Possible two-dimensional quasicrystal structures with a six-dimensional embedding space

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A discussion is given of all possible two-dimensional (2D) quasicrystal structures with a six-dimensional (6D) embedding space. Such quasicrystals have some properties different from the properties of known 2D quasicrystals. It is predicted that the diffraction pattern of the quasicrystal structures considered here can be indexed with integers using six rationally independent reciprocal basis vectors. They have seven-, nine-, fourteen-, and eighteenfold symmetries. The six hydrodynamic degrees of freedom in phases can be parametrized by three two-dimensional vector fields, and hence there are two types of phason fields in each case mentioned above. All quadratic invariants and elastic energy densities are also derived for such quasicrystals with use of group-representation theory.

I. INTRODUCTION

Since the detection of a quasicrystalline phase in the system Al-Mn by Shechtman *et al.*,[1] a number of other quasicrystals have been discovered.[2–4] The symmetries in the quasicrystals observed to date are icosahedral, decagonal (or perhaps pentagonal), octagonal, and dodecagonal. All quasicrystals except the icosahedral ones are quasi-two-dimensional. Many studies have been made on such quasicrystals both in theory and experiment.[5] These two-dimensional (2D) quasicrystals have a few essential properties in common. First, one needs a set of four rationally independent reciprocal basis vectors to index the diffraction pattern with integers. Second, these basis vectors can be considered to be a projection from a 4D embedding space (V) upon the 2D physical space (V_E). Third, the space V is the direct sum of V_E and V_I where V_I is the orthogonal complementary space. Finally, the four hydrodynamic degrees of freedom in phases can be parametrized by two two-dimensional vector fields. One of them is the phonon field (denoted by $\mathbf{u}$), and the other is the phason field (denoted by $\mathbf{w}$). One will naturally ask what is the next 2D quasicrystal structure (if it exists) which can be expected to be observed? From the study of the symmetry operations for quasiperiodic structures by Janssen[6] it follows that all the 2D quasicrystal structures with a 4D embedding space have already been discovered. The noncrystallographic orientational symmetries in such materials are only decagonal (or pentagonal), octagonal, and dodecagonal. The next 2D quasicrystal structures (if they exist) may have a 6D embedding space. The symmetries should be seven-, nine-, fourteen-, and eighteenfold. The higher-dimensional description and group structure of a possible quasicrystal with sevenfold symmetry have also been discussed by some authors;[7–9] but, nevertheless, the authors do not investigate the elastic and structural properties of quasicrystals with all the possible symmetries mentioned above. Although no physical systems with such symmetries have been discovered so far, it seems worthwhile to make theoretical predictions about their properties. This is the purpose of this paper.

In this paper we would like to investigate all 2D quasicrystals with a 6D embedding space. We analyze their structural and symmetry properties. We also derive all quadratic invariants and the expression for the elastic energy (density) f. Our results show that such quasicrystals have some properties different from the properties of known 2D quasicrystals. According to the phenomenological Landau theory,[10] the ordered phase can be described in terms of a Landau free energy F that can be expanded in a power series in the mass density $\rho(\mathbf{r})$. For example, the kth power of $\rho(\mathbf{r})$ gives rise to terms in F of the form

$$F^{(k)}=A_k\sum_{m_1m_2\cdots m_k}\int d\mathbf{r}\,\rho_{\mathbf{G}_{m_1}}\rho_{\mathbf{G}_{m_2}}\cdots\rho_{\mathbf{G}_{m_k}}\exp\Big(\sum_i \mathbf{G}_{m_i}\cdot\mathbf{r}\Big)$$
$$=VA_k\sum_{\mathbf{G}_m\in L_{\mathbf{R}}}\Delta\left[\sum_i \mathbf{G}_{m_i}\right]\cos\left[\sum_i \phi_{\mathbf{G}_{m_i}}\right]\prod_i|\rho_{\mathbf{G}_{m_i}}| , \tag{1.1}$$

