# Dynamical mean-field theory for a spring-block model of fracture

Jørgen Vitting Andersen
*Department of Physics, McGill University, Rutherford Building, 3600 University Street, Montréal, Québec, Canada H3A 2T8*
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In a recently proposed spring-block model of fracture, it was found that growth of domains of positive and negative components of the stress field before cracking sets in, was crucial for the pattern formation of the cracks. In this paper a mean-field theory is proposed to describe the dynamic behavior of the stress field in the spring-block model. Mean-field site and pair approximations are made for the system before cracking sets in. The single-site mean-field approximation gives steady-state densities of positive, negative, and zero components of the stress field, in quantitative agreement with the spring-block model. The pair approximation gives densities of the components of the stress field which are in close agreement with the simulation of the spring model, and predicts domain growth of the positive and negative components of the stress field as seen in the spring-block model.

In a recent paper[1] a spring-block model was introduced to study general features of the statics and dynamics of fracturing. The spring-block model contained only one parameter, the ratio between the threshold for block slips and the threshold for springs to break. It was found that growth of domains of positive and negative components of the stress field before cracking sets in was crucial for the pattern formation of the cracks. In this paper a mean-field theory is introduced to study the dynamic behavior of the components of the stress field of the model in Ref. 1.[2] Mean-field site and pair approximations are made for the system before cracking sets in, i.e., in the limit where the threshold for springs to break is infinity.

The model in Ref. 1 consists of a two-dimensional array of blocks displaced from equidistant equilibrium points, and connected to four nearest-neighbor blocks via springs with a time-dependent coupling constant $K(t)$. The time-dependent coupling constant was introduced in Ref. 1 as an increasing time-dependent Young's modulus of the system, but, since the force on a block is given as the coupling constant times the displacement, one can just as well interpret the constant $K(t)$ as a length scale of the system (assuming a constant Young's modulus), representing an expansion or contraction of the system. Initially, the blocks are randomly displaced and connected to nearest-neighbor blocks via springs. Free-boundary conditions are used so that all the blocks at the boundary are only connected to blocks within the bulk.

A block slip occurs when the magnitude of the force on a block exceeds a threshold value for slip, $F_s$. The moving block is assumed to slip to zero-force position, which redefines the forces on those nearest-neighbor blocks connected to the sliding block.[3] A block slip can in turn give rise to new block slips, creating a chain reaction of events. The system is driven by a time-dependent coupling constant $K(t)$, which is assumed to increase linearly with time, reflecting the presence of a constant, increasing, external driving field. Units are chosen such that $K(t)=t+1$. The increasing coupling constant $K(t)$ continuously triggers new block slips.

The total force $\mathbf{F}_{i,j}$ on block $(i,j)$ is a vector and can be written as

$$\begin{aligned}\mathbf{F}_{i,j}&=(F^x_{i,j},F^y_{i,j})\\&=K(t)(4\,dx_{i,j}-dx_{i-1,j}-dx_{i+1,j}-dx_{i,j-1}-dx_{i,j+1}\,,\\&\qquad 4\,dy_{i,j}-dy_{i-1,j}-dy_{i+1,j}-dy_{i,j-1}-dy_{i,j+1})\,,\end{aligned}\tag{1}$$

where $dx_{i,j},dy_{i,j}$ are the displacements of the block $i,j$ in the $x$ and $y$ directions, respectively.

Since I only consider the model without allowing the springs to break, the model can be mapped onto a cellular automaton, which obeys the rules:[1,3]

(1) Initialize all sites so that the magnitude $|\mathbf{F}_{i,j}|$ of the force at site $i,j$ has a random value between 0 and $F_s$.

(2) If any $|\mathbf{F}_{i,j}|\geq F_s$ then redistribute the force $\mathbf{F}_{i,j}$ to its neighboring blocks:

$$\mathbf{F}_{nn}\rightarrow\mathbf{F}_{nn}+\alpha\mathbf{F}_{i,j},\quad \mathbf{F}_{i,j}\rightarrow\mathbf{0}\,.\tag{2}$$

Here $\alpha$ is a measure for the conservation of the redistributed force in a block slip.[4]

(3) Repeat step 2 until every unstable block has slipped.

