Approximate analytical solution of the nonlinear Boltzmann equation with an electron-electron interaction

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(Received 4 August 1993)

The nonlinear Boltzmannn equation in which electron-electron interactions are taken into account and in the presence of an electric field is solved by an iterative strongly convergent procedure starting from the linearized solution. The numerical calculations are interpolated so as to have an explicit analytical Fokker-Planck equation whose steady-state solution is of the kind of a Chapman-Cowling-Davydov expression. The analytical solution is useful to treat the long time tails of the distribution function which would demand 10^7 years of calculation by a Monte Carlo method.

In the case of spatial uniformity, the electron probability density $f(\mathbf{r},\mathbf{v},t)=f(\mathbf{v},t)$ is governed by the nonlinear Boltzmann equation in which the electron-electron interactions are taken into account:[1,2]

$$\frac{\partial f}{\partial t}+\mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}}=\int d^2\Omega\left\{\sum_i\left[\int d^3\mathbf{V}_i' F_i(\mathbf{V}_i',t)f(\mathbf{v}',t)\frac{|\mathbf{v}'-\mathbf{V}_i'|^3}{|\mathbf{v}-\mathbf{V}_i'|^3}\nu_{ai}(|\mathbf{v}'-\mathbf{V}_i'|,\mu_s)-\int d^3\mathbf{V}_i F_i(\mathbf{V}_i,t)f(\mathbf{v},t)\nu_{ai}(|\mathbf{v}-\mathbf{V}_i|,\mu_s)\right]\right\}$$
$$+\int d^2\Omega\left\{\int d^3\mathbf{v}_*' f(\mathbf{v}_*',t)f(\mathbf{v}',t)\frac{|\mathbf{v}-\mathbf{v}_*'|^3}{|\mathbf{v}-\mathbf{v}_*'|^3}\nu_e(|\mathbf{v}'-\mathbf{v}_*'|,\mu_s)-\int d^3\mathbf{v}_* f(\mathbf{v}_*,t)f(\mathbf{v},t)\nu_e(|\mathbf{v}-\mathbf{v}_*|,\mu_s)\right\}, \quad (1)$$

where t denotes time, $\mathbf{a}=e\mathbf{E}/m$ the electron acceleration due to the electric field $\mathbf{E}$, $\mathbf{v}'$ and $\mathbf{V}_i'$ are the velocities of the electron and the atom of the ith species, respectively, immediately before a scattering event such that the corresponding velocities after collision are $\mathbf{v}$ and $\mathbf{V}_i$ at a scattering angle θ_s with $\mu_s=\cos\theta_s$, $d^2\Omega=\sin\theta_s d\theta_s d\varphi_s$, where φ_s is the azimuthal scattering angle, $F_i(\mathbf{V}_i,t)$ the distribution function of the ith species of scattering atoms, ν_{ai} and ν_e the ith species atom-electron and electron-electron differential collision frequencies, respectively.

Equation (1) is rigorously valid when $f(\mathbf{r},\mathbf{v},t)$ does not depend on $\mathbf{r}$. Its linear part (in f) is still valid in the form of Eq. (1) even when $f(\mathbf{r},\mathbf{v},t)$ depends on $\mathbf{r}$ provided $F_i(\mathbf{r},\mathbf{V}_i,t)=F_i(\mathbf{V}_i,t)$ is $\mathbf{r}$ independent as can be proved by integrating the complete Boltzmann equation over $\mathbf{r}$. The nonlinear part of Eq. (1) (given by the last two terms of the collision integral) is an acceptable approximation upon integration over $\mathbf{r}$ when its contribution is modest compared to that of the linear part as can be seen by the density-gradient expansion.[3]

Except in the trivial case of equilibrium, for which $f(\mathbf{v})\propto\exp(-mv^2/2kT)$, no exact solutions are known for Eq. (1). There are numerical solutions, for instance, obtained by the Monte Carlo method. These solutions are very useful in many cases but not when the long time tail of $f(\mathbf{v},t)$ is of interest. For example, the $1/\omega$ flicker noise for electrons has been measured down to $\omega_{\min}=(1\text{ month})^{-1}$. Taking into account that the flight time of electrons in semiconductors is $\simeq 10^{-13}$ s one should calculate the trajectories of $30\times 86\,400/10^{-13}\simeq 2.6\times 10^{19}$ free flights, demanding a time of $\simeq 10^7$ years if using a modern supercomputer.

