Localization properties of quasi-one-dimensional conductor networks in a random magnetic field

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We investigate the localization of electrons on a ladder-shaped quasi-one-dimensional network of clean wires, with a quenched random magnetic flux across each of its square plaquettes. In the weak-disorder regime, the localization length ξ is much larger than the side of the plaquettes. Using perturbative analytic techniques, we derive scaling laws of the form $\xi \sim 1/w^{\alpha}$, with w being the width of magnetic disorder. The critical exponent α assumes different values in various energy ranges: $\alpha=4$ when only one channel is open, $\alpha=2$ when both channels are open, $\alpha=1$ around external and internal band edges. The corresponding scaling functions and amplitudes are accurately determined by numerical simulations. Magnetic disorder and potential disorder thus pertain to different universality classes.

I. INTRODUCTION

There is a growing interest in the study of electron localization in the presence of a random magnetic field.[1–4] This problem has been suggested in the past, merely as an alternative kind of disorder.[5] Within the tight-binding approximation, unlike in the Anderson model, where disorder is implemented through random site energies, magnetic disorder is implemented via random phases in the hopping matrix elements. These phases are usually chosen as independent variables, drawn from a common probability distribution, characterized by its width w. Most efforts so far have been directed toward the study of two-dimensional (2D) systems. One of the main motivations for these investigations is the question of whether such a system exhibits a mobility edge. Beside its own academic interest, this problem is related to some recent attempts to interpret the $\nu=\frac{1}{2}$ anomaly observed in the fractional quantum Hall effect. A good exposition of these topics can be found in Ref. 6.

Much less effort has been directed toward the study of 1D or quasi-1D systems in the presence of random magnetic fluxes. The authors of Ref. 6 quote an unpublished work by some of them, on a ladder-shaped network with random fluxes.[7] As expected, all the eigenstates are found to be localized, and the conductance, as given by the Landauer formula, obeys a roughly log-normal distribution. Yet the presence of random magnetic fluxes is a novel kind of disorder, which can be expected to lead to new physics, with respect to the usual theory of localization, at least as far as quantitative aspects are concerned.

This is essentially the point of the present paper. We investigate the propagation of noninteracting electrons on a ladder-shaped network of quantum wires, with an independent random magnetic flux across each of its square cells, or *plaquettes*. This model is especially convenient for analytic perturbative calculations, since it can be recast in terms of an infinite produce of independent transfer matrices (as far as we know, there is no such formulation for tight-binding models with magnetic disorder). A novel kind of scaling behavior is predicted, concerning especially the divergence law of the localization length ξ as the strength of magnetic disorder w tends to zero. This paper is organized as follows. In Sec. II, we present our model for a network of quantum wires, which is adapted from previous works,[4,8] and we discuss its dispersion relation in the presence of a constant magnetic field. Section III contains our analytic and numerical results. We derive a weak-disorder expansion of both positive Lyapunov exponents. The outcome assumes several different forms in various energy ranges (one open channel, two open channels, external and internal band edges). These predictions are corroborated by accurate numerical data. Section IV closes up with a short summary and a discussion.

II. THE MODEL

A. Generalities

We consider a ladder-shaped network of clean conducting wires in a quenched random transversal magnetic field. We express magnetic fluxes in units of $\hbar/e$, so that the flux quantum reads $\Phi_0=2\pi$. We set the length of the wires, i.e., the side of the square plaquettes, equal to $a=1$. Each plaquette embraces a magnetic flux which we denote by $2\phi_n$. Figure 1 illustrates our notations and conventions. We choose the gauge such that the vector potential is parallel to the axis of the ladder; it is taken to be constant along each wire, and reads $A=\phi_n$ between the points P_{n-1} and P_n, and $A=-\phi_n$ between the points Q_{n-1} and Q_n.

Following our previous works,[4–8] we write the wave functions along the wires as follows:

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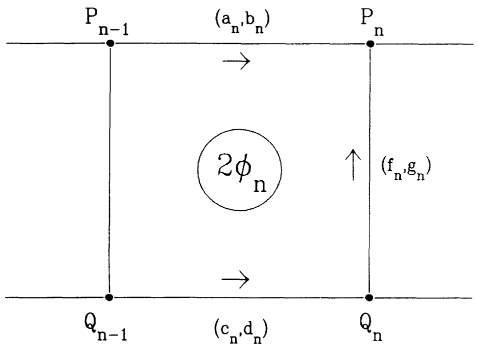

FIG. 1. Schema of a portion of the ladder network, showing the notations and conventions used in this paper.

from P_{n-1} to P_n:

$$\Psi(x)=e^{i\phi_n(x-n+1)}[a_n e^{ik(x-n+1)}+b_n e^{-ik(x-n+1)}] ,$$

from Q_{n-1} to Q_n:

$$\Psi(x)=e^{-i\phi_n(x-n+1)}[c_n e^{ik(x-n+1)}+d_n e^{-ik(x-n+1)}] , \tag{2.1}$$

from Q_n to P_n:

$$\Psi(y)=f_n e^{iky}+g_n e^{-iky} ,$$

where k is the momentum of an electron with energy $E=\hbar^2k^2/(2m)$.

The six amplitudes $\{a_n,b_n,c_n,d_n,f_n,g_n\}$ which describe the wave function in each plaquette are related by six linear equations, namely,

continuity at P_n:

$$e^{i\phi_n}(a_n e^{ik}+b_n e^{-ik})=f_n e^{ik}+g_n e^{-ik}$$
$$=a_{n+1}+b_{n+1} ,$$

continuity at Q_n:

$$e^{-i\phi_n}(c_n e^{ik}+d_n e^{-ik})=f_n+g_n=c_{n+1}+d_{n+1} \tag{2.2}$$

current conservation at P_n:

$$e^{i\phi_n}(a_n e^{ik}-b_n e^{-ik})+f_n e^{ik}-g_n e^{-ik}=a_{n+1}-b_{n+1}$$

current conservation at Q_n:

$$e^{-i\phi_n}(c_n e^{ik}-d_n e^{-ik})+g_n-f_n=c_{n+1}-d_{n+1} .$$

Two of these equations can be used to eliminate the transversal amplitudes f_n and g_n. It is advantageous to recast the remaining four relations among the longitudinal amplitudes in the following form