(4) Locate the block with the largest strain $|\mathbf{F}_{\max}|$. Multiply all the sites by $F_s/|\mathbf{F}_{\max}|$ and return to step 2.

In order to formulate a dynamic mean-field theory I will only consider the block slide of one central block, and make three assumptions: (i) It is assumed that the force components of the system can be represented by coarse-grained values, so that a block can only have three values: a positive force component (represented as black), a negative force component (represented as white), and a zero force (represented by a 0). (ii) When the central block slides the four neighbors will be assigned the color of the central block, and the central block becomes a 0 block. (iii) It is assumed that one block slide is attempted per time unit. The mean-field assumption (i) corresponds in the spring-block model to replacing the specific form for the distributions of the negative and pos-

 

itive force components of the blocks at a given time by a constant. Assumption (ii) corresponds to assuming that the force on the blocks of the system can be represented by coarse-grained values, so that a block can only have three values: a positive force component (represented as black), a negative force component (represented as white), and a zero force (represented by a 0); (ii) When the central block slides the four neighbors will be assigned the color of the central block, and the central block becomes a 0 block. (iii) It is assumed that one block slide is attempted per time unit. The mean-field assumption (i) corresponds in the spring-block model to replacing the specific form for the distributions of the negative and positive force components of the four nearest neighbors will always point in the same direction as the block that slid, whatever their value was before the central block slid. In the spring-block model the blocks slide due to the increasing coupling constant which defines the time scale by setting $K(t)=t+1$. In (iii) it is assumed that the increasing coupling constant gives rise to a constant number of block slides per time unit.

Let $N_B$ denote the numbers of black blocks, $N_W$ the numbers of white blocks, and $N_0=N-N_B-N_W$ the number of 0 blocks (in a lattice with a total of $N$ blocks). Let $P(i)=N_i/N$ denote the corresponding probabilities. Let $P_B^S$ denote the probability that a black block slides per time unit and $P_W^S=1-P_B^S$ denote the probability that a white block slides per time unit. In the site approximation, the change in the number of (say) black blocks if a black block slides is given by $4(1-P_B)$ (the probability that the four neighbors were not black) $-1$ (since the black block that slides becomes a 0 site). Likewise the change in the number of black blocks when a white block slides is given by $-4P_B$. One easily derives the following equations of motion for the numbers of the different blocks:

$$dN_B(t)/dt=4[1-P_B(t)]P_B^S(t)-P_B^S(t)-4P_B(t)P_W^S(t)$$
$$=3P_B^S(t)-4P_B(t)\ , \tag{3}$$

$$dN_0(t)/dt=1-4P_0(t)P_W^S(t)-4P_0(t)P_B^S(t)$$
$$=1-4P_0(t)\ , \tag{4}$$

$$N_W(t)=N-N_B(t)-N_0(t)\ . \tag{5}$$

In steady state the site approximation thus gives the probabilities:

$$P_B^{SS}=\tfrac{3}{3}P_B^{S_{SS}}\ , \tag{6}$$

$$P_W^{SS}=\tfrac{3}{4}P_W^{S_{SS}}\ , \tag{7}$$

$$P_0^{SS}=\tfrac{1}{4}\ , \tag{8}$$

where the index SS stands for steady state. In the simulations of the spring-block model,[1] the system was initialized so that the total force on the blocks was zero. Since the increasing coupling constant gives rise to the same change in the multiplicative factor on both the negative and positive force components, the total force of the system was therefore always zero. In the mean-field approximation, a total force of zero means that the probabilities for black and white blocks are symmetrical so that $P_B(t)=P_W(t)$; $P_B^S(t)=P_W^S(t)\equiv\frac{1}{2}$. The steady-state single-site approximation thus predicts $P_0^{SS}=\frac{1}{4}$, $P_B^{SS}=P_W^{SS}=\frac{3}{8}$. This is to be compared with the simulations of the spring-block model which give $P_0^{SS}\approx 0.10$, $P_B^{SS}\approx P_W^{SS}\approx 0.45$, independent of lattice size.