where ρ_G is the Fourier component of $\rho(\mathbf{r})$ with an amplitude $|\rho_G|$ and a phase Φ_G. The factor $\Delta(x)=\delta_{x,0}$ ensures that only terms where $\sum G_i=0$ contribute to the sum. The equilibrium ordered state is given by the (nonzero) values of $|\rho_G|$ and Φ_G that minimize F. In practice, it is sufficient to consider a small finite subset $\{\rho_G\}$ of the Fourier components ρ_G. Obviously, the subset $\{\rho_G\}$ must include Fourier components associated with the N reciprocal vectors that form a basis of the reciprocal lattice L_R. In addition, $\{\rho_G\}$ must include the inverse components $\{\rho_{-G}\}$ plus any vectors that can be obtained from the minimal set by point-symmetry operations associated with the scale of the orientational system. In d-dimensional quasicrystals, $N=n_i d$ where n_i is the number of incommensurate lengths associated with each lattice-vector direction. From Eq. (1.1) it is clear that for any set of G_i's that satisfies $\sum G_i=0$, minimization of F with respect to $|\rho_G|$ and Φ_G leads to a minimum-energy state with constraints on the Φ_G's. These constraints leave unspecified N Φ_G's. Because uniform shifts in these N phases leave F unchanged, they correspond to

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hydrodynamic variables in the theory. For the possible 2D quasicrystal structures considered in this article, six independent vectors can be used to construct the reciprocal lattice L_R, and the number of relatively incommensurate lengths is 3. For instance, in the case of heptagonal symmetry the seven vectors $\mathbf{G}_i = G[\cos(2\pi i/7), \sin(2\pi/7)]$, $i=0,1,2,\ldots,6$, determine the sevenfold symmetry and generate the reciprocal lattice where G sets the unique length scale of the system. These vectors are not, however, independent because $\sum \mathbf{G}_i = 0$, and any six of them can form a basis of L_R. Three collinear vectors $\mathbf{G}_0$, $\mathbf{G}_1+\mathbf{G}_6$, and $\mathbf{G}_2+\mathbf{G}_5$ are incommensurate. In the minimum-energy state the phases Φ_i associated with $\mathbf{G}_i$ satisfy $\sum \Phi_i = \text{constant}$, leaving six ($n_i = 3, N = 3d = 6$) independent components of Φ_i. Thus, we can parametrize these independent components with three two-component fields $\mathbf{u} = (u_x, u_y)$, $\mathbf{v} = (v_x, v_y)$, and $\mathbf{w} = (w_x, w_y)$. For example, in the heptagonal case

$$\phi_i = \mathbf{G}_i \cdot \mathbf{u} + \alpha \mathbf{G}_{\langle 2i \rangle_7} \cdot \mathbf{v} + \alpha' \mathbf{G}_{\langle 3i \rangle_7} \cdot \mathbf{w} , \tag{1.2}$$

where $\langle n \rangle_p$ means n mod$p$, and α and α' are scale factors. The higher-dimensional description of such quasicrystals requires a 6D embedding space. The vectors $\tilde{\mathbf{G}}_i$ with six components $(\tilde{\mathbf{G}}_i)_n$, $n=1,2,\ldots,6$, span a 6D lattice. Equation (1.2) can be written in the form

$$\phi_i = \tilde{\mathbf{G}}_i \cdot \tilde{\mathbf{u}} , \tag{1.3}$$

where $\tilde{\mathbf{u}}$ is the direct sum of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Since the dot product is a scalar, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{G}}$ must transform under the same representation of the point group associated with the quasicrystal considered. Meanwhile $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ can be identified as hydrodynamic variables. The phonon variable $\mathbf{u}$ transforms according to a vectorlike representation of the rotational symmetry group, whereas the phason variables $\mathbf{v}$ and $\mathbf{w}$ transform according to two other representations, neither of which is vectorlike. It should be noted that there are two types of phason field in this case. This means that there are 2D diffusive modes in addition to the modes found in conventional crystals. This characteristic seems to be unique to such quasicrystals. From the known transformation properties of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ under the point-symmetry operations, it is straightforward, with use of group theory, to construct all quadratic invariants involving gradients of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and calculate the elastic energy. In this way we find that the form of the terms that involve only one hydrodynamic variable in the elastic energy is the same for all symmetries, but the form of the terms that couple two variables is different for different symmetries. Moreover, there could be coupling terms not only between phonon and phason fields, but also between two different types of phason field.