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{bmatrix} = T(\phi_n) \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} , \tag{2.3}$$

where the 4×4 transfer matrix $T(\phi_n)$ reads

$$T(\phi)=\frac{1}{2i\sin k} \times \begin{bmatrix} (\cos k+2i\sin k)e^{i\phi+ik} & \cos k e^{i\phi-k} & -e^{-i\phi+ik} & -e^{-i\phi-ik} \\ -\cos k e^{i\phi+ik} & (-\cos k+2i\sin k)e^{i\phi-ik} & e^{-i\phi+ik} & e^{-i\phi-ik} \\ -e^{i\phi+ik} & -e^{i\phi-ik} & (\cos k+2i\sin k)e^{-i\phi+ik} & \cos k e^{-i\phi-ik} \\ e^{i\phi+ik} & e^{i\phi-ik} & -\cos k e^{-i\phi+ik} & -\cos k+2i\sin k)e^{-i\phi-ik} \end{bmatrix} . \tag{2.4}$$

It is worthwhile noticing that the transfer matrix $T(\phi_n)$ attached to the nth plaquette only involves the corresponding magnetic flux ϕ_n. The propagation across N cells of the network is therefore described by a product of N independent random matrices, of the form

$$\mathcal{T}_N=\prod_{n=1}^{N} T(\phi_n) . \tag{2.5}$$

B. Dispersion relation in a constant magnetic field

Let us first investigate the situation of a network in a constant magnetic field. All magnetic fluxes are then equal to the same number ϕ. We are thus led to diagonalize the constant transfer matrix $T(\phi)$. Using the notation $\lambda=e^t$ for its eigenvalues, we obtain the following relation

$$4\cosh^2 t-12\cos k\cos\phi\cosh t+9\cos^2 k+4\cos^2\phi-5=0 . \tag{2.6}$$

For further convenience, we suppose

$$\mathrm{Re}\,t\geq0, \quad |\mathrm{Im}\,t|\leq\pi . \tag{2.7}$$

The quadratic equation (2.6) has two roots, namely,

$$2\cosh t_1=3\cos k\cos\phi+(1+4\sin^2\phi-9\cos^2 k\sin^2\phi)^{1/2} \tag{2.8}$$
$$2\cosh t_2=3\cos k\cos\phi-(1+4\sin^2\phi-9\cos^2 k\sin^2\phi)^{1/2} ,$$

so that the four eigenvalues of the transfer matrix read e^{t_1}, e^{t_2}, e^{-t_1}, and e^{-t_2}.

The *dispersion relation* is obtained by substituting

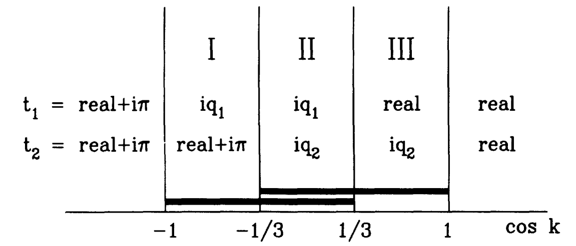

FIG. 2. Schema of the band structure of the network in the absence of a magnetic field, illustrating the definition of the various energy regions, and the ranges of the variables t_1 and t_2.

$t \to iq$ in Eq. (2.8), where q denotes the Bloch momentum. The spectrum thus consists of two intervals, namely a right band $(1-2|\cos\phi| \le 3\cos k \le 1+2|\cos\phi|)$, corresponding to having q_2 real, and a left band $(-1-2|\cos\phi| \le 3\cos k \le -1+2|\cos\phi|)$, corresponding to having q_1 real. Every real wave vector corresponds to the possibility of propagation through an *open transverse channel.* The two bands overlap for $|\cos\phi| \ge \frac{1}{2}$. It that case, two channels are open in the central region of the spectrum $(1-2|\cos\phi| \le 3\cos k \le -1+2|\cos\phi|)$.

In the absence of a magnetic field, the dispersion relation derived above assumes the form

$$2\cosh t_1 = 3\cos k + 1, \quad 2\cosh t_2 = 3\cos k - 1 \ . \tag{2.9}$$

The spectrum thus consists of two partially overlapping bands, namely $(-1 \le \cos k \le \frac{1}{3})$, and $(-\frac{1}{3} \le \cos k \le 1)$. This band structure, shown in Fig. 2, determines three regions, namely two lateral one-band regions (I) and (III), where only one channel is open, and a central region (II), where two channels are open.

III. THE LYAPUNOV EXPONENTS: ANALYTIC VERSUS NUMERICAL RESULTS

In order to model the presence of quenched disorder, the magnetic fluxes ϕ_n across the successive plaquettes are taken to be independent random variables, which we assume to be drawn from a common even-probability law $r(\phi)d\phi$. In this situation, the Furstenberg theory of matrix products[9] tells us that the main quantities of interest are the Lyapunov exponents of the product (2.5), in the thermodynamic limit of an infinitely long ladder. The system has two positive Lyapunov exponents, that we shall denote by $\gamma_1 \ge \gamma_2 \ge 0$, and two negative ones, namely their opposites $-\gamma_1$ and $-\gamma_2$. As a consequence, all the eigenstates are exponentially localized, and the localization length is given by

$$\xi = \frac{1}{\gamma_2} \ . \tag{3.1}$$

In the following, we shall be mostly interested in the quantitative behavior of the Lyapunov exponents in the regime where the random fluxes are small ($|\phi_n| \ll 1$). In this weak-disorder regime, physical quantities only depend on the first two even moments of the distribution of the random magnetic fluxes, namely

$$\mu_2 = \overline{\phi^2}, \quad \mu_4 = \overline{\phi^4} \ . \tag{3.2}$$

We also introduce for further reference the dimensionless ratio

$$K_4 = \frac{\mu_4}{\mu_2^2} \ge 1 \ , \tag{3.3}$$

which is a characteristic of the shape of disorder, sometimes referred to as the *kurtosis* of the distribution.

