing the total spin S. The theorem states further that the equality holds if and only if the interaction potential among the particles is pathological, i.e., if it contains hard cores. The real significance of the Lieb-Mattis theorem is that it reveals that ferromagnetism is very difficult to achieve. However, we note here especially that in the pathological case the ferromagnetism may appear in the ground states which are macroscopically degenerate, though from a practical point of view the situation is hardly ascribed to ferromagnetism.

The narrow-band limit of the Hubbard Hamiltonian considered by Nagaoka gives us a more important example for the existence of ferromagnetism. In the Nagaoka system (i) the intra-atomic Coulomb repulsion is first taken to be infinity (hard cores), (ii) transfer integrals are considered only between the nearest-neighbor sites, and (iii) N (the number of electrons) is taken to be equal to (the number of lattice points) -1. The result is that the ground state is ferromagnetic with $S=\frac{1}{2}N$ if the lattice is "bipartite" (Lieb's terminology). This theorem was first obtained by Thouless,[7] but his proof was regarded as incomplete.[6] However, Lieb[8] has recently shown how easily the Thouless proof can be made complete. Indeed, Lieb's proof is much simpler and clearer than Nagaoka's. In the Nagaoka system the existence of only a "single" hole is crucial, so that it would be difficult to extend his result to the thermodynamic limit. One may even claim that the ferromagnetism would perhaps be destroyed if there are two holes; each hole gives a spin alignment of a group of the medium electrons, but the two such groups could interact antiferromagnetically.

In view of these problems we will give up looking for good models for actual ferromagnetism, but we will simply "assume" that the ground state is ferromagnetic. For mathematical simplicity we consider here only the absolute zero temperature and only the systems satisfying the following conditions: (a) Relativistic effects are completely neglected, (b) the inversion symmetry is there (in order to avoid more lengthy analysis, another condition will be imposed, though it is not necessary, which says that the fermion system should be either in a perfect crystal or in a free space), (c) the ferromagnetic ground states are nondegenerate except for the intrinsic spin degeneracy for different eigenvalues of S_z (this is referred to as the condition of "quasinondegeneracy" throughout this paper), and (d) the thermodynamic limit exists.

A ferromagnetic ground state is supposed to be an eigenstate of S^2, i.e., total spin vector squared, as well as that of S_z. This is the most characteristic feature of ferromagnetism. By this feature the ferromagnetic ground state is found to be a pure quantum-mechanical eigenstate of the Hamiltonian, while other macroscopic states with long-range orders, such as the Néel state of antiferromagnetism, etc., cannot be energy eigenstates but are simply wave packets.

The spin-wave mode is said to be observed whenever, say, the inelastic small-angle scattering of thermal neutrons shows a sharp peak as a function of the energy loss ω of the bombarding neutrons. ($\hbar$ is taken to be unity throughout this paper.) Thus, to speak about the magnon mode exactly, we necessarily need to have some knowledge of the dynamical spin-spin correlation functions which are directly detected by neutron scatterings. For this purpose, sum rules for the dynamical correlation functions will be developed. Many authors[9–11] seem to have already reached sum rules that perhaps resemble the f-sum rules for the dynamical density-density correlation functions.[12] However, these conventional sum rules are not enough to lead to the required conclusions for the acoustic magnon mode. The difficulty comes, as will be shown later, from the contribution of Stoner excitations (namely, the individual-type excitations of electrons from up-spin states to down-spin ones if the direction of spontaneous magnetization is upward) to the transverse spin-spin correlation function. The Stoner excitations always have finite excitation energies as $q \to 0$, while the spin wave in this limit has vanishing excitation energy. Then, as we step forward to the first, second, and higher moments of the transverse spin-spin correlation function for small q, the contribution of the Stoner excitations becomes more and more dominant.

In order to exclude effectively the contributions from the Stoner excitations, we will try to develop new sum rules. Indeed, these sum rules take into account only a narrow frequency range that tends to zero as $q \to 0$. The sum rules take on especially simple closed forms if all the fermion spins are aligned in one direction in the ground state (such ferromagnetism will be referred to as complete ferromagnetism from now on). The main conclusion of this paper, which will be given in Sec. IV, is that the exchange stiffness D of the spin-wave spectrum $\omega = Dq^2$ in the long-wavelength limit is shown to vanish in complete ferromagnetism. Although this conclusion itself is derived rigorously and generally, it would be safest to say that its physical implications have not yet been derived with complete mathematical rigor. Our interpretation is that complete ferromagnetism can never appear in a stable (quasinondegenerate) ground state of any nonrelativistic system with inversion symmetry. As will be discussed later, this interpretation is the most plausible one. This is not in contradiction with the Lieb-Mattis conclusion of the possibility for complete ferromagnetism because the system they considered violates seriously the condition of quasinondegeneracy for the ground state.

Our result may not necessarily hold in artificial systems, where thermodynamic limits do not exist. Moreover, our conclusion can only be applied to general nonrelativistic many-particle systems, and may not be true for special models that are great simplifications of real systems. For example, the Heisenberg Hamiltonian and a restricted Hamiltonian (such as Hubbard's) given under the tight-binding approximation are beyond the scope of the present treatment. Indeed, the systems described by the Heisenberg Hamiltonian can be ferromagnetic at $T = 0$ °K whenever the exchange interactions have ferromagnetic sign. After all, our conclusion does not exclude the possibility of having all electron spins up in some valence or conduction bands while the core electrons do not show such complete spin polarization. Such a conclusion, when applied to the electrons in real crystals, would not be absurd at all, though it might not be significant from a practical point of view. It would be of some value, however, to say that the electron gas can never show such complete ferromagnetism, and also that an assembly of He^3 atoms, whether it is a solid or a liquid, can never be in the state of complete ferromagnetism, at least if all the relativistic effects are neglected.

The above conclusion presents these questions: Why can there be stable (incomplete) ferromagnetic ground states in nonrelativistic systems, and how can D be positive there? Although no rigorous mathematical proof has yet been given for the existence of such ferromagnetic systems satisfying the conditions (b), (c), and (d), there is little doubt of their existence. In Sec. V, an explanation is given for the possibility of $D > 0$ in partial ferromagnetism in spite of the conclusion of $D = 0$ for complete ferromagnetism.

For the sake of simplicity we will refer to the fermions simply as electrons.

II. BASIC FORMULAS FOR TRANSVERSE SPIN-SPIN CORRELATION FUNCTION AT $T = 0$ °K

The spin-spin correlation functions are defined by

$$S_{\mu\nu}(\vec{q}, \omega) = (1/2\pi) \int_{-\infty}^{\infty} dt\, e^{i\omega t - 0^+|t|} \langle \Psi_0 | M_\mu(\vec{q}, t) M_\nu(-\vec{q}) | \Psi_0 \rangle \, , \quad (1)$$

where Ψ_0 stands for the ground-state wave function, μ and ν indicate the coordinate axes ($x, y,$ or z), $M_\mu(\vec{q})$ is the Fourier transform of the μ component of the spin density, and $M_\mu(\vec{q}, t) = e^{iHt} M_\mu(\vec{q}) \times e^{-iHt}$, H being the Hamiltonian. Denoting the coordinates of the jth electron by $\vec{r}_j$ and the Pauli spin matrices

$$\sigma^x = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -\frac{1}{2}i \\ \frac{1}{2}i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

of the jth electron by $\vec{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$, we have

$$M(\vec{q}) = \sum_j e^{i\vec{q}\cdot\vec{r}_j} \vec{\sigma}_j \, . \quad (2)$$

The ground state belongs to a definite eigenstate of the operator

$$S^2 = M_x(0)^2 + M_y(0)^2 + M_z(0)^2 \, .$$

Let the eigenvalue be $M(M+1)$ so that

$$S^2 \Psi_0 = M(M+1)\Psi_0 \, .$$

Since the ground state is assumed to be ferromagnetic, M/N is nonvanishing even in the macroscopic limit ($N \to \infty$, V/N = fixed value, where N is the total number of electrons and V is the volume of the system). M is called the spontaneous magnetization. Let us take the z axis along the direction of the spontaneous magnetization so that

$$S_z \Psi_0 = M \Psi_0 \, ,$$

where

$$S_z \equiv M_z(0) = \sum_j \sigma_j^z \, .$$

Let us introduce the notations

$$\sigma_j^{\pm} \equiv \frac{1}{\sqrt{2}} (\sigma_j^x \pm i\sigma_j^y)$$

and

$$M_\pm(\vec{q}) = \frac{1}{\sqrt{2}}[M_x(\vec{q}) \pm iM_y(\vec{q})] = \sum_j e^{i\vec{q}\cdot\vec{r}_j}\sigma_j^\pm \qquad (2')$$

and define the transverse correlation functions by

$$S_{+-}(\vec{q},\omega) = (1/2\pi)\int_{-\infty}^{\infty} dt\, e^{i\omega t - O^+|t|}\langle\Psi_0|M_+(\vec{q},t)M_-(-\vec{q})|\Psi_0\rangle \qquad (3)$$

and

$S_{-+}(\vec{q},\omega)$ = (the subscripts + and − are interchanged in the above expression) .