Within the framework of the mean-field theory, one obtains a better approximation regarding bond probabilities, which is the next in a series of progressively more accurate cluster methods. Since every time a 0 block is created (by a block slide) it will have four black or white blocks as neighbors, one does not encounter 00 bonds. The model thus contains five different nearest-neighbor bond types, so the mean-field pair approximation takes the form of four coupled equations for the bond concentrations. Let $N_{ij}$ denote the number of $i$-$j$ nearest-neighbor bonds ($i,j=B,H,0$) and let $P_{ij}$ denote the corresponding probability for finding a bond of type $i$-$j$ (with, as mentioned above, $P_{00}=0$). In Fig. 1 is shown the process of a black block that slides, under the assumption that all four nearest-neighbor blocks take the color of the sliding block. The figure shows the 16 bonds that need to be considered in the pair approximation. If a black central block slides, the number of $BB$ bonds can be increased if one of the 12 bonds that are not nearest-neighbor bonds to the sliding block is a $BW$ or $B0$ bond. Only half of these cases will give rise to an increase in $BB$ bonds [namely, when the $W$ (0) block is nearest neighbor to the sliding block], thus giving the factors $6P_B^S(t)P_{BW}(t)$, $6P_B^S(t)P_{B0}(t)$ in the equation for $dN_{BB}(t)/dt$. Likewise the number of $BB$ bonds can decrease if a white block slides and if one of the 12 bonds that are not nearest-neighbor bonds to the sliding block is a BB bond. This gives a factor $-12P_W^S(t)P_{BB}(t)$ in the equation for $dN_{BB}(t)/dt$. Finally the number of $BB$ bonds can decrease if a black block slides and one of the four nearest neighbors is a black block. This is a conditional probability and can be calculated to be $-4P_B^S(t)P_{BB}(t)/2P_B(t)$. In general one has that the probability for a block to be of type $i$, given that one of its nearest neighbors is of type $j$, is $P(i|j)=P_{ij}/2P_j$.[2] Using similar arguments for the changes of the other bond numbers, the equations of motion within the pair approximation take the form

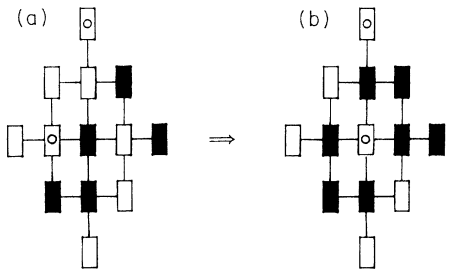


FIG. 1. Process of a black block that slides, under the assumption that all four nearest-neighbor blocks takes the color of the sliding block [Eqs. (9)–(12)]. The figure shows the 16 bonds that need to be considered in the pair approximation, Eqs. (9)–(16). (a) Before the block slides. (b) After the block slid.

$$dN_{BB}(t)/dt = 6P_B^S(t)P_{BW}(t) + 6P_B^S(t)P_{B0}(t) - 2P_B^S(t)\frac{P_{BB}(t)}{P_B(t)} - 12P_W^S(t)P_{BB}(t) , \quad (9)$$

$$dN_{WW}(t)/dt = 6P_W^S(t)P_{BW}(t) + 6P_W^S(t)P_{W0}(t) - 2P_W^S(t)\frac{P_{WW}(t)}{P_W(t)} - 12P_B^S(t)P_{WW}(t) , \quad (10)$$

$$dN_{BH}(t)/dt = 12P_B^S(t)P_{WW}(t) + 6P_B^S(t)P_{W0}(t) - 2P_B^S(t)P_{BW}(t)/P_B(t) - 6P_B^S(t)P_{BW}(t) + 12P_W^S(t)P_{BB}(t) + 6P_W^S(t)P_{B0}(t) - 2P_W^S(t)P_{BW}(t)/P_W(t) - 6P_W^S P_{BW}(t) , \quad (11)$$

$$DN_{B0}(t)/dt = 2P_B^S(t)[P_{BW}(t)/P_B(t) + P_{BB}(t)/P_B(T)] + 6P_B^S(t)P_{W0}(t) - 6P_B^S(t)P_{B0}(t) - 12P_W^S(t)P_{B0}(t) , \quad (12)$$

where $P_{W0}(t) = 1 - P_{BB}(t) - P_{WW}(t) - P_{BW}(t) - P_{B0}(t)$. The site probabilities are determined by $P_j = \frac{1}{2}(P_{jj} + \sum_i P_{ij})$.