A. Numerical procedure

The algorithm that we have chosen in order to evaluate numerically both positive Lyapunov exponents of the problem has been used extensively in the context of the Anderson model.[10] It consists in calculating two "infinite" sequences of four-component vectors $\{U_n\}$ and $\{V_n\}$, defined by the recursion relations

$$U_n = T(\phi_n)U_{n-1}, \quad V_n = T(\phi_n)V_{n-1} \ , \tag{3.4}$$

with some initial conditions U_0, V_0. The Lyapunov exponents γ_1 and γ_2 are then given by the following limits

$$\gamma_1 = \lim_{n\to\infty} \frac{1}{n} \ln\|U_n\|, \quad \gamma_1+\gamma_2 = \lim_{n\to\infty} \frac{1}{n} \ln\|U_n \times V_n\| \ , \tag{3.5}$$

where $\|U_n \times V_n\|$ denotes the area of the parallelogram spanned by the two vectors.

The Lyapunov exponents defined by Eq. (3.5) are *self-averaging quantities.* From a practical viewpoint, the accuracy of the numerical procedure is only limited by statistical errors, which decay as $n^{-1/2}$, by virtue of the law of large numbers. By performing an orthonormalization of the system $\{U_n, V_n\}$ at regular steps during the calculation, this algorithm can be run up to $n = 10^6 - 10^7$, without developing either overflows or appreciable rounding errors.

Most of the numerical data to be presented later on concern one of the following two types of disorder:

The *rectangular distribution*

$$r_{\text{rect}}(\phi) = \frac{1}{\pi w} \quad \text{for} \ -\frac{\pi w}{2} < \phi < \frac{\pi w}{2} \ , \tag{3.6}$$

which corresponds to

$$\mu_2 = \frac{\pi^2 w^2}{12}, \quad \mu_4 = \frac{\pi^4 w^4}{80}, \quad K_4 = \frac{9}{5} \ . \tag{3.7}$$

For the value $w=1$, we have a *uniform* distribution of the magnetic fluxes, which has been considered, e.g., in Ref. 4.

The *binary distribution*

$$r_{\text{bin}}(\phi) = \tfrac{1}{2}[\delta(\phi - W) + \delta(\phi + W)] \ , \tag{3.8}$$

which corresponds to

$$\mu_2 = W^2, \quad \mu_4 = W^4, \quad K_4 = 1 \ . \tag{3.9}$$

Figure 3 shows the global behavior of both Lyapunov exponents against $\cos k$, for a rectangular distribution with various values of the width w, going from $w=0$ (no magnetic field) to $w=1$ (uniform distribution of the magnetic fluxes). Various features visible on these plots will be explained by the quantitative results which follow.

B. Analytic weak-disorder expansion

In the weak-disorder regime, i.e., when the width of the distribution of the random fluxes is small, the Lyapunov exponents can be expanded in a systematic way in terms of the successive moments of this distribution. Weak-disorder expansions have been known for a long time, for the tight-binding Anderson model,[11] or the vibrational spectra of random harmonic chains;[12] a recent review of these topics can be found in Ref. 13. In the present case, we are going to derive such an expansion, up to the fourth order included, following closely the approach described in Ref. 14. This paper presents the basis for a perturbative expansion of the Lyapunov exponents of products of independent random matrices of any size, and gives explicit expressions of the results up to the fourth order.

We emphasize that this approach is *a priori* only meaningful out of the spectrum of the network. Predictions of physical interest concerning the Lyapunov exponents on the spectrum can only be reached by means of an *analytic continuation* of the outcome of the perturbative analysis. We shall come back to this important point later on.

We start by considering a momentum k such that $\cos k > 1$. In the limit of a vanishing disorder, we have

$$\gamma_1 = t_1, \quad \gamma_2 = t_2 , \tag{3.10}$$

using the variables defined in Eq. (2.9). The conditions (2.7) ensure that we have $\gamma_1 > \gamma_2 > 0$, as it should be. The expression (3.10) is, of course, the zeroth order of the weak-disorder expansion that we are going to derive.

Along the lines of Ref. 14, we first transform the random transfer matrix $T(\phi)$ in the basis where the unperturbed matrix $T(0)$ is diagonal. Using again the notation (2.9), we obtain

$$T(\phi) = M^{-1} D(\phi) M , \tag{3.11}$$

with

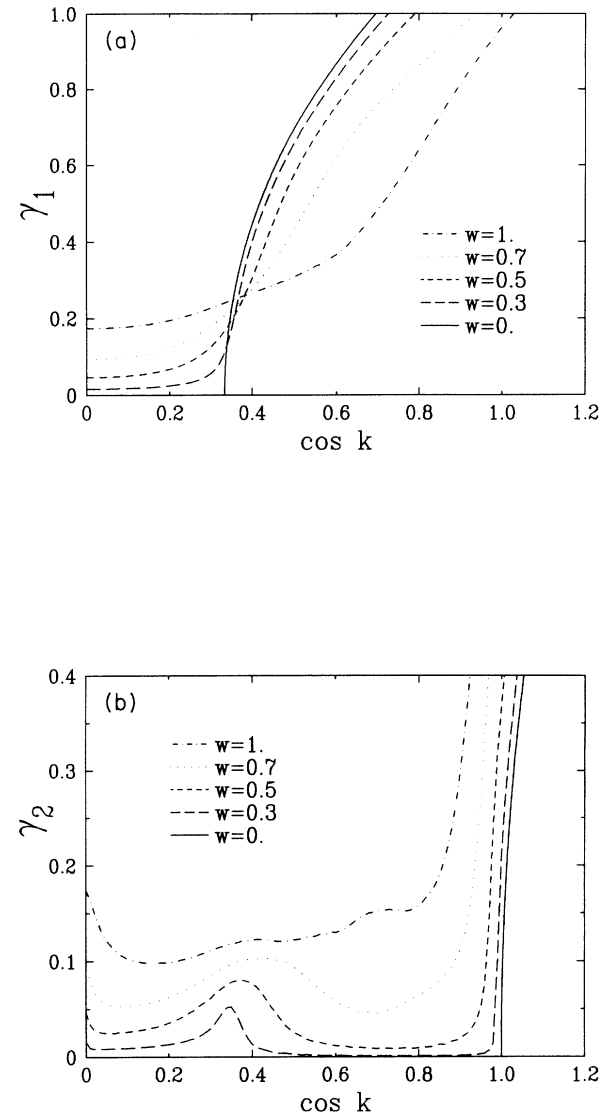

FIG. 3. Plot of the Lyapunov exponents γ_1 and γ_2 against $\cos k$, for several values of the width w of the rectangular distribution of the random fluxes.