These correlation functions appear in the cross section for magnetic neutron scattering, $d^2\sigma/d\Omega\, d\omega$, that is, the cross section for the scattering of neutrons in the direction represented by the solid angle $d\Omega$ with the energy losses lying between ω and $\omega + d\omega$. To be more specific, let us introduce the wave vector $\vec{k}$ of the incident neutrons and $\vec{k}'$ of the scattered ones. Then the momentum transferred from a neutron to the system is $\vec{q} = \vec{k} - \vec{k}'$, and the energy loss is $\omega = k^2/2m_N - k'^2/2m_N$. With this notation, the cross section under the Born approximation is expressed as

$$\frac{d^2\sigma}{d\Omega\, d\omega} \propto \frac{k'}{k}\sum_{\mu\nu}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)S_{\mu\nu}(\vec{q},\omega)$$
$$= \frac{k'}{k}\left\{\frac{1}{2}\left(1 + \frac{q_z^2}{q^2}\right)[S_{+-}(\vec{q},\omega) + S_{-+}(\vec{q},\omega)] + \left(1 - \frac{q_z^2}{q^2}\right)S_{zz}(\vec{q},\omega)\right\} , \qquad (4)$$

where the contribution from orbital magnetism of electrons has been ignored. In referring to the magnon excitations, we are interested in the component of the scattering related to the transverse correlation functions.

Let us introduce also a set of energy eigenfunctions Ψ_n ($n = 0, 1, 2, \ldots$), i.e.,

$$H\Psi_n = E_n\Psi_n .$$

Then from (3) the transverse correlation function is obtained as

$$S_{+-}(\vec{q},\omega) = \frac{1}{2\pi i}\sum_n |\langle\Psi_n|M_-(\vec{q})|\Psi_0\rangle|^2 \times\left[\frac{1}{\omega - (E_n - E_0) - iO^+} - \frac{1}{\omega - (E_n - E_0) + iO^+}\right]$$
$$= \sum_n |\langle\Psi_n|M_-(\vec{q})|\Psi_0\rangle|^2\delta(\omega - (E_n - E_0)) \qquad (5)$$

or

$$S_{+-}(\vec{q},\omega) = \frac{1}{\pi}\operatorname{Im}\langle\Psi_0|M_+(-\vec{q})\frac{1}{H - E_0 - \omega - iO^+}M_-(\vec{q})|\Psi_0\rangle$$
$$= \langle\Psi_0|M_+(-\vec{q})\delta(H - E_0 - \omega)M_-(\vec{q})|\Psi_0\rangle . \qquad (6)$$

On the other hand, the transverse dynamical spin susceptibility is defined by

$$\chi_{+-}(\vec{q},\omega + iO^+) = i\int_0^\infty dt\, e^{i\omega t - O^+|t|}\langle\Psi_0|[M_+(\vec{q},t), M_-(-\vec{q})]|\Psi_0\rangle . \qquad (7)$$

This describes the linear response of $M_+(q)$ against the circularly polarized transverse magnetic field,

$$H_x(\vec{r},t) + iH_y(\vec{r},t) = H_+(\vec{q},\omega)e^{i\vec{q}\cdot\vec{r} - i\omega t} .$$

The dynamical susceptibility defined above may be generalized to the case of complex values for the frequency variable by introducing

$$\chi_{+-}(\vec{q},z) = \left\langle\Psi_0\middle|M_+(\vec{q})\frac{1}{H - E_0 - z}M_-(-\vec{q})\middle|\Psi_0\right\rangle + \left\langle\Psi_0\middle|M_-(-\vec{q})\frac{1}{H - E_0 + z}M_+(\vec{q})\middle|\Psi_0\right\rangle . \qquad (8)$$

The expression (7) is just the special case of this generalized function. This is expressed also as

$$\chi_{+-}(\vec{q},z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A_{+-}(\vec{q},x)\,dx}{x - z} , \qquad (9)$$

where

$$A_{+-}(\vec{q},\omega) = \pi\langle\Psi_0|M_+(\vec{q})\delta(H - E_0 - \omega)M_-(-\vec{q})|\Psi_0\rangle \quad \text{for } \omega > 0$$
$$= -\pi\langle\Psi_0|M_-(-\vec{q})\delta(H - E_0 + \omega)M_+(\vec{q})|\Psi_0\rangle \quad \text{for } \omega < 0 .$$

A more concise expression for A is

$$A_{+-}(\vec{q},\omega) = \pi\{\langle\Psi_0|M_+(\vec{q})\delta(H - E_0 - \omega)M_-(-\vec{q})|\Psi_0\rangle - \langle\Psi_0|M_-(-\vec{q})\delta(H - E_0 + \omega)M_+(\vec{q})|\Psi_0\rangle\} \qquad (10)$$

or

$$A_{+-}(\vec{q},\omega) = \tfrac{1}{2}\int_{-\infty}^{\infty} dt\, e^{i\omega t - O^+|t|}\langle\Psi_0|[M_+(\vec{q},t), M_-(-\vec{q})]|\Psi_0\rangle . \qquad (11)$$

For this so-called spectral density function A we have the well-known sum rule

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}A_{+-}(\vec{q},\omega) = M . \qquad (12)$$

Throughout this paper we consider a nonrelativistic system of electrons described by the Hamiltonian

$$H = \sum_j p_j^2/2m + V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) , \qquad (13)$$

where the potential energy $V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ is assumed to have the inversion symmetry, i.e.,

$$V(-\vec{r}_1, -\vec{r}_2, \ldots, -\vec{r}_N) = V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$$

for an appropriate choice of the origin of the coordinates. Our condition is that there should be at least one such origin, i.e., the center for the inversion transformation. For mathematical simplicity we will confine ourselves to the consideration of the electrons in perfect crystals or in free space, where $V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ has the periodicity of the lattice or the translational invariance, though this condition is not inevitable.

Then we get the following equation of continuity:

$$[H, M_\pm(\vec{q})] = \sum_j e^{i\vec{q}\cdot\vec{r}_j}(\vec{q}\cdot\vec{p}_j/m)\sigma_j^\pm + (q^2/2m)\sum_j e^{i\vec{q}\cdot\vec{r}_j}\sigma_j^\pm \tag{14}$$

$$= \vec{q}\cdot\vec{J}_\pm(q) , \tag{15}$$

with

$$\vec{J}_\pm(q) = \frac{1}{2m}\sum_j (e^{i\vec{q}\cdot\vec{r}_j}\vec{p}_j + \vec{p}_j e^{i\vec{q}\cdot\vec{r}_j})\sigma_j^\pm \quad . \tag{16}$$

From (10) and (14) we obtain the sum rule[9–11]

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\, \omega A_{+-}(\vec{q}, \omega) = \frac{N}{4m} q^2 \tag{17}$$

for arbitrary values of q, where N is the total number of electrons. The proof of (17) is given by the following:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\, \omega A_{+-}(\vec{q}, \omega) &= \langle\Psi_0|M_+(\vec{q})(H-E_0)M_-(-\vec{q})|\Psi_0\rangle \\ &\quad + \langle\Psi_0|M_-(-\vec{q})(H-E_0)M_+(\vec{q})|\Psi_0\rangle \\ &= \langle\Psi_0|[M_+(\vec{q}),[H, M_-(-\vec{q})]]|\Psi_0\rangle \\ &= -(q^2/2m)M + (q^2/m)\sum_j \langle\Psi_0|\sigma_j^+\sigma_j^-|\Psi_0\rangle ,\end{aligned} \tag{17'}$$

$$\begin{aligned}\sigma_j^+\sigma_j^- &= \tfrac{1}{2}\{(\sigma_j^x)^2 + (\sigma_j^y)^2 - i[\sigma_j^x, \sigma_j^y]\} \\ &= \tfrac{1}{2}\{\sigma_j^2 - (\sigma_j^z)^2 + \sigma_j^z\} \\ &= \tfrac{1}{4} + \tfrac{1}{2}\sigma_j^z \quad .\end{aligned}$$

Substituting this into (17′), we arrive at (17).

Similarly we get

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} A_{-+}(\vec{q}, \omega) = -M \tag{12'}$$

and

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\, \omega A_{-+}(\vec{q}, \omega) = \frac{N}{4m} q^2 \tag{17''}$$

for any value of q.

III. CONVENTIONAL SUM RULES FOR SPIN-SPIN CORRELATION FUNCTION

The sum rules derived in Sec. II are perhaps the most conventional ones, but are not directly related to, say, the neutron diffraction cross sections. For the practical purpose of describing the neutron scattering we need to deal with the dynamical structure factor $S_{\mu\nu}(\vec{q}, \omega)$, instead of $\chi(\vec{q}, \omega)$ and $A(\vec{q}, \omega)$.

