Deformable superconductor model for the fluxon mass

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Outstanding difficulties in a deformable type-II superconductor model for the fluxon inertial mass per unit length μ_d are resolved. An identity for the inertial mass, valid for an arbitrary quasiparticle fraction when the ionic displacement field is irrotational, plays a critical role in the analysis. This approach avoids previously employed approximations, leading to qualitatively different results, including a fluxon mass which properly vanishes at the transition temperature and which has a greatly reduced magnitude. A framework for the solution of the elasticity equation for an isotropic superconductor is presented and the close relation between μ_d and the ionic strain field is shown.

The inertial mass of a vortex in a type-II superconductor can arise from a variety of mechanisms. Sources of the inertial mass include variation of the amplitude of the superconducting order parameter,[1] giving the "core" contribution, additional electromagnetic energy generated by the motion,[1-3] and mechanical stress and strain effects.[4-6] One of the latter mechanisms is investigated in this paper, in a model focusing on the difference between specific volumes in the normal and superconducting states.[4-6] There the vortex inertial mass is manifest in the kinetic energy of the ions of the crystal lattice in the strain field associated with the moving vortex. After a short review of the model, I describe a solution for the two-dimensional (2D) ionic displacement field and explicitly show the close relation between the vortex inertial mass per unit length μ_d and the ionic strain field. I then present an identity for evaluating the vortex mass.[7] This identity enables the vortex mass to be found directly from the superconductor dilation and the ionic displacement, thereby providing a convenient, compact expression.

Based upon different models of the superconductor quasiparticle fraction n, the resulting inertial mass is examined. Previous difficulties with the deformable superconductor model are discussed, including the behavior of μ_d near and at the transition temperature T_c. A newly derived expression for the fluxon mass is well behaved and has a high-temperature form which has a ready physical interpretation. The fluxon mass is numerically estimated and is no longer unusually large. In fact, the inertial mass arising from the elastic field mechanism is of the order of the electromagnetic contribution[1-3] or smaller. These results point up the importance of the identity derived to compute the inertial mass. Approximations employed earlier[6] do not yield the correct result.

I consider an infinite, isotropic superconductor whose elastic properties are characterized by Poisson ratio ν, ratio of difference of normal and superconducting volume to superconducting volume $\alpha=(V_n-V_s)/V_s$, and dilation αn,[4] where $n(\rho)$ is the fraction of quasiparticles about a flux line. The fraction n is normalized as 1 on the vortex axis. I follow a quasistatic approach to finding the strain field associated with vortex motion. Specifically, the vortex speed is restricted to well below the lattice sound speed v_a.[8,9] The discussion here is restricted to Abrikosov vortices in a continuous type-II superconductor. The elasticity equation for an isotropic superconductor with one vortex reads[6]

$$\frac{3(1-\nu)}{1+\nu}\nabla\nabla\cdot\mathbf{u}-\frac{3(1-2\nu)}{2(1+\nu)}\nabla\times\nabla\times\mathbf{u}=\alpha\nabla n\ , \tag{1}$$

where $\mathbf{u}$ is the ionic displacement. The right-hand side of this equation corresponds to the thermal analogy where the gradient of the dilation is replaced with the product of the coefficient of thermal expansion and the temperature gradient. Taking the divergence of Eq. (1) shows that the divergence of $\mathbf{u}$ generally satisfies the Poisson equation $\nabla^2(\nabla\cdot\mathbf{u})=\gamma\nabla^2 n$, where $\gamma\equiv\alpha(1+\nu)/3(1-\nu)$.

Of interest here is the subset of solutions of Eq. (1) for which

$$\nabla\times\mathbf{u}=0 \tag{2a}$$

and

$$\nabla\cdot\mathbf{u}=\gamma n+\text{const}\ . \tag{2b}$$

Then I can appeal to a theorem of vector analysis to the effect that a vector field is given once its divergence and curl are prescribed. (For a discussion of this theorem in 3D in electromagnetism, see Ref. 10.) When the displacement field is irrotational, it can be found as the gradient of a scalar potential ϕ,

$$\mathbf{u}=-\nabla\phi\ . \tag{3}$$

As $\nabla\cdot\mathbf{u}=-\nabla^2\phi$, the potential here is written in terms of the Green's function for the Laplacian in 2D,

$$\phi(\mathbf{x})=-\frac{1}{2\pi}\int_{R^2}(\nabla\cdot\mathbf{u})(\mathbf{x}')\ln|\mathbf{x}-\mathbf{x}'|d^2x'\ . \tag{4}$$