$$M = \begin{pmatrix} \dfrac{e^{-t_1} - e^{ik}}{2\sinh t_1} & \dfrac{e^{-t_1} - e^{-ik}}{2\sinh t_1} & \dfrac{e^{ik} - e^{-t_1}}{2\sinh t_1} & \dfrac{e^{-ik} - e^{-t_1}}{2\sinh t_1} \\ \dfrac{e^{-t_2} - e^{ik}}{2\sinh t_2} & \dfrac{e^{-t_2} - e^{-ik}}{2\sinh t_2} & \dfrac{e^{-t_2} - e^{ik}}{2\sinh t_2} & \dfrac{e^{-t_2} - e^{-ik}}{2\sinh t_2} \\ \dfrac{e^{ik} - e^{t_1}}{2\sinh t_1} & \dfrac{e^{-ik} - e^{t_1}}{2\sinh t_1} & \dfrac{e^{t_1} - e^{ik}}{2\sinh t_1} & \dfrac{e^{t_1} - e^{-ik}}{2\sinh t_1} \\ \dfrac{e^{ik} - e^{t_2}}{2\sinh t_2} & \dfrac{e^{-ik} - e^{t_2}}{2\sinh t_2} & \dfrac{e^{ik} - e^{t_2}}{2\sinh t_2} & \dfrac{e^{-ik} - e^{t_2}}{2\sinh t_2} \end{pmatrix} , \tag{3.12}$$

and

$$M^{-1} = \frac{1}{4i \sin k} \begin{pmatrix} e^{-ik} - e^{t_1} & e^{-ik} - e^{t_2} & e^{-ik} - e^{-t_1} & e^{-ik} - e^{-t_2} \\ e^{t_1} - e^{ik} & e^{t_2} - e^{ik} & e^{-t_1} - e^{ik} & e^{-t_2} - e^{ik} \\ e^{t_1} - e^{-ik} & e^{-ik} - e^{t_2} & e^{-t_1} - e^{-ik} & e^{-ik} - e^{-t_2} \\ e^{ik} - e^{t_1} & e^{t_2} - e^{ik} & e^{ik} - e^{-t_1} & e^{-t_2} - e^{ik} \end{pmatrix} , \tag{3.13}$$

and

$$D(\phi)=\begin{bmatrix} \cos\phi e^{t_1} & i\sin\phi\dfrac{e^{t_1+t_2}-1}{2\sinh t_1} & 0 & i\sin\phi\dfrac{e^{t_1-t_2}-1}{2\sinh t_1} \\ i\sin\phi\dfrac{e^{t_1+t_2}-1}{2\sinh t_2} & \cos\phi e^{t_2} & i\sin\phi\dfrac{e^{t_2-t_1}-1}{2\sinh t_2} & 0 \\ 0 & i\sin\phi\dfrac{1-e^{t_2-t_1}}{2\sinh t_1} & \cos\phi e^{t_1} & i\sin\phi\dfrac{1-e^{-t_1-t_2}}{2\sinh t_1} \\ i\sin\phi\dfrac{1-e^{t_1-t_2}}{2\sinh t_2} & 0 & i\sin\phi\dfrac{1-e^{-t_1-t_2}}{2\sinh t_2} & \cos\phi e^{-t_2} \end{bmatrix} . \tag{3.14}$$

The derivation of the weak-disorder expansion of both Lyapunov exponents up to fourth order is a lengthy application of the general approach exposed in Ref. 14. This calculation presents no peculiar difficulty, except the following point. The average of the matrix $D(\phi)$ over disorder is not $D(0)$, but rather

$$\overline{D(\phi)}=\begin{bmatrix} e^{t_1+\tau} & 0 & 0 & 0 \\ 0 & e^{t_2+\tau} & 0 & 0 \\ 0 & 0 & e^{-t_1+\tau} & 0 \\ 0 & 0 & 0 & e^{-t_2+\tau} \end{bmatrix} , \tag{3.15}$$

with

$$\tau=\ln(\overline{\cos\phi})=-\frac{\mu_2}{2}+\frac{1}{24}(\mu_4-3\mu_2^2)+\cdots . \tag{3.16}$$

Our final result is the following

$$\gamma_1=t_1-\frac{\mu_2}{2}-\frac{\mu_4}{12}-\mu_2^2\frac{(e^{t_1-t_2}-1)^4}{32\sinh^2 t_1\sinh^2 t_2 e^{2t_1}(e^{2t_1}-e^{-2t_2})}+\cdots , \tag{3.17}$$

$$\begin{aligned}\gamma_2={}&t_2+\mu_2\frac{\cosh t_1\cosh t_2-1}{2\sinh t_1\sinh t_2}+\mu_4\left[-\frac{1}{12}+\frac{(e^{t_1+t_2}-1)^2}{6\sinh t_1\sinh t_2 e^{t_1+t_2}}-\frac{(e^{t_1+t_2}-1)^4}{32\sinh^2 t_1\sinh^2 t_2 e^{2t_1+2t_2}}\right]\\ &-\mu_2^2\frac{1}{32\sinh^2 t_1\sinh^2 t_2}\left[\frac{(e^{t_2-t_1}-1)^4}{e^{2t_2}(e^{2t_2}-e^{-2t_1})}+\frac{2(e^{t_1+t_2}-1)^2(e^{t_1-t_2}-1)^2}{e^{2t_1+t_2}(e^{t_2}-e^{-t_2})}\right.\\ &\left.+\frac{2(e^{t_1+t_2}-1)^2(e^{t_2-t_1}-1)^2}{e^{t_1+2t_2}(e^{t_1}-e^{-t_1})}\right]+\cdots .\end{aligned} \tag{3.18}$$

We recall that this result has been established under the assumption that both t_1 and t_2 are real, and that these conditions are only met for $\cos k > 1$, i.e., out of the spectrum. In order to obtain information of physical interest, we have to reach the spectrum of the system, and therefore to perform an analytic continuation of the results (3.17), (3.18). The Lyapunov exponents are given by the *real parts* of the analytically continued expressions. In the next subsections, we describe various regimes in detail, and compare our analytic predictions to numerical data. As it turns out, five different cases have to be considered.