First of all, we notice the following sum rules for the transverse spin-spin correlation functions:

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega\, S_{+-}(\vec{q}, \omega) &= \langle\Psi_0|M_+(\vec{q})M_-(-\vec{q})|\Psi_0\rangle , \\ \int_{-\infty}^{\infty} d\omega\, S_{-+}(\vec{q}, \omega) &= \langle\Psi_0|M_-(\vec{q})M_+(-\vec{q})|\Psi_0\rangle .\end{aligned} \tag{18}$$

The ground state has the largest possible eigenvalue of S_z under the condition that it belongs to a definite eigenstate of S^2. Therefore,

$$M_+(0)|\Psi_0\rangle = 0 ,$$

and we expect that

$$\lim_{q\to 0} M_+(\vec{q})|\Psi_0\rangle = 0 .$$

Thus,

$$\lim_{q\to 0}\int_{-\infty}^{\infty} d\omega [S_{+-}(\vec{q}, \omega) + S_{-+}(\vec{q}, \omega)] = M . \tag{19}$$

Needless to say, the contribution from $S_{-+}(\vec{q}, \omega)$ to (19) is of the order of q^2.

From (6) we obtain also

$$\int_{-\infty}^{\infty} d\omega\, S_{+-}(\vec{q}, \omega)\omega = \langle\Psi_0|M_+(\vec{q})[H, M_-(-\vec{q})]|\Psi_0\rangle \tag{20}$$

and similarly

$$\int_{-\infty}^{\infty} \omega\, S_{-+}(-\vec{q}, \omega)\, d\omega = -\langle\Psi_0|[H, M_-(-\vec{q})]M_+(\vec{q})|\Psi_0\rangle .$$

Then we obtain the sum rule

$$\begin{aligned}\int_{-\infty}^{\infty} \omega[S_{+-}(\vec{q}, \omega) + S_{-+}(-\vec{q}, \omega)]\, d\omega \\ = \langle\Psi_0|[M_+(\vec{q}),[H, M_-(-\vec{q})]]|\Psi_0\rangle = Nq^2/4m ,\end{aligned} \tag{21}$$

where the algebra used for the derivation of (17′) has been adopted again.

As can be seen from (5), $S(\vec{q}, \omega)$ is always positive. From this fact and also from the sum rules (19) and (21), we can readily conclude that as $q\to 0$ the transverse correlation $[S_{+-}(\vec{q}, \omega) + S_{-+}(\vec{q}, \omega)]$ can have nonvanishing (and large) values only at small values of ω comparable with $q^2/2m$, while it is vanishingly small (of the order of q^2) for large values of $\omega\,[\gg(q^2/2m)]$. This fact can simply be expressed by the following result for the first moment $\langle\omega\rangle_q^0$:

$$\langle\omega\rangle_q^0 \equiv \frac{\int \omega[S_{+-}(\vec{q}, \omega) + S_{-+}(\vec{q}, \omega)]\, d\omega}{\int [S_{+-}(\vec{q}, \omega) + S_{-+}(\vec{q}, \omega)]\, d\omega} = Nq^2/4Mm \quad \text{(for small } q) . \tag{22}$$

From this result, however, we cannot yet conclude that there is a well-defined magnon mode for small q. It is not yet possible to say anything about the width of the magon spectrum in comparison with its peak value. Moreover, it is absolutely wrong to expect that Eq. (22) tells us about the exchange stiffness D of the acoustic spin waves. The difficulty is more remarkable when we consider the mean-square fluctuation of the transverse excitation energies defined by

$$\langle(\omega - \langle\omega\rangle_q^0)^2\rangle_q^0 \equiv \frac{\int (\omega - \langle\omega\rangle_q^0)^2 [S_{+-} + S_{-+}]\, d\omega}{\int [S_{+-}(\vec{q},\omega) + S_{-+}(\vec{q},\omega)]\, d\omega} \quad .$$

This is found to be of the order of q^2. However, one must not conclude from this result that the width of the magnon excitation energies {if it is defined by $[\langle(\omega - \langle\omega\rangle_q^0)^2\rangle_q^0]^{1/2}$} is larger than the mean excitation energy so that there are no magnon modes at all.

Actually the Stoner excitations contribute to sum rule (21) with importance almost equal to that of the magnon contribution. From (21) we can conclude that the integral

$$\int_a^\infty S_{+-}(\vec{q},\omega)\, d\omega = \sum_{n\,(E_n - E_0 > a)} |\langle\Psi_n | M_-(\vec{q}) | \Psi_0\rangle|^2 ,$$

with some finite positive energy value a, is of the order of q^2. Then the Stoner excitations are expected to give the dominant contribution to

$$\int_{-\infty}^\infty \omega^n [S_{+-}(\vec{q},\omega) + S_{-+}(\vec{q},\omega)]\, d\omega \quad ,$$

with $n = 2, 3, 4, \ldots$ as $q \to 0$. Indeed, these higher moments are always of the order of q^2, as can be proven easily.

Before closing this section, let us notice a simple theorem on the spin-current correlation functions defined by the diadic form

$$\overleftrightarrow{K}(q) = \left\langle \Psi_0 \left| \vec{J}_+(q) \frac{1}{h} \vec{J}_-(-q) \right| \Psi_0 \right\rangle + \left\langle \Psi_0 \left| \vec{J}_-(-q) \frac{1}{h} \vec{J}_+(q) \right| \Psi_0 \right\rangle \quad , \quad (23)$$

where $h = H - E_0$. It is noticed that Eqs. (20) and (21) with the help of Eq. (15) can also be expressed as

$$\begin{aligned}
\int \omega [S_{+-}(\vec{q},\omega) + S_{-+}(-\vec{q},\omega)]\, d\omega
&= \langle\Psi_0 | M_+(\vec{q}) \{-\vec{q}\cdot\vec{J}_-(-q)\} | \Psi_0\rangle \\
&\quad + \langle\Psi_0 | M_-(-\vec{q}) \{\vec{q}\cdot\vec{J}_+(q)\} | \Psi_0\rangle \\
&= \left\langle \Psi_0 \left| [M_+(\vec{q}), H] \frac{1}{h} \{-\vec{q}\cdot\vec{J}_-(-q)\} \right| \Psi_0 \right\rangle \\
&\quad + \left\langle \Psi_0 \left| [M_-(-\vec{q}), H] \frac{1}{h} \{\vec{q}\cdot\vec{J}_+(q)\} \right| \Psi_0 \right\rangle \\
&= \vec{q} \cdot \overleftrightarrow{K}(q) \cdot \vec{q} \; .
\end{aligned}$$

This transformation is possible because of the following reasons: First, consider the case of $q \neq 0$. Since the periodic or translational symmetry of the system has been assumed, the intermediate states must have a different translational symmetry from that of the ground state. Then the condition of quasinondegeneracy leads us to conclude $h > 0$ or $h \neq 0$. If $q = 0$, the intermediate states should have a different parity from that of the ground state, then again $h > 0$. The sum rule (21) tells us that

$$K_{\mu\nu}(q) = \frac{N}{4m} \delta_{\mu\nu} \; , \quad \mu, \nu = x, y, \text{ or } z \; . \quad (24)$$

This means that

$$K_{\mu\nu}(q) = K_{\mu\nu}(0) \quad (25)$$

for arbitrary values of q.

IV. SPIN WAVES IN COMPLETE FERROMAGNETISM

Since the conventional sum rules developed in the previous sections are not adequate for investigating the excitation energy and the energy width of a magnon with a small wave vector $\vec{q}$, some new transverse sum rules are formed here to investigate magnon excitations. For this purpose the following integral is considered:

$$I(q,\alpha) \equiv \int_{-\infty}^\infty d\omega\, e^{-\alpha\omega} [S_{+-}(\vec{q},\omega) + S_{-+}(-\vec{q},\omega)] \; , \quad (26)$$

where

$$\alpha = \alpha_q \equiv c/q^{2-\delta} \; , \quad 1 \gg \delta > 0 \; .$$

The constant c is chosen to be positive and is regarded as a variable independent of q. Thus

$$\alpha q^2/2m \to 0 \qquad \text{whenever } q \to 0 \; .$$

As can be seen quite easily, the contributions to the integral $I(q, \alpha)$ from all the high-energy states (i.e., the states with excitation energies $\gg q^2/2m$) are eliminated whenever q tends to zero.

The advantage of introducing the frequency cutoff $e^{-\alpha\omega}$ can be clearly seen when the first moment defined by

$$\langle\omega\rangle_q = \frac{\int_{-\infty}^\infty \omega e^{-\alpha\omega} [S_{+-}(\vec{q},\omega) + S_{-+}(\vec{q},\omega)]\, d\omega}{\int_{-\infty}^\infty e^{-\alpha\omega} [S_{+-}(\vec{q},\omega) + S_{-+}(\vec{q},\omega)]\, d\omega} \quad (27)$$

is carefully analyzed. Assume tentatively that there is only a single transverse collective mode among the long-wavelength transverse excitations and that the mode has the spectrum $\omega = Dq^2$. Then $S_{+-}(\vec{q},\omega) + S_{-+}(-\vec{q},\omega)$ for any ω larger than $\omega_1 \equiv c' q^{2-\delta'}$, $1 \gg \delta' > 0$, should be bounded by a maximum value S_1 while taking the limit of $q \to 0$. Similarly $S_{+-} + S_{-+}$ for any $\omega < \omega_2 \equiv c'' q^{2+\delta''}$, $1 \gg \delta'' > 0$, is bounded by a maximum value S_2 when taking that limit. Thus the background contribution to the numerator of the first moment (27) is seen to be

$$\left(\int_0^{\omega_2} d\omega + \int_{\omega_1}^\infty d\omega\right) \omega e^{-\alpha\omega} [S_{+-}(\vec{q},\omega) + S_{-+}(-\vec{q},\omega)]$$

$$< S_2 \int_0^{\omega_2} d\omega\, \omega\, e^{-\alpha\omega} + S_1 \int_{\omega_1}^{\infty} d\omega\, \omega\, e^{-\alpha\omega} .$$

As $q \to 0$, the right-hand side approaches a value $\sim S_1 c^{-1} c' q^{4-\delta-\delta'}$. Therefore, the background contribution becomes negligible as compared with the magnon contribution, which is expected to be

$$Dq^2 e^{-\alpha D q^2} \int d\omega\, [S_{+-}(\vec{q}, \omega) + S_{-+}(-\vec{q}, \omega)] = MDq^2$$

as $q \to 0$. Indeed, disappearance of the background contribution from suitably defined moments of frequency spectra is just what we intended to achieve.