The organization of this article is as follows. In Sec. II group theory is employed to obtain some structural and elastic properties of the quasicrystal with sevenfold symmetry. In Sec. III we investigate the quasicrystals with nine-, fourteen-, and eighteenfold symmetries in a similar manner, with an emphasis on new features. Finally, some remarks are made in Sec. IV.

II. SEVENFOLD SYMMETRY

We use the same method as is used in Ref. 11. Six rationally independent reciprocal vectors $\mathbf{G}_i = G[\cos(2\pi i/7), \sin(2\pi i/7)]$, $i=1,2,\ldots,6$, form the basis of the lattice for the 2D quasicrystal with sevenfold symmetry. The point group C_{7v} has 14 elements, five conjugacy classes, and five irreducible representations, two of which are one dimensional and three two dimensional (see Tables I–IV). Two generators are the sevenfold rotation α and the mirror operation β. The action of α and β on the basis vectors is given by

$$\Gamma(\alpha) = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{vmatrix} ,$$
$$\Gamma(\beta) = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} . \tag{2.1}$$

This matrix representation Γ is reducible. The reduction is

$$\Gamma = \Gamma_3 + \Gamma_4 + \Gamma_5 . \tag{2.2}$$

It can be seen that the rotations in V_E correspond to the irreducible component Γ_3, which is a vector representation, so the phonon field $\mathbf{u}$ transforms according to this representation Γ_3. The rotations in V_I correspond to the two other irreducible components Γ_4 and Γ_5, neither of which is vectorlike. This means that two types of phason field transform according to them. One, let us say $\mathbf{v}$, transforms under Γ_4, and the other transforms under Γ_5. Since the displacement gradients $\partial_j u_i$, $\partial_j v_i$, and $\partial_j w_i$ $(i,j=1,2)$ transform according to their respective direct-product representations, we can construct all quadratic invariants composed of these gradients and hence find the expression for the elastic energy to quadratic order with the help of group-representation theory.

By an argument very similar to that given in Ref. 11, for the phonon field, $\partial_j u_i$ transform under

$$\Gamma_3 \times \Gamma_3 = \Gamma_1 + \Gamma_2 + \Gamma_4 . \tag{2.3}$$

TABLE I. Character table for point groups C_{7v}. ρ is the primitive seventh root of unity ($\rho^7=1$).

	ε	α	α^2	α^3	β
Γ_1	1	1	1	1	1
Γ_2	1	1	1	1	-1
Γ_3	2	$\rho+\rho^{-1}$	$\rho^2+\rho^{-2}$	$\rho^3+\rho^{-3}$	0
Γ_4	2	$\rho^2+\rho^{-2}$	$\rho^3+\rho^{-3}$	$\rho+\rho^{-1}$	0
Γ_5	2	$\rho^3+\rho^{-3}$	$\rho+\rho^{-1}$	$\rho^2+\rho^{-2}$	0

TABLE II. Character table for point group C_{14v}. ρ is the primitive seventh root of unity ($\rho^7=1$).