C. The one-channel regions (I) ($-1 < \cos k < -\frac{1}{3}$) and (III) ($\frac{1}{3} < \cos k < 1$)

In these two lateral regions of the spectrum (see Fig. 2), one of the t variables is real, and the other is imaginary. In other words, only one transversal channel is open. We consider for definiteness region (III). We thus have t_1 real positive, whereas $t_2=iq_2$, with q_2 being the Bloch momentum of the single band.

The largest Lyapunov exponent is $\gamma_1=t_1$, up to small corrections, which can be directly read from Eq. (3.17). In order to obtain γ_2, and hence the localization length, we have to continue analytically the perturbative result (3.18). Since t_2 is imaginary, neither the term of order zero nor the term proportional to μ_2 contribute to γ_2. As a consequence, only both kinds of fourth-order terms, proportional to μ_2^2 and to μ_4, contribute to γ_2. We thus obtain

$$\gamma_2 \approx \mu_2^2 [A(\cos k) + K_4 A_4(\cos k)] . \quad (3.19)$$

Explicit, albeit not very illuminating, expressions for the amplitude functions A and A_4 can be derived by taking the real part of Eq. (3.18).

We thus arrive at the conclusion that the inverse localization length vanishes as w^4. Moreover, the prefactor of the $\xi \sim 1/w^4$ law is not universal, since it involves explicitly the kurtosis K_4 defined in Eq. (3.3), and thus depends on the shape of the distribution of the fluxes.

D. The two-channel region (II) ($-\frac{1}{3} < \cos k < \frac{1}{3}$)

In this central region (see Fig. 2), both bands of the system overlap in the absence of a magnetic field: both Bloch momenta q_1 and q_2 are real, i.e., both transversal channels are open. The general weak-disorder result (3.17), (3.18) does not yield directly any quantitative information about the Lyapunov exponents in this region. Indeed, perturbative techniques of this type are known to break down when two of the unperturbed eigenvalues, such as e^{t_1} and e^{t_2}, have equal moduli, and more generally when the width of disorder becomes comparable to the difference between these moduli. A more sophisticated degenerate perturbative approach is needed there, as explained, e.g., in Ref. 15 for the case of 2×2 matrices. In the present case of 4×4 matrices, we can assert from the general structure of the problem that both Lyapunov exponents vanish proportionally to μ_2, namely,

$$\gamma_l \approx \mu_2 B_l(\cos k) \quad (l=1 \text{ or } 2) , \quad (3.20)$$

without obtaining explicit expressions for the amplitude functions B_l.

E. The band center ($\cos k = 0$)

We can provide a quantitative derivation of the result (3.20) at the center of the spectrum ($\cos k = 0$). Indeed, the degenerate perturbation theory mentioned in the previous subsection assumes a simpler form at this special energy. The key point is that a judicious grouping of the random transfer matrices $D(\phi_n)$ reduces them to 2×2 blocks. The outcome of this analysis reads

$$B_l(0) = \frac{2}{3}\left[\frac{\sqrt{3}}{\ln(2+\sqrt{3})} - 1\right] = 0.210\,127 \quad (l=1 \text{ or } 2) . \quad (3.21)$$

The main lines of the derivation of the result (3.21) go as follows. At the center of the band, we have $t_1 = i\pi/3$ and $t_2 = 2i\pi/3$. As a consequence, $e^{6t_1} = e^{6t_2} = 1$. This observation suggests consideration of the product of six consecutive transfer matrices

$$\Pi_6 = D(\phi_6)D(\phi_5)D(\phi_4)D(\phi_3)D(\phi_2)D(\phi_1) . \quad (3.22)$$

It turns out that the product Π_6 decomposes, via a change of basis, into two 2×2 blocks. We have indeed

$$\Pi_6 = N^{-1}\begin{bmatrix} P & 0 \\ 0 & P^T \end{bmatrix} N , \quad (3.23)$$

with

$$N = N^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} , \quad (3.24)$$

and where P^T denotes the transpose of the matrix P. Furthermore, the blocks which enter Eq. (3.23) are close to the identity matrix for a weak disorder. More precisely, to first order in the magnetic fluxes, we have

$$P = \begin{bmatrix} 1-Z & -X+iY \\ X+iY & 1+Z \end{bmatrix} , \quad (3.25)$$

with

$$X = \frac{1}{\sqrt{3}}(\phi_1 + 2\phi_2 + \phi_3 - \phi_4 - 2\phi_5 - \phi_6) + \cdots$$
$$Y = \phi_1 - \phi_3 - \phi_4 + \phi_6 + \cdots \quad (3.26)$$
$$Z = \frac{1}{\sqrt{3}}(\phi_1 - \phi_2 + \phi_3 - \phi_4 + \phi_5 - \phi_6) + \cdots ,$$

where the dots represent higher-order terms in the magnetic fluxes. To the quadratic order, the statistics of these random variables is as follows

$$\overline{X} = \overline{Y} = \overline{Z} = 0, \quad \overline{XY} = \overline{XZ} = \overline{YZ} = 0 ,$$
$$\overline{X^2} = \overline{Y^2} = 4\mu_2, \quad \overline{Z^2} = 2\mu_2 . \quad (3.27)$$

Because of the decomposition (3.23), we have

$$\gamma_1 = \gamma_2 = \frac{\gamma}{6} , \quad (3.28)$$

where γ is the positive Lyapunov exponent of an infinite product of 2×2 matrices P_k. Equations (3.23) and (3.28) are exact results, which hold beyond perturbation theory. In order to estimate the Lyapunov exponent γ to lowest order in the weak-disorder regime, we use the following standard procedure.[11–13] Define a sequence of two-component vectors $W_k = (x_k, y_k)$, such that $W_k = P_k W_{k-1}$, and introduce the ratios, or Riccati variables, $S_k = x_k / y_k$. These complex numbers obey the following rational transform

$$S_k = \frac{(1-Z_k)S_{k-1} - X_k + iY_k}{(X_k + iY_k)S_{k-1} + 1 + Z_k} , \quad (3.29)$$

and we have

$$\gamma = -3\mu_2 + \langle\langle \ln|(X_k + iY_k)S_{k-1} + 1 + Z_k| \rangle\rangle , \quad (3.30)$$

where $\langle\langle \cdots \rangle\rangle$ denotes an average over the stationary distribution of the Riccati variables S_k, which is invariant under the transform (3.29). The explicit constant $-3\mu_2$ in Eq. (3.30) originates in the contribution to the matrix P which is quadratic in the fluxes, and has not been written explicitly in Eq. (3.26). This constant has been fixed by using the identity $\det P = 1$.