The sum rule (21) tells us that the transverse frequency spectrum for small values of q is peaked for small values of ω ranging from $\omega = 0$ to a value comparable with $q^2/2m$. Then it is evident that

$$\lim_{q \to 0} I(q, \alpha_q) = M . \tag{28}$$

Therefore, the first moment (27) is expected to give the eigenfrequency Dq^2 of the magnon mode with wave number q if we assume the existence of a single (only single) magnon mode. Actually this assumption is rather unnecessary in order for us to arrive at the results of this section. The assumption has just been tentatively introduced for the purpose of seeing the significance of the frequency cutoff.

Inserting (6) and the corresponding expression for $S_{-+}(-\vec{q}, \omega)$ into (26), we obtain

$$I(q, \alpha) = \langle \Psi_0 | M_+(\vec{q}) \exp(-\alpha h) M_-(-\vec{q}) | \Psi_0 \rangle + \langle \Psi_0 | M_-(-\vec{q}) \exp(-\alpha h) M_+(\vec{q}) | \Psi_0 \rangle , \tag{29}$$

with $h = H - E_0$.

Suppose tentatively that there is a nonrelativistic many-electron system showing complete ferromagnetism. This means that all electron spins are aligned in one direction in the ground state, namely, $M = \frac{1}{2}N$. Then

$$\sigma_j^+ | \Psi_0 \rangle = 0 , \quad j = 1, \ldots, N . \tag{30}$$

In this case the expression (29) is very much simplified as

$$I(q, \alpha) = \sum_j \langle \Psi_0 | \sigma_j^+ e^{i\vec{q}\cdot\vec{r}_j} \exp(-\alpha h) e^{-i\vec{q}\cdot\vec{r}_j} \sigma_j^- | \Psi_0 \rangle$$

$$= e^{-\alpha q^2/2m} \sum_j \langle \Psi_0 | \sigma_j^+ \exp[-\alpha(h - \vec{q}\cdot\vec{p}_j/m)] \sigma_j^- | \Psi_0 \rangle , \tag{31}$$

where the explicit form of Hamiltonian (13) and the formula

$$e^{i\vec{q}\cdot\vec{r}_j} F(\vec{r}_1, \ldots, \vec{r}_N; \vec{p}_1, \ldots, \vec{p}_N) e^{-i\vec{q}\cdot\vec{r}_j}$$

$$= F(\vec{r}_1, \ldots, \vec{r}_N; \vec{p}_1, \ldots, \vec{p}_{j-1}, \vec{p}_j - \vec{q}, \vec{p}_{j+1}, \ldots, \vec{p}_N)$$

have been made use of. On account of (30), expression (31) is further transformed as

$$I(q, \alpha) = e^{-\alpha q^2/2m} \langle \Psi_0 | S^+ \exp\{-\alpha [h - (\vec{q}\cdot\vec{J}_d)]\} S^- | \Psi_0 \rangle ,$$

where

$$S^{\pm} = \sum_j \sigma_j^{\pm} , \tag{32}$$

and

$$\vec{J}_d = \sum_j (\vec{p}_j/m)(\tfrac{1}{2} - \sigma_j^z)$$

is the current of the electrons with down spin.

Let us define the normalized wave function

$$|\overline{\Psi}_0\rangle = M^{-1/2} S^- | \Psi_0 \rangle ,$$

with energy eigenvalue E_0 or $h|\overline{\Psi}_0\rangle = 0$. Then

$$I(q, \alpha) = M e^{-\alpha q^2/2m} \langle \overline{\Psi}_0 | \exp[-\alpha(h - \mathcal{J})] | \overline{\Psi}_0 \rangle , \tag{32'}$$

where $\mathcal{J} \equiv \vec{q}\cdot\vec{J}_d$. The first moment defined by (27) is then given by

$$\langle \omega \rangle_q = -\frac{\partial I(q, \alpha)/\partial\alpha}{I(q, \alpha)}$$

$$= q^2/2m - F/G , \tag{33}$$

where

$$F \equiv \langle \overline{\Psi}_0 | \mathcal{J} \exp[-\alpha(h - \mathcal{J})] | \overline{\Psi}_0 \rangle$$

and

$$G \equiv \langle \overline{\Psi}_0 | \exp[-\alpha(h - \mathcal{J})] | \overline{\Psi}_0 \rangle .$$

The quantities F and G may be expanded as follows:

$$F = \sum_{n=1}^{\infty} f_{2n} , \tag{34}$$

where

$$f_{2n} = \int_0^{\alpha} du_1 \cdots \int_0^{u_{2n-2}} du_{2n-1} \times \langle \overline{\Psi}_0 | \mathcal{J} e^{-(\alpha - u_1)h} \mathcal{J} \cdots e^{(u_{2n-1} - u_{2n-2})h} \mathcal{J} | \overline{\Psi}_0 \rangle$$

and

$$G = 1 + \sum_{n=1}^{\infty} g_{2n} , \tag{35}$$

where

$$g_{2n} = \int_0^{\alpha} du_1 \cdots \int_0^{u_{2n-1}} du_{2n} \times \langle \overline{\Psi}_0 | \mathcal{J} e^{(u_2 - u_1)h} \mathcal{J} \cdots e^{(u_{2n} - u_{2n-1})h} \mathcal{J} | \overline{\Psi}_0 \rangle .$$

The first term in expansion (34) is evaluated as

$$f_2 = \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1 - e^{-\alpha h}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle . \tag{36}$$

Since the system is invariant for the inversion of space coordinates, $\overline{\Psi}_0$ must have a definite parity and, hence, $\mathcal{J}\overline{\Psi}_0$ should have the opposite parity. Owing to the condition of quasinondegeneracy of the ground state, $\mathcal{J}\overline{\Psi}_0$ can not contain any energy eigenstate that has the same energy as E_0. Therefore, $h > 0$ and $h \neq 0$ in the above expression.

Let us define the spectrum function

$$A(\omega)=\langle\bar{\Psi}_0|\mathcal{J}\delta(\omega-h)\mathcal{J}|\bar{\Psi}_0\rangle . \qquad (37)$$

In the thermodynamic limit this is expected to be a continuous function of ω. Indeed, what we meant by the existence of the thermodynamic limit is the existence of such $A(\omega)$. Then (36) may be expressed as

$$f_2=\int_0^\infty \frac{A(\omega)}{\omega}\,d\omega-\int_0^\infty \frac{e^{-\alpha\omega}}{\omega}A(\omega)\,d\omega . \qquad (36')$$

The first term in (36′) is evaluated as

$$\int\frac{A(\omega)}{\omega}d\omega=\left\langle\bar{\Psi}_0\middle|\mathcal{J}\frac{1}{h}\mathcal{J}\middle|\bar{\Psi}_0\right\rangle$$

$$=M^{-1}\sum_j\left\langle\Psi_0\middle|\sigma_j^+\frac{\vec{q}\cdot\vec{p}_j}{m}\frac{1}{h}\frac{\vec{q}\cdot\vec{p}_j}{m}\sigma_j^-\middle|\Psi_0\right\rangle$$

$$=\frac{1}{2M}\sum_j\left\langle\Psi_0\middle|\frac{\vec{q}\cdot\vec{p}_j}{m}\frac{1}{h}\frac{\vec{q}\cdot\vec{p}_j}{m}\middle|\Psi_0\right\rangle.$$

Now

$$(\vec{p}_j/m)|\Psi_0\rangle=\dot{\vec{x}}_j|\Psi_0\rangle=i^{-1}[\vec{x}_j,H]|\Psi_0\rangle=ih\vec{x}_j|\Psi_0\rangle$$

and

$$\langle\Psi_0|(\vec{p}_j/m)=-i\langle\Psi_0|\vec{x}_j h .$$

Therefore,

$$\left\langle\Psi_0\middle|\frac{\vec{q}\cdot\vec{p}_j}{m}\frac{1}{h}\frac{\vec{q}\cdot\vec{p}_j}{m}\middle|\Psi_0\right\rangle$$

$$=(i/2m)\langle\Psi_0|(\vec{q}\cdot\vec{p}_j)(\vec{q}\cdot\vec{x}_j)|\Psi_0\rangle$$

$$-(i/2m)\langle\Psi_0|(\vec{q}\cdot\vec{x}_j)(\vec{q}\cdot\vec{p}_j)|\Psi_0\rangle$$

$$=q^2/2m . \qquad (38)$$

Consequently,

$$\int_0^\infty\frac{A(\omega)}{\omega}d\omega=\frac{q^2}{2m} . \qquad (39)$$

This means that $A(\omega)\to 0$ as $\omega\to 0$. Therefore we have

$$A(\omega)\propto(q^2/m)\,\omega^\nu \quad \text{for small } \omega \qquad (40)$$

with a positive constant ν. The precise definition of ν is

$$\nu\equiv\lim_{\omega\to 0}\omega\frac{\partial\ln A(\omega)}{\partial\omega} .$$

In this convention, $\omega^\nu\ln\omega$, for instance, is simply denoted as ω^ν, and $e^{-\text{const}/\omega}$ is understood as $\nu=\infty$.