	ε	α	α^2	α^3	α^4	α^5	α^6	α^7	β	$\alpha\beta$
Γ_1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	1	1	1	1	1	-1	-1
Γ_3	1	-1	1	-1	1	-1	1	-1	1	-1
Γ_4	1	-1	1	-1	1	-1	1	-1	-1	1
Γ_5	2	$-(\rho^3+\rho^{-3})$	$\rho+\rho^{-1}$	$-(\rho^2+\rho^{-2})$	$\rho^2+\rho^{-2}$	$-(\rho+\rho^{-1})$	$\rho^3+\rho^{-3}$	-2	0	0
Γ_6	2	$(\rho^3+\rho^{-3})$	$\rho+\rho^{-1}$	$(\rho^2+\rho^{-2})$	$\rho^2+\rho^{-2}$	$(\rho+\rho^{-1})$	$\rho^3+\rho^{-3}$	2	0	0
Γ_7	2	$-(\rho+\rho^{-1})$	$\rho^2+\rho^{-2}$	$-(\rho^3+\rho^{-3})$	$\rho^3+\rho^{-3}$	$-(\rho^2+\rho^{-2})$	$\rho+\rho^{-1}$	-2	0	0
Γ_8	2	$(\rho+\rho^{-1})$	$\rho^2+\rho^{-2}$	$\rho^3+\rho^{-3}$	$\rho^3+\rho^{-3}$	$\rho^2+\rho^{-2}$	$\rho+\rho^{-1}$	2	0	0
Γ_9	2	$-(\rho^2+\rho^{-2})$	$\rho^3+\rho^{-3}$	$-(\rho+\rho^{-1})$	$\rho+\rho^{-1}$	$-(\rho^3+\rho^{-3})$	$\rho^2+\rho^{-2}$	-2	0	0
Γ_{10}	2	$(\rho^2+\rho^{-2})$	$\rho^3+\rho^{-3}$	$(\rho+\rho^{-1})$	$\rho+\rho^{-1}$	$\rho^3+\rho^{-3}$	$\rho^2+\rho^{-2}$	2	0	0

We have two quadratic invariants

$$(E_{11}+E_{22})^2\ ,\quad (E_{11}-E_{22})^2+(2E_{12})^2\ ,$$

or equivalently,

$$(E_{11}+E_{22})^2\ ,\quad (E_{11}E_{22}-E_{12}^2)\ , \tag{2.4}$$

where the notations $E_{ij}=(\partial_j u_i+\partial_i u_j)/2$ is used. This field behaves like the usual isotropic solids. There are two independent elastic constants: λ and μ. The part of the elastic energy due to the phonon field is

$$f^u=\tfrac{1}{2}\lambda(\nabla\cdot\mathbf{u})^2+\mu E_{ij}E_{ij}\ . \tag{2.5}$$

For the phason variable $\mathbf{v}$, four components of $\partial_j v_i$ transform under

$$\Gamma_3\times\Gamma_4=\Gamma_3+\Gamma_5\ . \tag{2.6}$$

Among them the pair $(\partial_1 v_1+\partial_2 v_2, \partial_1 v_2-\partial_2 v_1)$ spans the 2D irreducible representation Γ_3, and the pair $(\partial_1 v_1-\partial_2 v_2, \partial_1 v_2+\partial_2 v_1)$ spans the 2D irreducible representation Γ_5. We thus have two quadratic invariants:

$$\begin{aligned}&(\partial_1 v_1+\partial_2 v_2)^2+(\partial_1 v_2-\partial_2 v_1)^2\ ,\\ &(\partial_1 v_1-\partial_2 v_2)^2+(\partial_1 v_2+\partial_2 v_1)^2\ ,\end{aligned} \tag{2.7}$$

and then two independent elastic constants: C_1 and C_2. The part of the elastic energy arising from this variable is

$$\begin{aligned}f^v=&C_1[(\partial_1 v_1+\partial_2 v_2)^2+(\partial_1 v_2-\partial_2 v_1)^2]\\ &+C_2[(\partial_1 v_1-\partial_2 v_2)^2+(\partial_1 v_2+\partial_2 v_1)^2]\ .\end{aligned} \tag{2.8}$$

For the phason variable $\mathbf{w}$, four components of $\partial_j w_i$ transform under

$$\Gamma_3\times\Gamma_5=\Gamma_4+\Gamma_5\ . \tag{2.9}$$

Among them the pair $(\partial_1 w_1+\partial_2 w_2, \partial_1 w_2-\partial_2 w_1)$ spans the 2D irreducible representation Γ_4, and the pair $(\partial_1 w_1-\partial_2 w_2, \partial_1 w_2+\partial_2 w_1)$ spans the 2D irreducible representation Γ_5. It follows that there are two quadratic invariants:

$$\begin{aligned}&(\partial_1 w_1+\partial_2 w_2)^2+(\partial_1 w_2-\partial_2 w_1)^2\ ,\\ &(\partial_1 w_1-\partial_2 w_2)^2+(\partial_1 w_2+\partial_2 w_1)^2\ ,\end{aligned} \tag{2.10}$$

and two independent elastic constants: K_1 and K_2. The relevant contribution to the elastic energy is

$$\begin{aligned}f^w=&K_1[(\partial_1 w_1+\partial_2 w_2)^2+(\partial_1 w_2-\partial_2 w_1)^2]\\ &+K_2[(\partial_1 w_1-\partial_2 w_2)^2+(\partial_1 w_2+\partial_2 w_1)^2]\ .\end{aligned} \tag{2.11}$$

Finally, notice that the irreducible representation Γ_4 occurs in both of the reduction equations (2.3) and (2.9). This means that there exists an invariant

$$(E_{11}-E_{22})(\partial_1 w_1+\partial_2 w_2)+2E_{12}(\partial_1 w_2-\partial_2 w_1) \tag{2.12}$$

coupling the gradients of $\mathbf{u}$ and $\mathbf{w}$, and leading to a cross term between photon and phason fields with the form of

$$f^{uw}=R_1[(E_{11}-E_{22})(\partial_1 w_1+\partial_2 w_2)+2E_{12}(\partial_1 w_2-\partial_2 w_1)] \tag{2.13}$$

in the elastic energy. Similarly, since the irreducible representation Γ_5 occurs in both of the reduction equations (2.6) and (2.9), there exists an invariant

$$(\partial_1 v_1-\partial_2 v_2)(\partial_1 w_1-\partial_2 w_2)+(\partial_1 v_2+\partial_2 v_1)(\partial_1 w_2+\partial_2 w_1) \tag{2.14}$$

coupling the gradients of $\mathbf{v}$ and $\mathbf{w}$, and leading to a cross term between two different types of phason field as follows:

$$\begin{aligned}f^{vw}=&R_2[(\partial_1 v_1-\partial_2 v_2)(\partial_1 w_1-\partial_2 w_2)\\ &+(\partial_1 v_2+\partial_2 v_1)(\partial_1 w_2+\partial_2 w_1)]\ .\end{aligned} \tag{2.15}$$

The above term results from the fact that there are two types of phason field in the quasicrystals considered here.

TABLE III. ρ is the primitive ninth root of unity ($\rho^9=1$).

	ε	α	α^2	α^3	α^4	β
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	1	1	-1
Γ_3	2	$\rho+\rho^{-1}$	$\rho^2+\rho^{-2}$	-1	$\rho^4+\rho^{-4}$	0
Γ_4	2	$\rho^2+\rho^{-2}$	$\rho^4+\rho^{-4}$	-1	$\rho+\rho^{-1}$	0
Γ_5	2	$\rho^4+\rho^{-4}$	$\rho+\rho^{-1}$	-1	$\rho^2+\rho^{-2}$	0
Γ_6	2	-1	-1	2	-1	0

TABLE IV. Character table for point group C_{18v}. ρ is the primitive ninth root of unity ($\rho^9=1$).