The distribution of the variable $X_k + iY_k$ is isotropic in the complex plane, at least to leading order in the random fluxes, as a consequence of the expressions (3.27). The Riccati variables therefore also share this property. We thus set $\rho_k = |S_k|$, and we denote by $F(\rho)$ the invariant distribution of the moduli ρ_k.

Assume for a while that these moduli obey a recursion relation of the form

$$\rho_k = \rho_{k-1} + \epsilon(\rho_{k-1}) , \tag{3.31}$$

where the $\epsilon(\rho_{k-1})$ are small random variables, whose distribution depends on ρ_{k-1}. Their distribution is characterized by its first two moments $\overline{\epsilon(\rho)} = 0$, and $\overline{\epsilon(\rho)^2} = \Delta(\rho) \ll 1$. The stationary probability density $F(\rho)$, associated with the random process (3.31), has a well-defined limiting behavior for a vanishingly small disorder, which can be evaluated in several ways. One can, for example, write a linear integral relation following Dyson and Schmidt,[13] expressing the invariance of $F(\rho)$ under the transformation (3.31), perform a gradient expansion of this integral equation, and finally integrate the differential equation thus obtained. We skip the details, and only mention the following outcome

$$F(\rho) = \frac{C}{\Delta(\rho)} , \tag{3.32}$$

where the constant C is to be fixed by normalization.

Going back to the present problem, we expand Eq. (3.29) to first order in the random variables X_k, Y_k, and Z_k, and average over the phases of $(X_k + iY_k)$ and of S_{k-1}, using the fact that they are statistically independent, and that each of them has a uniform density over the circle. We are thus left with an equation for the moduli ρ_k of the type (3.31). The result (3.32) yields after some algebra the following expression

$$F(\rho) = \frac{C}{1 + 4\rho^2 + \rho^4} \tag{3.33}$$

for the invariant distribution of the moduli of the Riccati variables, with

$$\frac{1}{C} = \int_0^\infty \frac{\rho\, d\rho}{1 + 4\rho^2 + \rho^4} = \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} . \tag{3.34}$$

The last step consists in performing the average in the right-hand side of Eq. (3.30). This cannot be done by a naive expansion in powers of the disorder, since the second moment $\langle\langle \rho^2 \rangle\rangle$ of the law (3.33) diverges logarithmically. One can instead average in the first place over the uniform phase θ_k of the product of complex variables $(X_k + iY_k)S_k$, by means of the identity

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \ln|a + be^{i\theta}| = \max(\ln|a|, \ln|b|) . \tag{3.35}$$

This leads at once to the result

$$\gamma = 2(C-2)\mu_2 , \tag{3.36}$$

which is equivalent to the announced formula (3.21).

F. Scaling law around external band edges ($\cos k = \pm 1$)

We now look at the behavior of the Lyapunov exponents near the external band edges of the spectrum of the unperturbed network, i.e., for $\cos k \to \pm 1$ (see Fig. 2). Consider the upper edge for definiteness. We have $t_1 = \ln(2+\sqrt{3})$ for $\cos k = 1$, whereas $t_2 = iq_2$, where the Bloch momentum q_2 vanishes according to $q_2^2 \approx 3(1-\cos k) \approx 3k^2/2$.

It turns out that the weak-disorder expansion (3.18) for γ_2 is singular for $t_2 \to 0$, i.e., $\cos k \to 1$. By keeping only the most singular part of each term as $t_2 \to 0$, we get

$$\gamma_2 \approx t_2 \left[1 + \frac{\mu_2}{2\sqrt{3}t_2^2} - \frac{\mu_2^2}{24t_2^4} + \cdots \right] . \tag{3.37}$$

This form strongly suggests the existence of a scaling region, defined by the conditions that both t_2 and μ_2 are small, where γ_2 obeys a scaling behavior of the form

$$\gamma_2 \approx t_2 G_0 \left[\frac{\mu_2}{t_2^2} \right] . \tag{3.38}$$

The scaling law (3.38) can be recast in the following equivalent form

$$\gamma_2 \approx \sqrt{\mu_2} F_0 \left[\frac{\cos k - 1}{\mu_2} \right] , \tag{3.39}$$

which is more convenient for the reason that the scaling function $F_0(x)$ is everywhere smooth, even at the origin.

For $x \to +\infty$, i.e., deep outside the band, the result (3.37) implies

$$F_0(x) = \sqrt{3x} + \frac{1}{6\sqrt{x}} - \frac{\sqrt{3}}{216x^{3/2}} + \cdots \quad (x \to +\infty) . \tag{3.40}$$

Conversely, for $x \to -\infty$, i.e., deep in region (III), the second Lyapunov exponent vanishes as the fourth power of the strength of disorder, according to Eq. (3.19). This behavior corresponds to nonuniversal corrections to the scaling law (3.39), masking the true falloff of $F_0(x)$, which we believe to be exponential.