This result is illustrated for the case of cylindrical symmetry with polar coordinates (ρ,θ). In so doing one uses the relation

$$(x-x')^2+(y-y')^2=\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')\ . \tag{5}$$

We are assuming that n is independent of θ in which case we can conveniently set $\theta=0$. When the resulting angular integration in Eq. (4) is performed[11,12] we have

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$$\phi(\rho)=-\gamma\left[\int_{\rho}^{\infty}\ln\rho' n(\rho')\rho' d\rho'+\ln\rho\int_{0}^{\rho}n(\rho')\rho' d\rho'\right], \quad (6)$$

where we have put the constant of integration in Eq. (2b) to zero. Then from $u_\rho=-\partial\phi/\partial\rho$ we have

$$u_\rho(\rho)=\frac{\gamma}{\rho}\int_0^\rho n(\rho')\rho' d\rho' . \quad (7)$$

From the solution for the ionic displacement, Eq. (7) [or for the scalar potential, Eq. (6)], we can readily evaluate the strain and stress fields accompanying vortex motion. We do so here continuing under the restriction of cylindrical symmetry. The components of the strain field are found from[13,14]

$$\epsilon_\rho=\frac{\partial u_\rho}{\partial\rho} , \quad (8a)$$

$$\epsilon_\theta=\frac{u_\rho}{\rho} , \quad (8b)$$

giving

$$\epsilon_\rho=-\frac{\gamma}{\rho^2}\int^\rho n(\rho')\rho' d\rho'+\gamma n(\rho) \quad (9a)$$

and

$$\epsilon_\theta=\frac{\gamma}{\rho^2}\int^\rho n(\rho')\rho' d\rho' . \quad (9b)$$

As a check we have $\epsilon_\rho+\epsilon_\theta=\gamma n=\nabla\cdot\mathbf{u}$. Due to the cylindrical symmetry and $u_\theta=0$ the shearing strain $\gamma_{\rho\theta}$ vanishes.

The stress field can be computed according to the boundary conditions in the z direction.[13,14] If the ends of the sample are restrained in such a way that $\epsilon_z=0$, we treat a plane-strain problem. If, on the other hand, the ends are free to expand the stress component $\sigma_z=0$ and we have a plane-stress problem. In writing the stress components, Young's modulus E is introduced.

In the deformable superconductor model the vortex mass per unit length is found from the kinetic energy of the ions perturbed by the elastic inhomogeneity represented by the vortex, $\mu_d=2T_{\text{ion}}/v^2$, where $v \ll v_a$ is the vortex speed. We have[6]

$$\mu_d=\pi\rho_m\int_a^\infty\left[\left[\frac{\partial u_\rho}{\partial\rho}\right]^2+\left[\frac{u_\rho}{\rho}\right]^2\right]\rho d\rho , \quad (10)$$

where ρ_m is the ionic mass density and a, the lower cutoff, is taken as the interatomic spacing. Using Eqs. (8) we have

$$\mu_d=\pi\rho_m\int_a^\infty(\epsilon_\rho^2+\epsilon_\theta^2)\rho d\rho , \quad (11)$$

showing the direct connection between the strain field and inertial mass. The square of the magnitude of the strain field serves as the areal density for μ_d.

A useful identity for the fluxon inertial mass can be derived from Eq. (2b). By writing this equation in the radial coordinate ρ, squaring both sides, and integrating over ρ, I obtain

$$\frac{\mu_d}{\pi\rho_m}=\gamma^2\int_a^\infty n^2(\rho)\rho d\rho+u_\rho^2(a) . \quad (12)$$

In writing Eq. (12) the constant in Eq. (2b) has been ignored and the boundary condition $u_\rho(\infty)=0$ has been assumed. The quasiparticle fraction n due to the vortex can be written in terms of the superconducting gap $\Delta(\rho)$ as $n(\rho)=1-|\Delta(\rho)/\Delta(\infty)|^2$.

The fluxon mass can be simply found in the London limit of Ginzburg-Landau (GL) theory, where n is nonzero only for $0\le\rho\le\xi$, and ξ is the coherence length. In this limit we have $u_\rho(\rho)=\gamma\xi^2/2\rho$ and

$$\frac{\mu_d}{\pi\rho_m}=\frac{\gamma^2}{2}\left[\xi^2-a^2+\frac{\xi^4}{2a^2}\right] . \quad (13)$$

In London theory the ionic displacement becomes singular on the vortex axis. This is just one of the unrealistic features of the London model, as I presently discuss. This case is of special importance because the result of Ref. 6 is of the order of the London limit result.