Approximate solutions are obtainable by the reduction of Eq. (1) to a Fokker-Planck equation. Chandrasekar[4] obtained it for the isotropic component f_0 of f and then he solved it numerically. Spitzer and collaborators[5,6] have extended this calculation by retaining the first two components of the Legendre expansion

$$f(\mathbf{v},t)=f_0(v,t)+\mu f_1(v,t)\,, \quad (2)$$

where $\mu=\mathbf{E}\cdot\mathbf{v}/Ev$. Then Rosenbluth and collaborators[7] further developed Spitzer's method in the case of an inverse-square force in the collision operator which generates logarithmic divergences at small scattering angles. To suppress the divergences they introduce a natural cutoff where the relative kinetic energy of the particles is approximated by $\frac{3}{2}kT$. They say that "it would probably not be justified in any event to consider the argument of $\ln D$ as better determined than this." Actually, this approximation is acceptable for $v^2>\langle v^2\rangle$ but not for $v^2\ll\langle v^2\rangle$ which is an important speed interval for the study of the flicker noise.

The complicated expressions appearing in this method are to be solved numerically and no explicit analytical solution is obtained. This method is therefore unable of producing the long time tail of $f(\mathbf{v},t)$.

Rosenbluth's method[7] has been simplified and improved by Gurevich[8] and by Kagan and Lyagushchenko[9] who introduce both the atom and electron temperatures (T and T_e, respectively) in the Coulomb logarithm. However, Gurevich[8] restricts his treatment to a small degree of ionization and to assumed Maxwellian distributions. This second point is criticized by Kagan and Lyagushchenko[9] who find an approximate expression for $f_0(v)$ in the steady-state case. No time dependence is considered

in Refs. 8 and 9 and experimental values for T_e are used in Ref. 9 in order to obtain tractable results.

Neither of the above approximations are used in our new method consisting of the use of Eq. (2) and of an iterative procedure for the nonlinear electron-electron interaction. In spite of our heavy use of the computer, we succeed to obtain an explicit approximate expression for the Fokker-Planck equation and its solution in the steady-state case by analytical interpolation of the partial numerical results.

For the electron-atom interactions we use the Davydov approximation,[10] i.e., $F(\mathbf{V},t)=\delta^3(\mathbf{V})$ and the substitution of f_0 for

$$\bar{f}_0=f_0+(kT/mv)(\partial f_0/\partial v) . \tag{3}$$

Multiplying the resulting equation by the jth Legendre polynomial ($P_0=1$ and $P_1=\mu$) and integrating over μ gives[1]

$$\frac{\partial f_0}{\partial t}+\frac{a}{3v^2}\frac{\partial}{\partial v}(v^2 f_1)=B_0 . \tag{4}$$

$$\frac{\partial f_1}{\partial t}+a\frac{\partial f_0}{\partial v}=B_1 , \tag{5}$$

where

$$B_j=\sum_i \int d^2\Omega\{f_j(v',t)(v'/v)^3\nu_{ai}(v',\mu_s)P_j(\mu_s)-f_j(v,t)\nu_{ai}(v,\mu_s)\}$$
$$+\frac{4Ne^4}{m^2}\int d^2\Omega_c(1-\mu_c)^{-2}\left\{\int d^3v'_*|\mathbf{v}-\mathbf{v}'_*|^{-3}f_0(v'_*,t)f_j(v',t)P_j(\mu_s)-\int d^3v_*|\mathbf{v}-\mathbf{v}_*|^{-3}f_0(v_*,t)f_j(v,t)\right\} , \tag{6}$$

μ_s being the scattering angle in the laboratory system.

To obtain Eq. (6) we have used the physical consideration that the number of electrons scattered inside an elementary solid angle is the same in any reference system, i.e., $\nu_e d^2\Omega=\nu_{ec}d^2\Omega_c$ in which the subscript c denotes the center-of-mass system and ν_{ec} is given by the Rutherford formula[11] (in Gauss system)

$$\nu_{ec}=\frac{4Ne^4}{m^2|\mathbf{v}'_c-\mathbf{v}'_{*c}|^3(1-\mu_c)^2} , \tag{7}$$

where N is the free-electron concentration and $\mu_c=\cos\theta_c$ the cosine of the scattering angle θ_c in the center-of-mass system. Moreover, we have also used the obvious relation $|\mathbf{v}'-\mathbf{v}'_*|=|\mathbf{v}'_c-\mathbf{v}'_{*c}|$ representing the nonrelativistic invariance of the relative velocities.