Numerical integration of Eqs. (9)–(12) in the case of symmetrical probabilities for the white and black blocks gives $P_0^{SS} \approx 0.18$, $P_B^{SS} = P_W^{SS} \approx 0.41$, which gives a better agreement with the spring-block model than does the single-site approximation. One can now make a direct comparison with the "excess energy" $\Delta E(t)$ in the spring-block model, which was introduced to give a measure of the domain sizes of domains of equal force components at a given time $t$. $\Delta E$ is a measure of the total length of the domain boundary network[5–7] and scales as $\Delta E \sim \bar{R}(t)^{-1}$, where $\bar{R}(t)$ is a length scale of the ordered domains. In the pair approximation, one has $\Delta E(t) = E(t) - E^{SS}$ where $E(t) = P_B B(t) + P_{WW}(t) - P_{BW}(t)$ and $E^{SS} = \lim_{t\to\infty} E(t)$. In Fig. 2 are plotted the different bond probabilities together with $E(t)$ for a simulation with $N = 500$ blocks and assuming a time-independent probability for a block slip $P_B^S = P_W^S \equiv \frac{1}{2}$. The simulation was done with a bias in the initial distribution of black and white blocks. Initial distributions with symmetrical probabilities between black and white blocks, and black and white bonds, gave similar trends, with the same steady-state values for $E(t)$ and the bond probabilities. The fact that $E(t)$ is a positive and increasing function for initial values of time $t$ means that the simple mean-field assumptions actually are able to predict a domain growth of domains of equal force components, like the one that was observed in the spring-block model. Comparing the steady-state values of the bond probabilities in the mean-field assumption: $P_{BB}^{SS} = P_{WW}^{SS} \approx 0.17$; $P_{BW} \approx 0.28$; $P_{W0}^{SS} = P_{B0}^{SS} \approx 0.19$, with the steady-state values of the bond probabilities for simulations of the spring-block model: $P_{BB}^{SS} \approx P_{WW}^{SS} \approx 0.38$; $P_{BW} \approx 0.04$; $P_{W0}^{SS} \approx P_{B0}^{SS} \approx 0.10$ (for a lattice with 500 sites), one notices that the domains of equal force components are smaller and the interfaces between different force components are larger in the mean-field approximation. In Ref. 1 one found that $\Delta E(t) \propto t^{-\beta}$ with $\beta \approx \frac{1}{3}$, whereas the pair approximation predicts an exponential behavior $\Delta E(t) \propto \exp[-(t/\alpha)]$ with $\alpha = N$. It is not strange that the mean-field picture fails to predict the right dynamical behavior, especially because of the very crude approximation (ii) that assumes that one block slide is attempted per time unit. In the spring-block model the block slides occur due to the increasing coupling constant. It is not true that a new block slide occurs every time the coupling constant is increased by 1—rather the activity of block slides has a tendency to cluster so that time periods with a high activity are often followed by time periods without any activity. Besides, it was shown in Ref. 1 that the probability that a single block slide would result in $s$ other instantaneous block slides was a power-law distribution $D(s) \propto s^{-1.2}$ for the steady state. Even though $D(s)$ might have an exponential form for early times before the system has reached the steady state, it is clear that the total neglect of any temporal correlations in the mean-field assumption (ii) is at best a very crude approximation. As an attempt to include the above-mentioned temporal correlations, one could let the probabilities that a black or white block slides, $P_B^S$ and $P_W^S$, depend on, e.g., $1/P_B(t)$ and $1/P_W(t)$ instead of being constants, as was used for the simulation in Fig. 2. This would reflect the fact that the force in the spring-block model starts to accumulate on fewer blocks as the number of blocks with a zero force grows, so that the blocks with a force different from zero should be more likely to slide. Incorporating this into Eqs. (9)–(12) one again finds an exponentially fast domain growth with the same steady-state values as for the simulation in Fig. 2.