On account of (40) the second term in (36′) is evaluated as

$$\int_0^\infty\frac{e^{-\alpha\omega}}{\omega}A(\omega)\,d\omega\propto\alpha^{-\nu}\frac{q^2}{m} . \qquad (40')$$

Then

$$\int_0^\infty\frac{e^{-\alpha\omega}}{\omega}A(\omega)\,d\omega\Big/\int_0^\infty\frac{A(\omega)}{\omega}d\omega\propto\alpha^{-\nu}\to 0$$

as $q\to 0$. Therefore,

$$f_2=q^2/2m+O(\alpha^{-\nu}q^2) . \qquad (41)$$

The next term in expansion (34) is analyzed as

$$f_4=\int_0^\alpha du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\,\langle\bar{\Psi}_0|\mathcal{J}e^{-(\alpha-u_1)h}\mathcal{J}e^{(u_2-u_1)h}\mathcal{J}e^{(u_3-u_2)h}\mathcal{J}|\bar{\Psi}_0\rangle$$
$$=f_4'+f_4'' , \qquad (42)$$

where

$$f_4'\equiv\int_0^\alpha du_1\cdots\int_0^{u_2}du_3\langle\bar{\Psi}_0|\mathcal{J}e^{-(\alpha-u_1)h}\mathcal{J}|\bar{\Psi}_0\rangle\langle\bar{\Psi}_0|\mathcal{J}e^{(u_3-u_2)h}\mathcal{J}|\bar{\Psi}_0\rangle$$

and

$$f_4''\equiv\int_0^\alpha du_1\cdots\int_0^{u_2}du_3\langle\bar{\Psi}_0|\,\mathcal{J}e^{-(\alpha-u_1)h}\mathcal{J}Pe^{(u_2-u_1)h}\mathcal{J}e^{(u_3-u_2)h}\mathcal{J}|\bar{\Psi}_0\rangle ,$$

with

$$P\equiv 1-|\bar{\Psi}_0\rangle\langle\bar{\Psi}_0| .$$

Now

$$f_4'=\int_0^\alpha du_2\int_{u_2}^\alpha du_1\int_0^{u_2}du_3\,\langle\bar{\Psi}_0|\mathcal{J}e^{-(\alpha-u_1)h}\mathcal{J}|\bar{\Psi}_0\rangle\langle\bar{\Psi}_0|\mathcal{J}e^{(u_3-u_2)h}\mathcal{J}|\bar{\Psi}_0\rangle$$

$$=\int_0^\alpha du_2\left\langle\bar{\Psi}_0\middle|\mathcal{J}\frac{1-e^{-(\alpha-u_2)h}}{h}\mathcal{J}\middle|\bar{\Psi}_0\right\rangle\left\langle\bar{\Psi}_0\middle|\mathcal{J}\frac{1-e^{-u_2h}}{h}\mathcal{J}\middle|\bar{\Psi}_0\right\rangle$$

$$=\int_0^\alpha du\int_0^\infty d\epsilon\int_0^\infty d\epsilon'\,\frac{1-e^{-(\alpha-u)\epsilon}}{\epsilon}\,\frac{1-e^{-u\epsilon'}}{\epsilon'}A(\epsilon)\,A(\epsilon') . \qquad (43)$$

Since $1 \geqq 1-e^{-(\alpha-u)\epsilon} \geqq 0$, $1 \geqq 1-e^{-u\epsilon'} \geqq 0$, and $A(\epsilon) \geqq 0$, the following inequality is seen to hold:

$$0 < f_4' < \int_0^\alpha du \int_0^\infty d\epsilon \int_0^\infty d\epsilon' \frac{A(\epsilon)}{\epsilon}\frac{A(\epsilon')}{\epsilon'} = \alpha(q^2/2m)^2 . \tag{43'}$$

For the analysis of f_4'', let us define

$$\mathcal{G}^* = \sum_{n,m} |\overline{\Psi}_n\rangle\, |\mathcal{G}_{n,m}|\, \langle\overline{\Psi}_m| ,$$

where

$$\mathcal{G}_{n,m} = \langle\overline{\Psi}_n|\mathcal{G}|\overline{\Psi}_m\rangle$$

and $\overline{\Psi}_n$'s satisfy the following conditions:

$$H|\overline{\Psi}_n\rangle = E_n|\overline{\Psi}_n\rangle ,$$

$$S_z|\overline{\Psi}_n\rangle = (\tfrac{1}{2}N-1)|\overline{\Psi}_n\rangle ,$$

and

$$\langle\overline{\Psi}_n|\overline{\Psi}_m\rangle = \delta_{n,m} .$$

In the definition of f_4'' all the exponential functions are always positive. Therefore,

$$|f_4''| < \int_0^\alpha du_1 \int_0^{u_1} du_2 \int_0^{u_2} du_3 \langle\overline{\Psi}_0|\mathcal{G}^* e^{-(\alpha-u_1)h}\mathcal{G}^* P e^{(u_2-u_1)h}\mathcal{G}^* e^{(u_3-u_2)h}\mathcal{G}^*|\overline{\Psi}_0\rangle$$

$$= \int_0^\alpha du_1 \int_0^{u_1} du_2 \left\langle\overline{\Psi}_0\left|\mathcal{G}^* e^{-(\alpha-u_1)h}\mathcal{G}^* P e^{(u_2-u_1)h}\mathcal{G}^* \frac{1-e^{-u_2h}}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle$$

$$\leqq \int_0^\alpha du_1 \int_0^{u_1} du_2 \left\langle\overline{\Psi}_0\left|\mathcal{G}^* e^{-(\alpha-u_1)h}\mathcal{G}^* P e^{(u_2-u_1)h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle$$

$$= \int_0^\alpha du_1 \left\langle\overline{\Psi}_0\left|\mathcal{G}^* e^{-(\alpha-u_1)h}\mathcal{G}^* P \frac{1-e^{-u_1h}}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle \leqq \int_0^\alpha du_1 \left\langle\overline{\Psi}_0\left|\mathcal{G}^* e^{-(\alpha-u_1)h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle$$

$$= \left\langle\overline{\Psi}_0\left|\mathcal{G}^* \frac{1-e^{-\alpha h}}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle \leqq \left\langle\overline{\Psi}_0\left|\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle \propto q^4 . \tag{44}$$

In the above analysis the condition of quasinondegeneracy imposed on our ground state plays the crucial role. As the result of this condition, there exists the formal expression containing P/h.

From (43′) and (44) we may safely conclude that

$$|f_4| < \alpha(q^2/2m)^2 + \left\langle\overline{\Psi}_0\left|\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle$$

for sufficiently small values of q. The above reasoning can easily be extended to the similar analysis of the higher terms in the expansion (34). Then we conclude that

$$|f_{2n}| < \left[1 + \left\langle\overline{\Psi}_0\left|\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle \times \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\alpha}\alpha\frac{\partial}{\partial\alpha}\right) + \cdots\right] \frac{1}{(n-1)!}\alpha^{n-1}\left(\frac{q^2}{2m}\right)^n$$

for small values of q. (The smallness does not depend on n.) Let us define

$$F_{n_0} = \sum_{n=1}^{n_0} |f_{2n}| .$$

Then

$$0 < F_{n_0} < \frac{q^2}{2m}\left[1 + \left\langle\overline{\Psi}_0\left|\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^* \frac{P}{h}\mathcal{G}^*\right|\overline{\Psi}_0\right\rangle \times \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\alpha}\alpha\frac{\partial}{\partial\alpha}\right) + \cdots\right] \sum_{n=0}^{n_0-1} (1/n!)(\alpha q^2/2m)^n$$

for small values of q. Since the right-hand side converges to a finite value $\sim (q^2/2m)e^{\alpha q^2/2m}$ as $n_0 \to \infty$ and $F_n < F_{n+1}$ for all n, $\lim F_n$ as $n \to \infty$ should exist. Therefore, it is concluded that the expansion (34) constitutes a convergent series. The convergency becomes more and more rapid as q becomes smaller and smaller, especially,

$$F \to f_2 = \int_0^\infty d\epsilon A(\epsilon)\frac{1-e^{-\alpha\epsilon}}{\epsilon} \quad \text{as } q \to 0 . \tag{45}$$

Let us consider now the expansion (35) of G. The first term is

$$g_2 = \int_0^\alpha du_1 \int_0^{u_1} du_2 \langle\overline{\Psi}_0|\mathcal{G} e^{(u_2-u_1)h}\mathcal{G}|\overline{\Psi}_0\rangle$$

$$= \int_0^\alpha du \left\langle\overline{\Psi}_0\left|\mathcal{G}\frac{1-e^{-uh}}{h}\mathcal{G}\right|\overline{\Psi}_0\right\rangle$$

$$= \int_0^\alpha du \int_0^\infty d\epsilon A(\epsilon)\frac{1-e^{-u\epsilon}}{\epsilon} . \tag{46}$$

Thus

$$0 < g_2 < \int_0^\alpha du \int_0^\infty d\epsilon A(\epsilon)/\epsilon = \alpha(q^2/2m) . \tag{46'}$$

An immediate conclusion of (46′) is that $g_2 \to 0$ like $O(q^6)$ as $q \to 0$.