The temperature dependence of $\mu_d(T)$ can be examined by taking illustrative models of the thermodynamic parameter α, approximately related to the bulk thermodynamic critical field H_c and its pressure derivative by $\alpha\simeq(H_c/4\pi)(\partial H_c/\partial p)_T$, and of the coherence length.[7] In making this examination the Poisson ratio, ionic mass density, and interatomic spacing can be assumed to vary little with temperature.[7] I find that μ_d as calculated in Ref. 6 and in the London limit does not vanish at the transition temperature T_c. Such a nonzero vortex mass in the normal state appears to be unphysical. Further discussion of this point and the relation between α and many other measurable quantities, including the coefficient of thermal expansion and bulk modulus, is given in Ref. 7.

The unrealistic result of the London model is connected with the abrupt change in the order parameter and quasiparticle fraction with distance. This behavior, in turn, leads to singular ionic displacement at $\rho=0$. The use of a quasiparticle fraction which decreases sufficiently rapidly with distance removes this difficulty.

Suppose that $n(\rho)=\exp(-\rho^2/\xi^2)$. This quasiparticle fraction is easy to work with analytically and satisfies the normalization condition $2\pi\int_0^\infty n(\rho)\rho d\rho=\pi\xi^2$. The resulting ionic displacement,

$$u_\rho(\rho)=\frac{\gamma\xi^2}{2\rho}(1-e^{-\rho^2/\xi^2}) , \quad (14)$$

has $u_\rho(0)=u_\rho(\infty)=0$ and a maximum which occurs at a distance of order ξ from the vortex axis. The fluxon mass found from Eq. (12) is

$$\frac{\mu_d}{\pi\rho_m}=\gamma^2\frac{\xi^2}{4}e^{-2a^2/\xi^2}+\frac{\gamma^2\xi^4}{4a^2}(1-e^{-a^2/\xi^2})^2 . \quad (15)$$

At high temperature, or whenever $a^2\ll\xi^2$, the fluxon mass per unit length takes the asymptotic form

$$\mu_d|_{a^2\ll\xi^2}=\pi\rho_m\frac{\gamma^2}{4}(\xi^2-a^2)\sim\pi\xi^2\rho_m\gamma^2 . \quad (16)$$

When ξ diverges with reduced temperature $t\equiv T/T_c$ as $(1-t^2)^{-1/2}$ and γ goes to zero as[7] $1-t^2$, it is clear μ_d in Eq. (16) properly reduces to zero as $t\to 1$. Figure 1

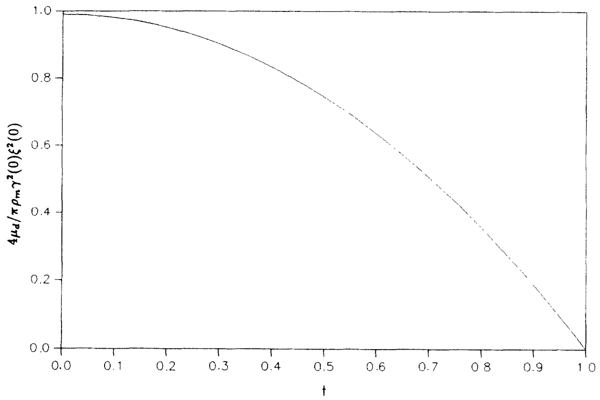

FIG. 1. Plot of the normalized vortex mass per unit length $4\mu_d/\pi\rho_m\gamma^2(0)\xi^2(0)$ from Eq. (15) vs reduced temperature $t \equiv T/T_c$. The ratio $a/\xi(0)$ of interatomic spacing a to the zero-temperature coherence length has been taken as 0.1.

shows a plot of the normalized vortex mass from Eq. (15) versus t.