Another change in the mean-field approximation that one could make is to introduce a limited probability for invasion of a neighboring white block when a black block slides, and vice versa, instead of the simple ansatz where all four neighboring block always get the same color as the sliding block. One way to do this is to include a factor of $2P_W(t)$ for the probability that a white block becomes a black block when a black block slides, and a factor of $2P_B(t)$ for the probability that a black block be-

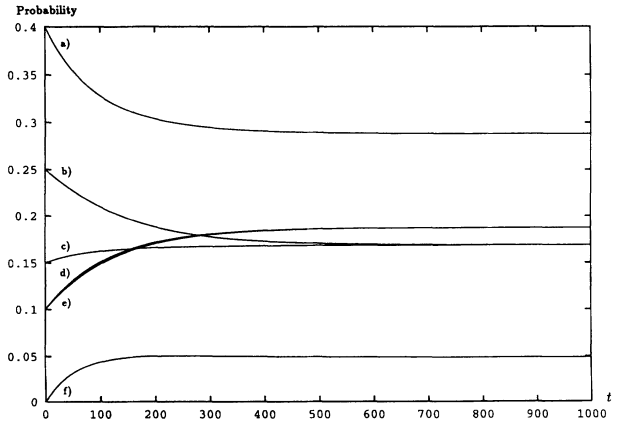


FIG. 2. Bond probabilities versus time $t$ under the assumption that all four nearest-neighbor blocks takes the color of the sliding block, Eqs. (9)–(12). (a) $P_{BB}(t)$, (b) $P_{WW}(t)$, (c) $P_{BW}(t)$, (d) $P_{W0}(t)$, and (e) $P_{B0}(t)$. (f) is the energy density defined by $E(t) \equiv P_{BB}(t) + P_{WW}(t) - P_{BW}$.

comes a white block when a white block slides. Starting with $P_W(0)=\frac{1}{2}$, this probability will be 1 as in the assumption (iii), but as time elapses and $P_W(t)$ becomes less than one-half, it will be more difficult for a black sliding block to invade a white neighboring block. This seems to be a natural assumption, since as noted above the force in the spring-block model starts to accumulate on fewer blocks as time elapses, thereby making the blocks more stable towards changes in sign when a neighboring block redistributes $\frac{1}{4}$ of it value. The spring-block model as noted before also contained the behavior of having instantaneous avalanches, meaning that for example a black block that slid eventually in a later block slip of the instantaneous avalanche again would become black.[8] With the same idea as before of a limited probability for invasion, one can then include a factor of $2P_i(t)$ $(i=W,B)$ for the probability that there will be any change when a block slides next to a block of the same color. With these remarks the modifications of Eqs. (9)–(12) take the form:

$$\frac{dN_{BB}(t)}{dt}=12P_B^S(t)P_{BW}(t)P_W(t)+6P_B^S(t)P_{B0}(t) -4P_B^S(t)P_{BB}(t)-24P_W^S(t)P_{BB}(t)P_B(t)\ , \quad (13)$$

$$\frac{dN_{WW}(t)}{dt}=12P_W^S(t)P_{BW}(t)P_B(t)+6P_W^S(t)P_{W0}(t) -4P_W^S(t)P_{WW}(t) -24P_B^S(t)P_{WW}(t)P_W(t)\ , \quad (14)$$

$$\frac{dN_{BH}(t)}{dt}=24P_B^S(t)P_{WW}(t)P_W(t)+6P_B^S(t)P_{W0}(t) -4P_B^S(t)\frac{P_{BW}(t)}{P_B(t)}P_W(t)-12P_B^S(t)P_{BW}(t)P_W(t) +24P_{W_S}(t)P_{BB}(t)P_B(t)+6P_W^S(t)P_{B0}(t) -4P_W^S(t)\frac{P_{BW}(t)}{P_W(t)}P_B(t)-12P_W^S P_{BW}(t)P_B(t)\ , \quad (15)$$