The second term in expansion (35), i.e., g_4, is divided into two parts as

$$g_4 = g_4' + g_4'' ,$$

where

$$g_4' \equiv \int_0^\alpha du_1 \ldots \int_0^{u_3} du_4$$

$$\times \langle \overline{\Psi}_0 | \mathcal{J} e^{(u_2 - u_1)h} \mathcal{J} | \overline{\Psi}_0 \rangle \langle \overline{\Psi}_0 | \mathcal{J} e^{(u_4 - u_3)h} \mathcal{J} | \overline{\Psi}_0 \rangle$$

and

$$g_4'' = \int_0^\alpha du_1 \ldots \int_0^{u_3} du_4$$

$$\times \langle \overline{\Psi}_0 | \mathcal{J} e^{(u_2 - u_1)h} \mathcal{J} P e^{(u_3 - u_2)h} \mathcal{J} e^{(u_4 - u_3)h} | \overline{\Psi}_0 \rangle ,$$

$$0 < g_4' = \int_0^\alpha du_1 \ldots \int_0^{u_2} du_3 \langle \overline{\Psi}_0 | \mathcal{J} e^{(u_2 - u_1)h} \mathcal{J} | \overline{\Psi}_0 \rangle \times \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1 - e^{-u_3 h}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle$$

$$< \int_0^\alpha du_1 \ldots \int_0^{u_2} du_3 \langle \overline{\Psi}_0 | \mathcal{J} e^{(u_2 - u_1)h} \mathcal{J} | \overline{\Psi}_0 \rangle \times \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle$$

$$= \int_0^\alpha du_2 \int_{u_2}^\alpha du_1 \, u_2 \langle \overline{\Psi}_0 | \mathcal{J} e^{(u_2 - u_1)h} \mathcal{J} | \overline{\Psi}_0 \rangle \times \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle$$

$$= \int_0^\alpha du \, u \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1 - e^{-(\alpha - u)h}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle \times \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle$$

$$< \int_0^\alpha du \, u \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle^2 .$$

Therefore

$$0 < g_4' < (1/2!)(\alpha q^2/2m)^2 \propto q^{2\delta}$$

as $q \to 0$. In the above expression Eq. (39) has been used.

We can prove

$$|g_4''| < \left\langle \overline{\Psi}_0 \middle| \mathcal{J}^* \frac{P}{h} \mathcal{J}^* \frac{P}{h} \mathcal{J}^* \frac{P}{h} \mathcal{J}^* \middle| \overline{\Psi}_0 \right\rangle \propto q^4$$

in the same way as has been done in deriving (44).

The above argument can easily be extended to the analysis of general higher terms in expansion (35). Then we get rigorously

$$|g_{2n}| < \left[1 + \left\langle \overline{\Psi}_0 \middle| \mathcal{J}^* \frac{P}{h} \mathcal{J}^* \frac{P}{h} \mathcal{J}^* \frac{P}{h} \mathcal{J}^* \middle| \overline{\Psi}_0 \right\rangle \times \frac{\partial}{\partial \alpha} \left(\frac{\partial}{\partial \alpha} \alpha \frac{\partial}{\partial \alpha} \right) + \cdots \right] (1/n!)(\alpha q^2/2m)^n$$

for small q. Then the series

$$\sum_{n=1}^\infty |g_{2n}|$$

is shown to converge. Therefore, it is concluded that expansion (35) constitutes a convergent series and

$$G \to 1 \quad \text{as } q \to 0 .$$

Thus we have obtained a rigorous proof of sum rule (28). It has been confirmed again that only the low-lying excitations with vanishing excitation energies [$\sim O(q^2/2m)$] can contribute to the transverse spin-spin correlation function for a small fixed value of q.

From (33), (36′), (43), and (46), we obtain

$$\lim_{q \to 0} \langle \omega \rangle_q = \frac{q^2}{2m} - \left(\int_0^\infty d\epsilon \, A(\epsilon) \frac{1 - e^{-\alpha \epsilon}}{\epsilon} + \int_0^\alpha du \int_0^\infty d\epsilon \int_0^\infty d\epsilon' \, A(\epsilon) A(\epsilon') \frac{1 - e^{-(\alpha - u)\epsilon}}{\epsilon} \frac{1 - e^{-u\epsilon'}}{\epsilon'} \right) \Big/ \left(1 + \int_0^\alpha du \int_0^\infty d\epsilon \, A(\epsilon) \frac{1 - e^{-u\epsilon}}{\epsilon} \right) \to \frac{q^2}{2m} - \int_0^\infty d\epsilon \, A(\epsilon) \frac{1 - e^{-\alpha \epsilon}}{\epsilon} + b ,$$

in the limit of small q, where

$$a \equiv \int_0^\infty d\epsilon \, A(\epsilon) \frac{e^{-\alpha \epsilon}}{\epsilon} .$$

Equation (40′) states that

$$a \propto \alpha^{-\nu} q^2 \propto q^{2(1+\nu) - \nu\delta} \quad \text{for small } q .$$

We have also

where

$$b \equiv \int_0^\alpha du \int_0^\infty d\epsilon \int_0^\infty d\epsilon' \, e^{-(\alpha - u)\epsilon} A(\epsilon) A(\epsilon') \times \frac{1 - e^{-u\epsilon}}{\epsilon} \frac{1 - e^{-u\epsilon'}}{\epsilon'} .$$

Applying (39) to the above, we get

$$\langle \omega \rangle_q = a + b$$

$$0 < b < \int_0^\alpha du \int_0^\infty d\epsilon \int_0^\infty d\epsilon' \, e^{-(\alpha-u)\epsilon} \frac{A(\epsilon)}{\epsilon} \frac{A(\epsilon')}{\epsilon'}$$

$$= \text{const} \times \int_0^\alpha du \, (\alpha - u)^{-\nu} (q^2/2m)^2$$

$$= \text{const} \times \alpha^{1-\nu} (q^2/2m)^2$$

$$\propto q^{2(1+\nu)+\delta(1-\nu)} \qquad \text{if } \nu < 1 .$$

For $\nu = 1$ we have

$$0 < b < \text{const} \times \int_0^{\alpha-\alpha^{-1}} du \, (\alpha - u)^{-1} (q^2/2m)^2 + \int_{\alpha-\alpha^{-1}}^{\alpha} du \Big[\int_0^\infty d\epsilon \, A(\epsilon)/\epsilon \Big]^2$$

$$\sim \text{const} \times (\ln\alpha)(q^2/2m)^2 + \text{const} \times \alpha^{-1} (q^2/2m)^2 .$$

Note that $\nu \leqq 1$.[13] Therefore, $b/a \to 0$, as $q \to 0$. Consequently,

$$\langle \omega \rangle_q \propto \alpha^{-\nu} q^2 \propto q^{2(1+\nu)-\nu\delta} \tag{47}$$

for small values of q. Note that the q^2 terms in $\langle \omega \rangle_q$ have completely canceled out each other.

If such a result as

$$\langle \omega \rangle_q = Dq^2 + O(\alpha^{-\nu} q^2)$$

with a positive constant D were obtained, then we could have concluded that there were the transverse magnon modes with the spectrum $\omega = Dq^2$. The term $O(\alpha^{-\nu} q^2)$ would then be ascribed simply to the background contribution. On the contrary, we have now only the background contribution while $D = 0$. From this it is rigorously concluded that there cannot be the magnon mode that has the typical dispersion $\omega = Dq^2$.

One might think then that the magnons in complete ferromagnetism would show another dispersion relation, say, $\omega = Bq^4$. Assume for a moment that this is true and there is no other transverse collective mode. Then let us take $\alpha = c/q^{4-\delta}$ with $1 \gg \delta > 0$. As a result of this assumption, $S_{+-}(\vec{q}, \omega)$ for any ω not smaller than $\omega_1 \equiv \text{const}\, q^{4-\delta'}$, $1 \gg \delta' > 0$, should be bounded in the limit $q \to 0$. In this way we could conclude

$$\langle \omega \rangle_q = Bq^4 + [\text{terms of at most } O(\alpha^{-1} q^{4-\delta'})]$$

or at least we could assert that the magnon excitation gives a dominant contribution to $\langle \omega \rangle_q$.