Figure 4 shows a plot of the scaling function F_0, obtained from data corresponding to rectangular distributions of magnetic fluxes, with various small values of their widths w. An accurate collapsing of the data is observed, as well as the behavior (3.40) for a large positive scaling variable x, and the fast decay for a large negative x. Figure 5 illustrates the convergence of $\gamma_2(\cos k = 1)$ toward the scaling behavior (3.39), as a function of the width of disorder, for both rectangular and binary disorder. Both series of data accurately converge toward the common value

$$F_0(0) \approx 0.76 . \tag{3.41}$$

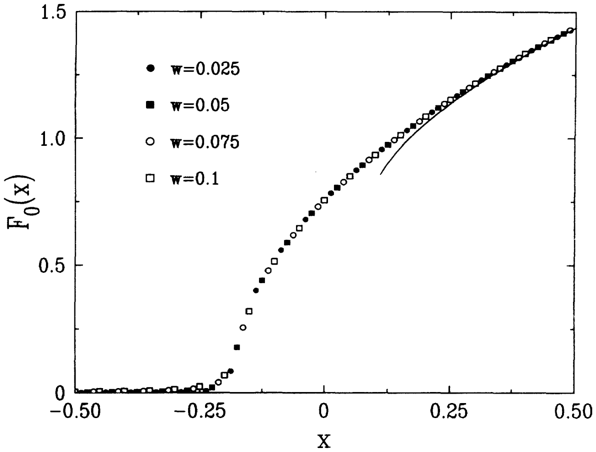

FIG. 4. Plot of the scaling function $F_0(x)$ describing the behavior of the Lyapunov exponent γ_2 near external band edges. The data correspond to rectangular distributions of the random fluxes, with various values of the width w. The continuous line for positive x shows the sum of the three terms given in Eq. (3.40).

G. Scaling law around internal band edges ($\cos k = \pm\frac{1}{3}$)

For $\cos k \to \pm\frac{1}{3}$, the spectrum of the network exhibits internal band edges, which demarcate the two-channel region (II) from the one-channel regions (I) or (III) (see Fig. 2). Consider $\cos k = \frac{1}{3}$ for definiteness. At this special point, t_1 vanishes according to $t_1^2 \approx 3\cos k - 1$, whereas we have $t_2 = i\pi/2$.

The weak-disorder expansion (3.17), (3.18) for both Lyapunov exponents is again singular. In analogy with the previous case, we are led to write down the following scaling laws

$$\gamma_l \approx t_1 G_l \left[\frac{\mu_2}{t_1^2} \right] \quad (l = 1 \text{ or } 2) , \qquad (3.42)$$

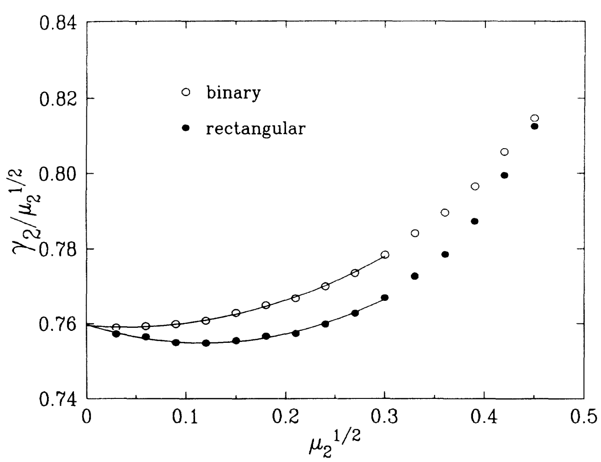

FIG. 5. Plot of the ratio $\gamma_2/\sqrt{\mu_2}$ against $\sqrt{\mu_2}$, for both binary and rectangular distributions of disorder, at the upper band edge ($\cos k = 1$). The common weak-disorder limit of both series of data is given in Eq. (3.41). The full lines show parabolic fits to the data points with $\mu_2 < 0.1$.

or equivalently

$$\gamma_l \approx \sqrt{\mu_2} F_l \left[\frac{\cos k - \frac{1}{3}}{\mu_2} \right] \quad (l = 1 \text{ or } 2) . \qquad (3.43)$$

Here again, both scaling functions $F_l(x)$ are expected to be everywhere smooth, including at the origin $x = 0$.

Equations (3.17), (3.18) only allow prediction of the asymptotic behavior of the scaling functions for $x \to +\infty$, i.e., deep in region (III). We thus obtain

$$F_1(x) \approx (3x)^{1/2}, \quad F_2(x) \approx \frac{1}{8(3x)^{3/2}} \quad (x \to +\infty) . \qquad (3.44)$$

On the other hand, for $x \to -\infty$, i.e., deep in region (II), Eq. (3.20) predicts that both Lyapunov exponents vanish as μ_2. Hence we have

$$F_l(x) \approx C_l(-x)^{-1/2} \quad (l = 1 \text{ or } 2, \; x \to -\infty) , \qquad (3.45)$$

where the C_l are two constants, related to the divergence law as $\cos k \to \frac{1}{3}$ of the amplitude functions $B_l(\cos k)$, which enter Eq. (3.20), namely,

$$B_l(\cos k) \approx C_l(\tfrac{1}{3} - \cos k)^{-1/2} \quad (l = 1 \text{ or } 2, \; \cos k \to \tfrac{1}{3}) . \qquad (3.46)$$

Figure 6 shows a plot of both scaling functions $F_1(x)$ and $F_2(x)$, obtained from data corresponding to rectangular distributions of magnetic fluxes, with various small values of their widths w. A clear collapse of the data is again observed, as well as the expected behaviors (3.44) and (3.45) for large positive or negative values of the scaling variable x. The fitted $(-x)^{-1/2}$ laws shown on the plots lead to the following values

$$C_1 \approx 0.17, \quad C_2 \approx 0.10 \qquad (3.47)$$

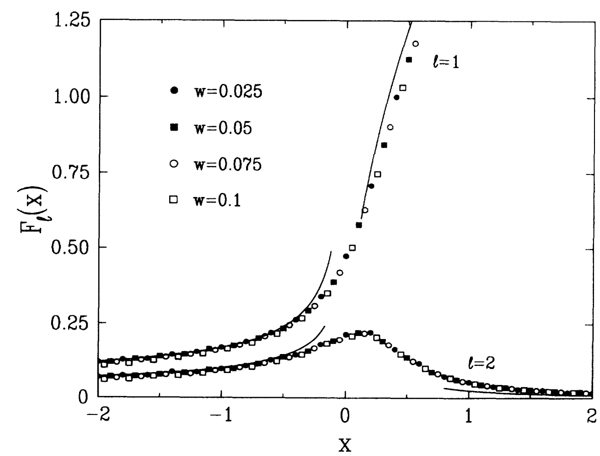

FIG. 6. Plot of the scaling functions $F_1(x)$ and $F_2(x)$ describing the behavior of both Lyapunov exponents near internal band edges. The data correspond to rectangular distributions of the random fluxes, with various values of the width w. The continuous lines for positive x show the asymptotic laws (3.44), whereas those for negative x are fitted to the functional form (3.46), with the constants given in Eq. (3.47).