	ε	α	α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	β	$\alpha\beta$
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	1	1	1	1	1	1	1	-1	-1
Γ_3	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
Γ_4	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1
Γ_5	2	$-(\rho^4+\rho^{-4})$	$\rho+\rho^{-1}$	1	$\rho^2+\rho^{-2}$	$-(\rho^2+\rho^{-2})$	-1	$-(\rho+\rho^{-1})$	$\rho^4+\rho^{-4}$	-2	0	0
Γ_6	2	$\rho^4+\rho^{-4}$	$\rho+\rho^{-1}$	-1	$\rho^2+\rho^{-2}$	$\rho^2+\rho^{-2}$	-1	$\rho+\rho^{-1}$	$\rho^4+\rho^{-4}$	2	0	0
Γ_7	2	$-(\rho+\rho^{-1})$	$\rho^2+\rho^{-2}$	1	$\rho^4+\rho^{-4}$	$-(\rho^4+\rho^{-4})$	-1	$-(\rho^2+\rho^{-2})$	$\rho+\rho^{-1}$	-2	0	0
Γ_8	2	$\rho+\rho^{-1}$	$\rho^2+\rho^{-2}$	-1	$\rho^4+\rho^{-4}$	$\rho^4+\rho^{-4}$	-1	$\rho^2+\rho^{-2}$	$\rho+\rho^{-1}$	2	0	0
Γ_9	2	$-(\rho^2+\rho^{-2})$	$\rho^4+\rho^{-4}$	1	$\rho+\rho^{-1}$	$-(\rho+\rho^{-1})$	-1	$-(\rho^4+\rho^{-4})$	$\rho^2+\rho^{-2}$	-2	0	0
Γ_{10}	2	$\rho^2+\rho^{-2}$	$\rho^4+\rho^{-4}$	-1	$\rho+\rho^{-1}$	$\rho+\rho^{-1}$	-1	$\rho^4+\rho^{-4}$	$\rho^2+\rho^{-2}$	2	0	0
Γ_{11}	2	1	-1	-2	-1	1	2	1	-1	-2	0	0
Γ_{12}	2	-1	-1	2	-1	-1	2	-1	-1	2	0	0

Therefore the 2D quasicrystal with sevenfold symmetry has eight independent elastic constants in all. The total elastic energy is

$$f=f^u+f^v+f^w+f^{uw}+f^{vw}\ , \tag{2.16}$$

where f^u, f^v, f^w, f^{uw}, and f^{vw} are given by Eqs. (2.5), (2.8), (2.11), (2.13), and (2.15), respectively.

III. NINE-, FOURTEEN-, AND EIGHTEENFOLD SYMMETRIES

It is straightforward to extend the mathematical treatment of sevenfold symmetry given in the preceding section to nine-, fourteen, and eighteenfold symmetries. This allows a very similar calculation to be illustrated in all the cases mentioned above, which greatly simplifies the analysis.

Six rationally independent reciprocal vectors $\mathbf{G}_i=G[\cos(2\pi i/9),\sin(2\pi i/9)]$, $i=\pm1,\pm2,\pm3$, form the basis of the lattice for the 2D quasicrystal with ninefold symmetry. The point group C_{9v} has 18 elements, six conjugacy classes, and six irreducible representations, two 1D and four 2D (see Table III). The action of a ninefold rotation α and a mirror β on the basis vectors is given by

$$\Gamma(\alpha)=\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & -1 & -1 & 0 & 0 \end{vmatrix}\ ,$$
$$\Gamma(\beta)=\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}\ . \tag{3.1}$$

The matrix representation Γ reduces to

$$\Gamma=\Gamma_3+\Gamma_4+\Gamma_5\ . \tag{3.2}$$

This means that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ transform under Γ_3, Γ_4, and Γ_5, respectively. For the phonon field, $\partial_j u_i$ transform under

$$\Gamma_3\times\Gamma_3=\Gamma_1+\Gamma_2+\Gamma_4\ . \tag{3.3}$$

It follows that there are two quadratic invariants, taking the same form as those in Eq. (2.4) and leading to f^u of the same form as those in Eq. (2.5). For the two types of phason field, $\partial_j v_i$ transform under

$$\Gamma_3\times\Gamma_4=\Gamma_3+\Gamma_6\ , \tag{3.4}$$

whereas $\partial_j w_i$ transform under

$$\Gamma_3\times\Gamma_5=\Gamma_5+\Gamma_6\ . \tag{3.5}$$

Thus it can be seen that we have two quadratic invariants

$$\begin{gathered}(\partial_1 v_1+\partial_2 v_2)^2+(\partial_1 v_2-\partial_2 V_1)^2\ ,\\ (\partial_1 v_1-\partial_2 v_2)^2+(\partial_1 v_2+\partial_2 v_1)^2\end{gathered} \tag{3.6}$$