By assuming $\alpha = c/q^{4-\delta}$, let us see whether we can really arrive at the above result. From

$$\langle \omega \rangle_q = \frac{q^2}{2m} - \frac{f_2 + f_4}{1 + g_2}$$

we see that

$$\langle \omega \rangle_q = b - f_4'' + O(\alpha^{-\nu} q^2) .$$

In the definition of b it is easily seen that the main contribution to the u integration comes from the region $u \sim \alpha$. Then we could conclude that

$$b \cong \int_0^\alpha du \int_0^\infty d\epsilon \int_0^\infty d\epsilon' \, e^{-(\alpha-u)\epsilon} \frac{A(\epsilon)}{\epsilon} \frac{A(\epsilon')}{\epsilon'}$$

$$= \text{const} \times \alpha^{1-\nu} (q^2/2m)^2 ,$$

with the convention

$$\alpha^0 = \ln\alpha .$$

Since $|f_4''| < \text{const} \times q^4$, we have

$$\langle \omega \rangle_q = b + O(\alpha^{-\nu} q^2)$$

for small values of q. Therefore, it has been said that the background contribution is again dominant even if α is chosen to be $c/q^{4-\delta}$.

The significance of all the above results could be clearly seen in the following conjecture which cannot be said to have been proved rigorously but, nevertheless, seems to give the only interpretation for the rigorously derived results. Our conjecture is that $S_{+-}(\vec{q}, \omega)$ as a function of ω ($\geqq 0$) has a peak at $\omega = 0$ or $\omega \cong 0$ with the frequency width $\Delta\omega \propto q^{2+x}$, $x > 0$. In this case the sum rule (21) in the long-wavelength limit should be exhausted by the Stoner excitations, though sum rule (19) must be exhausted by the "spin-wave" excitation at $\omega \sim 0$. The existence of spin waves with vanishing frequencies implies that the assumed ground state of complete ferromagnetism cannot be stable: There should be a very high degeneracy in the ground states which can have various values of the total spin multiplicity. This is not consistent with the condition of quasinondegeneracy.

The whole difficulty comes from the assumption of the existence of complete ferromagnetism satisfying conditions (a), (b), (c), and (d). Complete ferromagnetism could only exist in systems that violate one or more of these conditions. One example is the one-dimensional fermion gas interacting via a hard-core potential, considered by Lieb and Mattis.[5] In this case there is a macroscopic degeneracy of the ground states: The lowest energy state among the energy eigenstates with any fixed value of S should have the same energy as that of another such state for another fixed value of S S ($S \leqq \frac{1}{2}N$). The Nagaoka theorem might be regarded as another example, though the Hubbard Hamiltonian cannot be regarded as an example of our general Hamiltonian.[13] It seems to us that the possible lack of a thermodynamic limit in the Nagaoka system, rather than the special nature of the Hubbard Hamiltonian, is likely to be the basic reason that the present conclusion is compatible with the existence of complete ferromagnetism in his case.

V. GENERAL CONSIDERATIONS OF MAGNONS

Let us consider the following integral:

$$\int_0^\infty d\omega\, e^{-\beta\omega}\omega[S_{+-}(\vec{q},\omega)+S_{-+}(\vec{q},\omega)]$$

$$=\langle\Psi_0|M_+(\vec{q})e^{-\beta h}hM_-(-\vec{q})|\Psi_0\rangle + \langle\Psi_0|M_-(-\vec{q})e^{-\beta h}hM_+(\vec{q})|\Psi_0\rangle$$

$$=\left\langle\Psi_0\left|[M_+(\vec{q}),H]\frac{e^{-\beta h}}{h}[H,M_-(-\vec{q})]\right|\Psi_0\right\rangle + \left\langle\Psi_0\left|[M_-(-\vec{q}),H]\frac{e^{-\beta h}}{h}[H,M_+(\vec{q})]\right|\Psi_0\right\rangle$$

$$=\vec{q}\cdot\overleftrightarrow{K}(q,\beta)\cdot\vec{q}\,, \tag{48}$$

where the diadic $\overleftrightarrow{K}(q,\beta)$ is defined by

$$\overleftrightarrow{K}(q,\beta)=\left\langle\Psi_0\left|\vec{J}_+(q)\frac{e^{-\beta h}}{h}\vec{J}_-(-q)\right|\Psi_0\right\rangle + \left\langle\Psi_0\left|\vec{J}_-(-q)\frac{e^{-\beta h}}{h}\vec{J}_+(q)\right|\Psi_0\right\rangle\,. \tag{49}$$

It is seen that

$$\overleftrightarrow{K}(q,0)=\overleftrightarrow{K}(q)\,,$$

where $\overleftrightarrow{K}(q)$ was defined by (23). Let us define $\overleftrightarrow{\Delta}(q,\beta)$ by

$$\overleftrightarrow{K}(q,\beta)=\overleftrightarrow{K}(0,\beta)+\overleftrightarrow{\Delta}(q,\beta)\,. \tag{50}$$

From (25) it is evident that

$$\overleftrightarrow{\Delta}(q,0)=0 \quad \text{for any } q$$

and

$$\overleftrightarrow{\Delta}(0,\beta)=0\,.$$

$\overleftrightarrow{\Delta}(q,\alpha)$ may be expressed as

$$\overleftrightarrow{\Delta}(q,\alpha) =\left\langle\Psi_0\left|[\vec{J}_+(q)+\vec{J}_-(-q)]\frac{e^{-\alpha h}}{h}[\vec{J}_-(-q)+\vec{J}_+(q)]\right|\Psi_0\right\rangle - \left\langle\Psi_0\left|[\vec{J}_+(0)+\vec{J}_-(0)]\frac{e^{-\alpha h}}{h}[\vec{J}_+(0)+\vec{J}_-(0)]\right|\Psi_0\right\rangle\,. \tag{51}$$

Let us define further a new spectrum function by

$$\overleftrightarrow{A}_J(\omega)=\overleftrightarrow{A}_{+-}(\omega)+\overleftrightarrow{A}_{-+}(\omega)\,,$$

with

$$\overleftrightarrow{A}_{+-}(\omega)=\langle\Psi_0|\vec{J}_+(0)\delta(h-\omega)\vec{J}_-(0)|\Psi_0\rangle$$

and

$$\overleftrightarrow{A}_{-+}(\omega)=\langle\Psi_0|\vec{J}_-(0)\delta(h-\omega)\vec{J}_+(0)|\Psi_0\rangle\,.$$

In the case of complete ferromagnetism the relation between $A_J(\omega)$ and $A(\omega)$ defined by (37) is

$$\vec{q}\cdot\overleftrightarrow{A}_J(\omega)\cdot\vec{q}=\vec{q}\cdot\overleftrightarrow{A}_{+-}(\omega)\cdot\vec{q}=MA(\omega)\,,$$

then

$$\overleftrightarrow{K}(0,\beta)=\int_0^\infty\frac{e^{-\beta\omega}}{\omega}\overleftrightarrow{A}_J(\omega)d\omega\,. \tag{52}$$

We know that

$$\int_0^\infty\frac{\overleftrightarrow{A}_J(\omega)}{\omega}d\omega=\overleftrightarrow{K}(0)=\overleftrightarrow{K}(q)=(N/2m)\overleftrightarrow{1}\,, \tag{53}$$

where

$$(\overleftrightarrow{1})_{\mu,\nu}=\delta_{\mu,\nu} \quad \text{for } \mu,\nu=x,y,\text{ or } z.$$

Since the integral of (53) exists, we must have

$$A_J(\omega)=a\omega^\nu \quad \text{for small } \omega,$$

where a is a positive constant, ν (> 0) is defined by

$$\nu=\lim_{\omega\to 0}\omega\frac{d\ln A_J(\omega)}{d\omega}\,,$$

and $A_J(\omega)$ is the diagonal element of $\overleftrightarrow{A}_J(\omega)=A_J(\omega)\cdot\overleftrightarrow{1}$. In the above, such a convention as that used in the definition of ν by (40) has again been adopted. For $\beta\gg 1$ it is concluded that

$$\int_0^\infty[e^{-\beta\omega}A(\omega)/\omega]d\omega=a\int_0^\infty\omega^{-(1-\nu)}e^{-\beta\omega}d\omega\propto\beta^{-\nu}. \tag{54}$$

Let us consider again $I(q,\alpha_q)$ defined in Sec. IV. Consider especially

$$-\left(\frac{\partial I(q,\alpha)}{\partial\alpha}\right)_{\alpha=\alpha_q}=\vec{q}\cdot\overleftrightarrow{K}(q,\alpha_q)\cdot\vec{q}\,,$$

with $\alpha_q=c/q^{2-\delta}$. Equation (54) tells us that

$$\lim_{q\to 0}K(0,\alpha_q)=0 \quad \text{like } O(q^{\nu(2-\delta)}). \tag{55}$$

Therefore, it is seen that the magnon mode with the $\omega=Dq^2$ spectrum is obtained if and only if we have the following theorem:

$$\lim_{q\to 0}\overleftrightarrow{\Delta}(q,\alpha_q)=\rho_s\overleftrightarrow{1}\,,$$

with positive constant ρ_s independent of c and $\delta(1\gg\delta>0)$. Then we could obtain

$$Dq^2=\lim_{q\to 0}\left[-\left(\frac{\partial I(q,\alpha)}{\partial\alpha}\right)_{\alpha=\alpha_q}\Big/I(q,\alpha_q)\right]$$

$$=\lim_{q\to 0}\vec{q}\cdot\overleftrightarrow{\Delta}(q,\alpha_q)\cdot\vec{q}/M=(\rho_s/M)q^2$$

or

$$D=\rho_s/M\,.$$

Thus it has been found that the very singular term $\overleftrightarrow{\Delta}(q,\alpha)$ is responsible for the magnon excitation energy, while the $\overleftrightarrow{K}(0,\alpha)$ term has nothing to do with the magnon mode. Indeed, $\overleftrightarrow{K}(0,\alpha)$ contains only the background contribution, mostly due to the Stoner excitations. Any effort trying to relate D with $\langle\Psi_0|\vec{J}_+(0)[1/h]\vec{J}_-(0)|\Psi_0\rangle$ is thus seen to be irrelevant.[14] We could be more radical and say that the conventional sum rule (21) or (24) is irrelevant[15] to the present subject of finding D. More detailed analysis of $\overleftrightarrow{\Delta}(q, \alpha)$ in general ferromagnetism of nonrelativistic systems will be given in a separate paper.