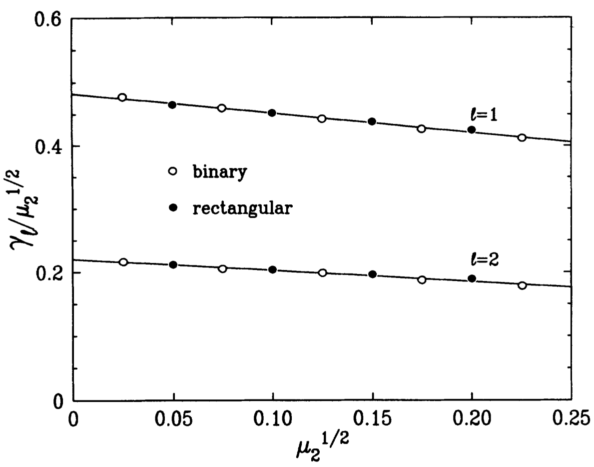

FIG. 7. Plot of the ratios $\gamma_l/\sqrt{\mu_2}$ against $\sqrt{\mu_2}$, for both binary and rectangular distributions of disorder, at the internal band edge ($\cos k=\frac{1}{3}$). The weak-disorder limits are the constants $F_l(0)$ given in Eq. (3.48). The straight lines have been obtained through least-squares fits.

for the constants C_l. Finally, Fig. 7 shows the convergence of both Lyapunov exponents exactly at the internal band edge ($\cos k=\frac{1}{3}$), as a function of the width of disorder, for both rectangular and binary disorder. Both series of data converge toward the common values

$$F_1(0)\approx 0.48,\quad F_2(0)\approx 0.22\ , \tag{3.48}$$

whereas the first corrections to scaling are proportional to the width of disorder in both cases.

IV. SUMMARY AND DISCUSSION

We have investigated the localization of electrons on a ladder-shaped network of quantum wires, with a random magnetic flux in each plaquette. The magnetic fluxes are assumed to be independent, and drawn from a common even distribution. As in any model of 1D disordered conductors, all eigenstates are exponentially localized. We have therefore focused our attention on the weak-disorder regime, where the localization length $\xi=1/\gamma_2$ is much larger than the side of the plaquettes, so that universal features can be expected. In this regime, the disorder is characterized by its width $w \ll 1$, and more precisely by its first two moments, $\mu_2 \sim w^2$ and $\mu_4 \sim w^4$. We have examined in detail localization properties in this weak-disorder regime, and derived a systematic expansion of both positive Lyapunov exponents, and checked the outcome of our analytic approach against data obtained by means of extensive numerical simulations.

The most important result is that magnetic disorder induces a novel universality class, in the sense that the divergence laws of the localization length ξ in the weak-disorder regime ($w\to 0$), in various ranges of energy, are different from those observed with potential disorder, like, for example, in the usual tight-binding Anderson model. Indeed, we find the divergence $\xi\sim 1/w^4$ in the lateral regions (I) and (III) of the spectrum, where the carriers are offered only one channel, and $\xi \sim 1/w^2$ in the central region (II), where both transverse channels are open. For potential disorder, $\xi\sim 1/w^2$ always holds inside the spectrum of quasi-1D systems, irrespective of their band structure. In regions (I) and (III), the law (3.19) is not universal, since it involves the kurtosis K_4 of the disorder, besides its width w. In region (II), the absolute prefactors of the power law (3.20) have only been determined at the band center [see Eq. (3.21)].

Still more interesting is the scaling behavior of the Lyapunov exponents near internal or external band edges, i.e., at the special values of energy where one new channel is just opening. In the present case of a ladder-shaped network, the maximal number of channels is $M=2$, i.e., the width of the network. There are two external band edges, and two internal ones. We have obtained the law $\xi\sim 1/w$ at each of these points, as well as the existence of scaling laws (3.39), (3.43) in their vicinity. The corresponding scaling functions have been determined numerically in an accurate way. These results are again in contrast with the $\xi\sim 1/w^{2/3}$ law, characteristic of band edges with potential disorder.[11]

To summarize the discussion, if we go continuously from outside the spectrum toward its interior, we meet the following sequence of exponents for the law of divergence of the localization length ($\xi\sim 1/w^{\alpha}$)

$$\alpha=0,\ 1,\ 4,\ 1,\ \text{and}\ 2\ . \tag{4.1}$$

This sequence is richer than the one observed in the case of potential disorder, like, for example, the Anderson model, namely

$$\alpha=0,\ \tfrac{2}{3},\ \text{and}\ 2\ . \tag{4.2}$$

The present form of magnetic disorder leads therefore to a new universality class of localization properties. This makes such model systems very appealing. Among further open questions of physical relevance, we can mention strip-shaped networks, where the number M of coupled wires is greater than 2. The number $0\le m(\cos k)\le M$ of open channels depends on energy in a more complicated way;[8] the question of how the divergence law of ξ depends on M and m should be addressed.

Another direction of research concerns the combined effects of potential and magnetic disorder. One of the most interesting questions concerns the influence of various kinds of randomness on the localization length. Let us denote them as follows: ξ_V with potential disorder, in the absence of magnetic disorder, ξ_B with magnetic disorder, in the absence of potential disorder, and ξ_{V+B} in the presence of both kinds of disorder. In a strong uniform magnetic field, it has been shown[16,17] that the violation of time-reversal symmetry T leads to $\xi_{V,B_0}=2\xi_V$, at least at weak disorder. With quenched disorder in the magnetic field, the T symmetry is still violated, but not in such a simple way. It is interesting to notice that, in the insulating regime of quasi-1D systems, the results of the present work rather take the form $\ln\xi_B \sim 2\ln\xi_V$, at least in the one-channel regime, since we have $\xi_B \sim 1/w^4 \gg \xi_V \sim 1/w^2$. How does ξ_{V+B} compare with ξ_B and ξ_V? We hope to come back to these questions in the future.

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