coupling $\mathbf{v}$ to $\mathbf{v}$, and two quadratic invariants

$$\begin{gathered}(\partial_1 w_1+\partial_2 w_2)^2+(\partial_1 w_2-\partial_2 w_1)^2\ ,\\ (\partial_1 w_1-\partial_2 w_2)^2+(\partial_1 w_2+\partial_2 w_1)^2\end{gathered} \tag{3.7}$$

coupling $\mathbf{w}$ to $\mathbf{w}$. Consequently, the part of the elastic energy due to the phason fields is the sum of f^v and f^w, which are given by Eqs. (2.8) and (2.9), respectively. Moreover, since the irreducible representation Γ_6 occurs in both of the reduction equations (3.4) and (3.5), we thus have an invariant

$$(\partial_1 v_1-\partial_2 v_2)(\partial_1 w_1+\partial_2 w_2)+(\partial_1 v_2+\partial_2 v_1)(\partial_1 w_2-\partial_2 w_1) \tag{3.8}$$

which yields a coupling term of the form

$$\begin{aligned}f^{vw}=R_2[&(\partial_1 v_1-\partial_2 v_2)(\partial_1 w_1+\partial_2 w_2)\\ &+(\partial_1 v_2+\partial_2 v_1)(\partial_1 w_2-\partial_2 w_1)]\ .\end{aligned} \tag{3.9}$$

It should be noted that there are no phonon-phason coupling terms in the elastic energy. Therefore, the total elastic energy is

$$f=f^u+f^v+f^w+f^{vw} \tag{3.10}$$

TABLE V. Comparison of results for different symmetries.

	Sevenfold	Ninefold	Fourteenfold	Eighteenfold
No. of phonon elastic constants	2	2	2	2
No. of phason elastic constants	4	4	4	4
Phonon-phason coupling	1	0	1	0
Phason-phason coupling	1	1	1	1
Total no. of elastic constants	8	7	8	7

for a 2D quasicrystal with ninefold symmetry, where f^u, f^v, f^w, and f^{vw} are given by Eqs. (2.5), (2.8), (2.11), and (3.9), respectively.

Similarly, for fourteenfold symmetry the total elastic energy is

$$f=f^u+f^v+f^w+f^{uw}+f^{vw}\ , \tag{3.11}$$

where

$$f^{vw}=R_2[(\partial_1 v_1+\partial_2 v_2)(\partial_1 w_1-\partial_2 w_2)+(\partial_1 v_2-\partial_2 v_1)(\partial_1 w_2+\partial_2 w_1)]\ , \tag{3.12}$$

f^u, f^v, f^w, and f^{uw} are given by Eqs. (2.5), (2.8), (2.11), and (2.13), respectively. For eighteenfold symmetry the total elastic energy is

$$f=f^u+f^v+f^w+f^{vw}\ , \tag{3.13}$$

where f^u, f^v, f^w, and f^{vw} are given by Eqs. (2.5), (2.8), (2.11), and (3.12), respectively.

From the results it can be seen that the terms in the total elastic energy, f^u, f^v, and f^w, take the same form for all symmetries, but the forms of the coupling terms and the number of independent elastic constants are different for different symmetries (see Table V).

IV. CONCLUDING REMARKS

We have investigated some structural and elastic properties of all 2D quasicrystals with a 6D embedding space, derived their quadratic invariants, and calculated the elastic energies to quadratic order. Here we conclude that such quasicrystals have some properties different from the properties of known 2D quasicrystals. The diffraction pattern of the quasicrystals considered here can be indexed with integers using six rationally independent reciprocal basis vectors, which can be regarded as a projection from a 6D embedding space upon the 2D physical space. The six hydrodynamic degrees of freedom in phases can be parametrized by three two-dimensional vectors, **u**, **v**, and **w**, which are related to three different representations of the rotational symmetry group. In this case the orthogonal complementary space is no longer irreducible because it can be decomposed into a direct sum of two invariant subspaces. With each of them is associated one type of phason field. The form of the terms that involve only one hydrodynamic variable in the elastic energy is the same for all symmetries, but the forms of the coupling terms are different for different symmetries.

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