We approximated $f(\mathbf{v}_*,t)\simeq f_0(v_*,t)$ in Eq. (6) since $\mu f(v_*,t)$ is much smaller than f_0 for free electrons in gases and semiconductors when $E<E_{\text{in}}$, where E_{in} is the electric field value at which the inelastic collisions begin to be appreciable compared to the elastic ones. Obviously we cannot neglect $f_1(v,t)$ in Eq. (6) because f_1 appears as a factor and is not added to the leading term $f_0(v,t)$.

For $E<E_{\text{in}}$ and electron-atom scattering it is[12]

$$v'-v\simeq v(1-\mu_s)m/M_i , \tag{8}$$

where m and M_i are the masses of the electron and of the ith species atoms, respectively, so that the following first-order expansion is sufficient:

$$v'^3 f_j(v',t)\nu_{ai}(v',\mu_s)\simeq v^3 f_j(v,t)\nu_{ai}(v,\mu_s)+\frac{m}{M_i}v(1-\mu_s)\frac{\partial}{\partial v}[v^3 f_j(v,t)\nu_{ai}(v,\mu_s)] . \tag{9}$$

The scatterings between free electrons are always elastic (to within the soft radiated photons which, on the average, are compensated by the absorbed soft photons) but the recoil of the collided electrons can be large. However, because of Eq. (7), the scatterings with small θ_c (hence with $\mu_c=\cos\theta_c\simeq 1$) are much more effective. They imply small speed variations so that

$$f_j(v',t)\simeq f_j(v,t)+(v'-v)\frac{\partial}{\partial v}f_j(v,t) . \tag{10}$$

Substituting Eqs. (9) and (10) in Eq. (6) gives

$$B_0\simeq\sum_i\frac{m}{M_i}\frac{1}{v^2}\frac{\partial}{\partial v}[v^3\bar{f}_0(v,t)\nu_{ami}(v)]+A(v)\frac{\partial}{\partial v}\bar{f}_0(v,t) , \tag{11}$$

where

$$\nu_{ami}=\int d^2\Omega(1-\mu_s)\nu_{ai}(v,\mu_s) \tag{12}$$

is the collision frequency for momentum transfer for electrons with atoms and ions. Moreover,

$$A(v)=\frac{4Ne^4}{m^2}\int d^3\mathbf{v}'_*\int d^2\Omega_c(1-\mu_c)^{-2}\times\frac{v'-v}{|\mathbf{v}-\mathbf{v}'_*|^3}f_0(v'_*,t) . \tag{13}$$

For $j=1$ the second term on the rhs of Eq. (9) can be neglected (usual approximation) since the first term does not cancel. We get

$$B_1\simeq-\left[\sum_i\nu_{ami}+\langle\nu_{em}\rangle\right]f_1(v,t) , \tag{14}$$

where the average value of the electron-electron collision frequency ν_{em} for momentum transfer turns out to be

$$\langle\nu_{em}\rangle=\frac{4Ne^4}{m^2}\int d^3\mathbf{v}_*\int d^2\Omega_c(1-\mu_c)^{-2}|\mathbf{v}-\mathbf{v}_*|^3\times f_0(v_*,t)(1-\mu_s) . \tag{15}$$

As usual we put $\partial f_1/\partial t=0$ in Eq. (5) so that it becomes, by the use of Eq. (14),

$$a\frac{\partial f_0}{\partial v}=-\left[\langle \nu_{em}\rangle+\sum \nu_{ami}\right]f_1(v,t)\ . \tag{16}$$

Substituting Eq. (11) in Eq. (4) gives

$$\frac{\partial f_0}{\partial t}-\frac{a^2}{3v^2}\frac{\partial}{\partial v}\left[v^2\left[\langle \nu_{em}\rangle+\sum \nu_{ami}\right]^{-1}\frac{\partial f_0}{\partial v}\right]$$
$$=\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^3\bar{f}_0\sum\frac{m}{M_i}\nu_{ami}\right]+A\frac{\partial \bar{f}_0}{\partial v}\ . \tag{17}$$