$$\frac{dN_{B0}(t)}{dt}=2P_B^S(t)\left[\frac{P_{BW}(t)}{P_B(t)}2P_W(t)+2P_{BB}(t)P_B(t)\right] +12P_B^S(t)P_{W0}(t)P_W(t)-6P_B^S(t)P_{B0}(t) -12P_W^S(t)P_{B0}(t)P_B(t)-6P_W^S(t)P_{B0}(t) +2P_W^S\frac{P_{BW}(t)}{P_W(t)}[1-2P_B(t)]\ . \quad (16)$$

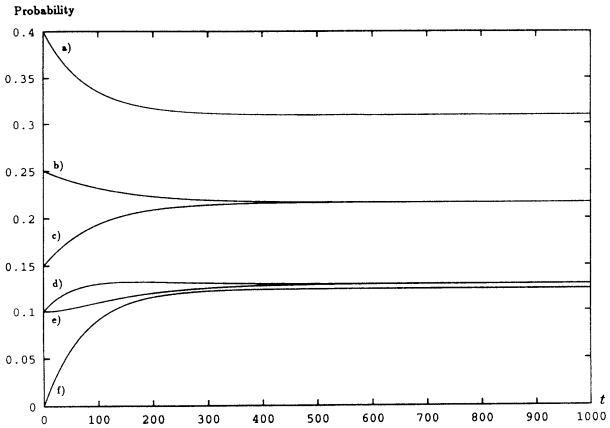


FIG. 3. Bond probabilities and energy density versus time $t$ under the assumption of a limited probability for change of color of the four neighboring blocks when a block slides, Eqs. (13)–(16). (a) $P_{BB}(t)$, (b) $P_{WW}(t)$, (c) $P_{BW}(t)$, (d) $P_{W0}(t)$, (e) $P_{B0}(t)$, and (f) $E(t)$.

In Fig. 3 are plotted the different bond probabilities together with $E(t)$ in the case where one has a limited probability for invading the neighboring block under a block slip, Eqs. (13)–(16). The initial distributions were as for the simulations shown in Fig. 2 and it is assumed that $P_B^S=P_W^S\equiv\frac{1}{2}$.

The steady-state values for the different probabilities were found to be: $P_0^{SS}\approx 0.12$, $P_B^{SS}=P_W^{SS}\approx 0.44$, $P_{BB}^{SS}=P_{WW}^{SS}\approx 0.22$, $P_{BW}\approx 0.31$, and $P_{W0}^{SS}=P_{B0}^{SS}\approx 0.13$. One notices that the pair approximation with limited probability for invasion gives close agreement with the densities of negative and positive components that were seen in the simulation of the spring-block model. As one would expect, the bond probabilities with a limited invasion probability give better agreement with the simulations of the spring-block model, but the mean-field theory still predicts too high a value for the probability of having interfaces between different force components. The dynamics of the domain growth is still found to have an exponential dependence.

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[3]Z. Olami *et al.*, Phys. Rev. Lett. **68**, 1244 (1992); K. Christensen and Z. Olami, Phys. Rev. A **45**, 665 (1992).

[4]Reference 1 only considered the case of a conservative redistribution, implying $\alpha=\frac{1}{4},\frac{1}{3},\frac{1}{2}$ for blocks in the bulk, at the sides, and at the corners, respectively.

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[6]O. G. Mouritsen, in *Kinetics of Ordering and Growth at Surfaces,* Vol. 59 of *NATO Advanced Study Institute, Series B: Physics,* edited by M. G. Lagally (Plenum, New York, 1990), p. 1.

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[8]The probability that an initially sliding (say) black block would become white at a later stage of a propagating avalanche is zero, since an avalanche of sliding blocks always is constrained to blocks with the same sign of force component. A black block sliding next to a white block makes the white block more stable.