Similarly, when $n(\rho)=2e^{-\rho/\xi}-e^{-2\rho/\xi}$, corresponding to an order parameter $\Delta(\rho)/\Delta(\infty)\approx 1-\exp(-\rho/\xi)$, it can be shown that[7] the asymptotic form $\mu_d \sim \pi\xi^2\rho_m\gamma^2$ again results when $a^2 \ll \xi^2$. This asymptotic form for $t \sim 1$ is a major result of this paper as it helps to resolve the outstanding problem of the temperature dependence of the fluxon mass per unit length. Moreover, the high-temperature form of μ_d has a ready physical interpretation. It is the mass per unit length of the ions contained in a region of the size of the vortex core, multiplied by the square of a thermodynamic quantity describing the superconducting-normal-state transition. The latter quantity (γ^2), proportional near T_c to the square of the bulk thermodynamic critical field, provides an envelope temperature dependence which ensures that μ_d vanishes at the transition temperature. Further, using the above approximate relation between α and H_c, μ_d can be written as the condensation energy per unit length $\xi^2H_c^2/8\pi$ divided by the square of a characteristic speed. Since the square of the longitudinal sound speed is $c_l^2=(3/\rho_m B)(1-\nu)/(1+\nu)$, where B is the reciprocal bulk modulus, this characteristic speed is related to the speed of sound in the superconductor.

In Ref. 6, the fluxon mass in the deformable superconductor model was calculated by omitting the contribution of ϵ_θ, the angular component of the strain field. This is a deficiency[7] as described above, giving a misleading result for μ_d.

Estimation of the vortex mass is of importance in describing dynamic vortex phenomena including rf response[3,15] and quantum tunneling.[16] (If it were possible to attain the necessary conditions of low temperature and very weak pinning, quantum tunneling of vortices may occur.) A suitable function for describing vortex response is the complex-valued dynamic mobility.[3,17,18] The mobility can simultaneously include the effects of inertia, pinning, flux flow, and flux creep. Knowledge of the vortex mass is desirable in determining whether to include an inertial term in the equation of motion or dynamic mobility. The vortex mobility can be written in the limiting case when the viscous drag force vanishes, where inertial effects should be pronounced.[3]

I estimate μ_d for low temperature based on Eq. (15) for a high-T_c superconductor with material parameters $\rho_m \simeq 6\times10^3$ kg/m^3; $\xi \simeq 1.5\times10^{-9}$ m, $\gamma \approx 10^{-5}$ ($H_c \approx 1$ T, $\nu \simeq 0.25$), and $a/\xi \simeq 0.1$. The first term of Eq. (15) provides the dominant contribution, $\mu_d(T=0)\approx 10^{-24}$ kg/m, or $\mu_d \approx 10^4\ m_e$/cm, where m_e is the electronic mass. This result is qualitatively different from that of Ref. 6, the discrepancy being 5 or 6 orders of magnitude. Because of the small ratio a^2/ξ^2, the second term in Eq. (15) does not contribute significantly.

Magneto-optics[19] or other measurements of the lattice strain field associated with vortex motion could provide important information on elastic mechanisms contributing to the total vortex mass. Knowledge of the detailed temperature dependence of the dilation or crystal strain field produced by fluxon motion could be useful in determining that of the resulting fluxon mass. Further discussion of such magneto-optics experiments is given elsewhere.[7]

In summary, outstanding difficulties in a deformable type-II superconductor model for the fluxon inertial mass μ_d have been resolved. The novel identity, Eq. (12), has played a critical role in the analysis. This identity permits μ_d to be easily evaluated given the quasiparticle fraction n. Specifically, the fluxon mass need not be approximated by ignoring the contribution of the angular component of the strain field.[6] As described in this paper, such an approximation has a qualitative effect on the result. The differences can also be seen in the result for the London limit compared to more realistic models of the variation of the magnitude of the order parameter and associated fraction n. By deriving an exact expression I developed a form of μ_d with at least two major features: a physically better temperature dependence in that $\mu_d(T=T_c)=0$ and a physically more reasonable numerical value at low temperature. For if the vortex inertial mass were as large as claimed previously its effects might well have been expected to be manifest in existing experimental data. The result for μ_d obtained in this paper, typified by Eq. (15), is of the order of 10^4 electronic masses per cm at low temperature for a high-T_c superconductor, being 5 or 6 orders of magnitude smaller than that of Ref. 6.

The result for the vortex inertial mass in the deformable superconductor model, like most of its kind, holds under a large number of restrictions, including very low fluxon speed,[8,9] and assumptions on the ionic strain field. The latter assumptions include an elastically isotropic superconductor and a radially symmetric displacement field. The present work provides a suitable basis for further extension and improvement of the deformable superconductor model.

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