The unknown function f_0 appears in A and $\langle \nu_{em}\rangle$ and is the cause of the nonlinearities in Eqs. (11) and (14). The physical idea to eliminate these nonlinearities is to replace f_0 in Eqs. (13) and (15) for its solution f_0' in the linear case, then solve again the system of Eqs. (4) and (5) finding a second-order solution f_0'' which is used to recompute Eqs. (13) and (15). Finally we check, that A and $\langle \nu_{em}\rangle$ are slightly changed by the new f_0''. In other words we use an iterative procedure and verify that it strongly converges so that the second step is already sufficient.

In order to obtain this iterative procedure, we recall the definition of the cosine of the scattering angle θ_c:

$$\mu_c=(\mathbf{v}'-\mathbf{v}_*')\cdot(\mathbf{v}-\mathbf{v}_*)|\mathbf{v}'-\mathbf{v}_*'|^{-1}|\mathbf{v}-\mathbf{v}_*|^{-1}\ , \tag{18}$$

where $\frac{1}{2}(\mathbf{v}-\mathbf{v}_*')$ and $\frac{1}{2}(\mathbf{v}-\mathbf{v}_*)$ are the velocities in the center-of-mass system immediately before and after a collision, respectively.

The 13 scalar quantities μ_c and the components of $\mathbf{v}',\mathbf{v}_*',\mathbf{v},\mathbf{v}_*$ are related by Eq. (18) and by other four scalar equations, one for the kinetic-energy conservation $v'^2+v_*'^2=v^2+\mathbf{v}_*^2$ and three for the components of the momentum $v_i'+v_{*i}'=v_i+v_{*i}$ with $i=1,2,3$.

Solving numerically the system of equations by standard mathematical routines and by the software package *Mathematica* we get

$$\begin{aligned} v_y'&=v_y'(\mathbf{v}_*',v_x',v,\mu,\varphi,\mu_c)\ ,\\ v_z'&=v_z'(\mathbf{v}_*',v_x',v,\mu,\varphi,\mu_c)\ ,\\ \mathbf{v}_*&=\mathbf{v}_*(\mathbf{v}_*',v_x',v,\mu,\varphi,\mu_c)\ . \end{aligned} \tag{19}$$

The integrands of Eqs. (13) and (15) do not depend on the azimuthal angle φ_c (in the center-of-mass system) so that we have $\int d^2\Omega_c=-2\pi\int d\mu_c$. The integrals (13) and (15) seem to be performed only on four variables $\mathbf{v}_*',v_x'$. Consequently, $A(v)$ and ν_{em} should be expressed as parametrically dependent on $\mathbf{v}=(v,\mu,\varphi)$ and v_x', which are the remaining four independent variables. In order to have a parametric dependence only on $|\mathbf{v}|$, we integrate Eqs. (13) and (15) on the additional variables μ,φ,v_x'. We get for $A(v)$

$$A(v)=\frac{4Ne^4}{m^2}\int d^3\mathbf{v}_*'f_0(v_*',t)\int_{-\infty}^{+\infty}dv_x'f_0(v_x',t)\int_{-1}^{1}\tfrac{1}{2}d\mu\int_0^{2\pi}d\varphi\int_{-1}^{\mu_{\rm CM}}d\mu_c(1-\mu_c)^{-2}\frac{v'(\mathbf{v}_*',v_x',v,\mu,\varphi,\mu_c)-v}{|\mathbf{v}(\mathbf{v}_*',v_x',v,\mu,\varphi,\mu_c)-\mathbf{v}_*'|^3}\ . \tag{20}$$

$A(v)$ is numerically computed for every v by means of the Monte Carlo method for a sevenfold integral, after having put Eq. (20) in an adimensional form with respect to the velocities $\mathbf{v}_*',v_x',v$. The extra integrations on v_x',μ,φ appearing on Eq. (20) are the counterpart of the physical fact that the scattering is characterized by two internal variables, namely, the impact parameter and an intrinsic angle. Note again that the values chosen for the *dependent* variables $\mathbf{v}_*',v_x',\mu,\varphi,\mu_c$ are not completely arbitrary but must constitute a *scattering compatible* subset, i.e., they have to be able to generate corresponding values for $v_y',v_z',v_{*x},v_{*y},v_{*z}$ obeying to the momentum and energy conservation. Notice also the integration over μ_c with the well-known logarithmic divergence at small scattering angles, which is usually limited by introducing a natural cutoff for $\theta_c\simeq\theta_{c\,\min}$, corresponding to a maximum for $\mu_c\simeq\mu_{\rm CM}$.