It would be interesting to see how $\overleftrightarrow{\Delta}(q, \alpha)$ vanishes in complete ferromagnetism ($M = \frac{1}{2}N$), though the mathematics given below is essentially equivalent to that used previously for the derivation of sum rule (47). In this case expression (51) becomes

$$\vec{q} \cdot \overleftrightarrow{\Delta}(\vec{q}, \alpha) \cdot \vec{q} = \left\langle \Psi_0 \middle| [\vec{q} \cdot \vec{J}_+(q)] \frac{e^{-\alpha h}}{h} [\vec{q} \cdot \vec{J}_-(-q)] \middle| \Psi_0 \right\rangle - \left\langle \Psi_0 \middle| [\vec{q} \cdot \vec{J}_+(0)] \frac{e^{-\alpha h}}{h} [\vec{q} \cdot \vec{J}_-(0)] \middle| \Psi_0 \right\rangle$$

$$= -\langle \Psi_0 | M_+(\vec{q}) e^{-\alpha h} [\vec{q} \cdot \vec{J}_-(-q)] | \Psi_0 \rangle - \langle \overline{\Psi}_0 | \mathcal{J} \frac{e^{-\alpha h}}{h} \mathcal{J} | \overline{\Psi}_0 \rangle , \quad (56)$$

where $\overline{\Psi}$ and $\mathcal{J}$ were defined in Sec. IV. The first term on the right-hand side of (56) becomes

$$-\sum_j \langle \Psi_0 | \sigma_j^+ e^{i\vec{q}\cdot\vec{r}_j} e^{-\alpha h} \tfrac{1}{2} (\xi_j e^{-i\vec{q}\cdot\vec{r}_j} + e^{-i\vec{q}\cdot\vec{r}_j} \xi_j) \sigma_j^- | \Psi_0 \rangle$$

$$= -\sum_j \langle \Psi_0 | \sigma_j^+ \exp[-\alpha(h - \xi_j + q^2/2m)] \times (\xi_j - q^2/2m) \sigma_j^- | \Psi_0 \rangle ,$$

where $\xi_j = \vec{q} \cdot \vec{p}_j / m$. This is further written as

$$-M \langle \overline{\Psi}_0 | \exp[-\alpha(h - \mathcal{J} + q^2/2m)] (\mathcal{J} - q^2/2m) | \overline{\Psi}_0 \rangle .$$

Thus (56) is expressed as

$$\vec{q} \cdot \overleftrightarrow{\Delta}(q, \alpha) \cdot \vec{q} = M \langle \overline{\Psi}_0 | \exp[-\alpha(h - \mathcal{J} + q^2/2m)] \times (q^2/2m - \mathcal{J}) | \overline{\Psi}_0 \rangle - \langle \overline{\Psi}_0 | \mathcal{J} (e^{-\alpha h}/h) \mathcal{J} | \overline{\Psi}_0 \rangle$$

$$= M e^{-\alpha q^2/2m} \Big[q^2/2m + \int_0^\alpha du \times \langle \overline{\Psi}_0 | \mathcal{J} e^{-uh} [q^2/2m - \mathcal{J}] | \overline{\Psi}_0 \rangle + \int_0^\alpha du_1 \int_0^{u_1} du_2 \langle \overline{\Psi}_0 | \mathcal{J} e^{-(u_1 - u_2)h} \mathcal{J} e^{-u_2 h} \times (q^2/2m - \mathcal{J}) | \overline{\Psi}_0 \rangle + \cdots - \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{e^{-\alpha h}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle \Big]$$

$$= M e^{-\alpha q^2/2m} \Big[q^2/2m - \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1 - e^{-\alpha h}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle + \int_0^\alpha du \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1 - e^{-uh}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle q^2/2m + \cdots - \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{e^{-\alpha h}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle \Big] .$$

Applying (39) to the above, we obtain

$$0 < \vec{q} \cdot \overleftrightarrow{\Delta}(q, \alpha) \cdot \vec{q} = M e^{-\alpha q^2/2m} \int_0^\alpha du \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1 - e^{-uh}}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle q^2/2m$$

$$< M e^{-\alpha q^2/2m} \int_0^\alpha du \left\langle \overline{\Psi}_0 \middle| \mathcal{J} \frac{1}{h} \mathcal{J} \middle| \overline{\Psi}_0 \right\rangle q^2/2m$$

$$< M \alpha (q^2/2m)^2$$

for small q. Therefore, it is concluded that

$$\lim_{q \to 0} \overleftrightarrow{\Delta}(q, \alpha_q) = 0$$

for complete ferromagnetism.

ACKNOWLEDGMENTS

This work was started while the author was staying at the Theoretical Physics Division, AERE, Harwell, Berkshire, England. The author would like to express his sincere thanks to Dr. W. Marshall, Dr. A. B. Lidiard, and Dr. J. Hubbard for giving him that opportunity. The author is grateful to Professor P. N. Argyres for inviting him for the visiting professorship at Northeastern University and also to the Theoretical Solid-State Physics Group there for useful discussions. The final version of the manuscript was written during the author's stay at Service de Physique Théorique, C.E.N. de Saclay. He is grateful to Dr. C. De Dominicis for his hospitality and interest in this work. He would like to express his sincere gratitude to Professor E. Lieb for valuable communications and especially for informing him of the pathological case of the Lieb-Mattis theorem. Finally he would like to express his thanks to his colleagues at the Institute of Physics, College of General Education, University of Tokyo, for allowing him to take this sabbatical leave.

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*This work was started while the author was staying at the Theoretical Physics Division, AERE, Harwell, Berkshire, England, during the summer of 1969.

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[13]If $\nu > 1$, the integral $\int_0^\infty (A(\epsilon)/\epsilon^2)\, d\epsilon$ is finite. However, this integral does not exist in the thermodynamic limit, as can be seen from the following argument:

$$\int_0^\infty \frac{A(\epsilon)}{\epsilon^2}\, d\epsilon = \left\langle \bar{\Psi}_0 \left| \mathcal{J} \frac{1}{h^2} \mathcal{J} \right| \bar{\Psi}_0 \right\rangle$$

$$= \frac{1}{2M} \sum_j \left\langle \Psi_0 \left| \frac{\vec{q}\cdot\vec{p}_j}{m} \frac{1}{h^2} \frac{\vec{q}\cdot\vec{p}_j}{m} \right| \Psi_0 \right\rangle$$

$$= \frac{1}{N} \sum_j \langle \Psi_0 | (\vec{q}\cdot\vec{x}_j)^2 | \Psi_0 \rangle \to \infty$$

in the thermodynamic limit where $N \to \infty$ with $V/N = \text{const}$. Presumably ν will be unity in interacting many-particle systems.

[14]This comment is applied, for instance, to the author's own report circulated with the same title as that of the present paper. The so-called most general expression for D was given there, but it is actually seen to be vanishing. Whatever excuse could be given for its derivation, it was essentially based on the following treatment:

$$D\, q^2 = M^{-1} \langle \Psi_0 | [\vec{q}\cdot\vec{J}_+(q)]\, (e^{-\alpha h}/h)\, [\vec{q}\cdot\vec{J}_-(-q)] | \Psi_0 \rangle$$

$$+ M^{-1} \langle \Psi_0 | [\vec{q}\cdot\vec{J}_-(-q)]\, (e^{-\alpha h}/h)\, [\vec{q}\cdot\vec{J}_+(q)] | \Psi_0 \rangle$$

$$\to M^{-1} \langle \Psi_0 | [\vec{q}\cdot\vec{J}_+(0)]\, (e^{-\alpha h}/h)\, [\vec{q}\cdot\vec{J}_-(0)] | \Psi_0 \rangle$$

$$+ M^{-1} \langle \Psi_0 | [\vec{q}\cdot\vec{J}_-(0)]\, (e^{-\alpha h}/h)\, [\vec{q}\cdot\vec{J}_+(0)] | \Psi_0 \rangle \propto q^2 \alpha^{-\nu},$$

which is probably wrong.

[15]This comment cannot readily be extended to criticism against the Feynman relation for the Landau phonons in condensed liquid He^4. There it would be possible to show that the contribution to the density-density correlation function from high-energy excitations is at most $O(q^{2+x})$ with $x > 0$.