For the sake of simplicity we consider tentatively the most simple and rough method of introducing the cutoff, i.e., the Conwell-Weisskopf collision model we have studied in a preceding paper.[2] In that paper we limited to the case of slightly ionized helium and we derived a simple formula for $\mu_{\rm CM}(v)$ based on the well-known Conwell-Weisskopf expression for electron-ion interaction[2] $\nu=NZ^2e^4[m^2v^3(1-\mu_s)^2]^{-1}$. Since now the Rutherford formula (7) has to be used instead, we see that v^3 corresponds to $\frac{1}{4}|\mathbf{v}'-\mathbf{v}_*'|^3$ and the formula for $\mu_{\rm CM}(v)$, in the case of Si of our interest, can be extended in a straightforward way to

$$\mu_{\rm CM}=\frac{4^{-4/3}C|\mathbf{v}'-\mathbf{v}_*'|^4-1}{4^{-4/3}C|\mathbf{v}'-\mathbf{v}_*'|^4+1}\ , \tag{21}$$

where C is a constant[2] given by

$$C=10^{-13}(2.23N^{-0.775}-1.5\times10^{-14})\ , \tag{22}$$

v being measured in cm/s and N in cm^{-3}.

We now express Eq. (15) in a similar way. To compute the integrals (13) and (15) in a parallel way, it is better to have the integration for (15) over the same variables as in Eq. (20). We use the relation

$$d^3\mathbf{v}_*=d^3\mathbf{v}_*'|\mathbf{v}'-\mathbf{v}_*'|^3|\mathbf{v}-\mathbf{v}_*'|^{-3}$$

(valid for elastic collisions) in Eq. (15). Taking into account that for particles with the same mass undergoing elastic collisions it is $|\mathbf{v}'-\mathbf{v}_*'|=|\mathbf{v}-\mathbf{v}_*|$ we get

$$\langle \nu_{em}\rangle=\frac{2Ne^4}{m^2}\int d^3\mathbf{v}_*'\int_{-\infty}^{+\infty}dv_x'f_0(v_x',t)\int_{-1}^{1}d\mu\int_0^{2\pi}d\varphi\int_{-1}^{\mu_{\rm CM}}d\mu_c f_0[v_*(\mathbf{v}_*',v_x',v,\mu,\varphi,\mu_c),t](1-\mu_c)^{-2}$$
$$\times\frac{1-\mathbf{v}_*'\cdot\mathbf{v}(v'v)^{-1}}{|\mathbf{v}-\mathbf{v}_*'|^3}\ , \tag{23}$$

where v_* in $f_0(v_*,t)$ now has to be expressed as function of the eight variables assumed as independent.

We use for $f_0(v'_*,t)$ the solution of the linearized problem. We treat here the most important case which is the asymptotic or time-independent problem where f_0 is given by the Chapman-Cowling-Davydov expression.[10,13,2] By this f_0 we numerically get the value of $\langle \nu_{em} \rangle(v_r)$ and $A(v_r)$ as functions of the reduced speed

$$v_r = v/\langle v^2 \rangle^{1/2}$$

$$\langle \nu_{em} \rangle(v_r) = (2.178\times 10^{13} N^*)/(0.2523 v_r + 0.0909)^2 \quad (24)$$

and

$$A(v_r) = (1.2102\times 10^{14} N^*)/v_r^2 , \quad (25)$$

where N^* is a reduced density measured in units of 10^{15} cm^{-3}.

Because of Eq. (25) (valid in a wide v_r range) the last term of Eq. (17) may be written as $A(v)\partial \bar{f}_0/\partial v = v^{-2}\partial[v^2 A(v)\bar{f}_0]/\partial v$ so that Eq. (17) acquires the more simple final form [in which Eq. (3) has been used]

$$\frac{\partial f_0}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left\{ v^2 \left[\frac{a^2}{3}\left[\langle \nu_{em} \rangle + \sum \nu_{ami} \right]^{-1} \frac{\partial f_0}{\partial v} + \left[f_0 + \frac{kT}{mv}\frac{\partial f_0}{\partial v} \right] \left[A(v) + mv \sum \nu_{ami}/M_i \right] \right] \right\} , \quad (26)$$

where $\langle \nu_{em} \rangle$ and $A(v)$ have been calculated in the steady-state case $\partial f_0/\partial t = 0$.

When $\partial f_0/\partial t = 0$ the quantity inside the curly bracket must be a null constant since it vanishes for $v \to 0$. Consequently, we get

$$\frac{\partial f_0}{\partial v}\left[\frac{kT}{m} + \frac{a^2}{3\nu_{eq}^2} \right] + v f_0 = 0 , \quad (27)$$

where

$$\nu_{eq}^2 = \left[\langle \nu_{em} \rangle + \sum \nu_{ami} \right] \left[v^{-1} A(v) + m \sum \frac{\nu_{ami}}{M_i} \right] , \quad (28)$$

with $A(v)$ given by Eq. (25).

The solution of Eq. (27) is

$$f_0(v) \propto \exp \int_0^v - \frac{mv\, dv}{kT + ma^2(3\nu_{eq}^2)^{-1}} , \quad (29)$$

which is a kind of Chapman-Cowling-Davydov equation.

We calculate the second-order approximation of $\langle \nu_{em} \rangle$ and $A(v_r)$ by Eq. (29). We have found that $\langle \nu_{em} \rangle_2$ differs slightly from the value $\langle \nu_{em} \rangle$ of the first approximation and is

$$\langle \nu_{em} \rangle_2 = \frac{1.9315\times 10^{13}}{(0.1671 v_r + 0.1055)^2} . \quad (30)$$

Similarly the second-order approximation A_2 slightly differs from the first-order approximation A and is

$$A_2 = (1.54\times 10^{14} N^*)/v_r^2 . \quad (31)$$

We have therefore verified that our iterative procedure converges very rapidly so that already the second step is sufficient.

The explicit solution of Eq. (26) when $\partial f_0/\partial t = 0$ is given by Eq. (29). We clearly see the effect of the electron-electron interactions in ν_{eq} [given by Eq. (28)], which is the geometrical average of two factors. The first factor is simply the sum of the electron-ion, electron-neutral, and electron-electron interactions. The electron contribution in the second factor is larger since $v^{-1}A(v)$ is practically equal to $\langle \nu_{em} \rangle$ for $v_r \simeq \langle v_r^2 \rangle^{1/2}$ [as can be seen from Eqs. (30) and (31)] but the second term inside this second factor is strongly reduced by the ratio m/M_i. When the electron concentration is comparable with those of the ions and of the neutral molecules, it is $\langle \nu_{em} \rangle \simeq \sum \nu_{ami}$ but $v^{-1}A(v) \gg m\sum \nu_{ami}/M_i$. Consequently, Eqs. (28) and (29) are very different in the linear case (in which the electron-electron interactions are neglected) and in the nonlinear case (electron-electron interactions included). In spite of this $\langle \nu_{em} \rangle_2 \simeq \langle \nu_{em} \rangle$ and $A(v) \simeq A_2(v)$. The insensitivity of $\langle \nu_{em} \rangle$ and $A(v)$ from the shape of a normalized $f_0(v)$ implies an independence of $f_0(v,t)$ since at any given time t an $f_0(v,t)$ has a particular shape that could be that of a steady-state $f_0(v)$. It follows that Eqs. (30) and (31) may be used even in Eq. (26) with $\partial f_0/\partial t \neq 0$. Obviously a slightly better approximation should be obtained by $\langle \nu_{em} \rangle$ and $A(v)$ calculated by the linearized time-dependent solution.

Concluding, the explicit analytical Fokker-Planck equation is given by Eq. (26) with $\langle \nu_{em} \rangle$ and A is given by Eqs. (30) and (31), respectively. This has been achieved by the use of modern symbolic and numerical software tools. The analytical equation (26) is very useful for the study of the long time tails of f_0 as demanded for the electric noise and, in particular, for that having a spectral power density inversely proportional to the angular frequency ω.

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