Transition from Sharvin to Drude resistance in high-mobility wires

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The resistance of a wire is calculated from the ballistic up to the diffusive transport regime through a semiclassical transmission approach. A formula is derived which describes the transition from the Sharvin resistance to the Drude resistance when the mean free path becomes comparable to the wire length. The exact expression differs less than 2.5% from an interpolation formula which simply adds the two resistances. Good agreement with a recent experiment by Tarucha *et al.* is obtained.

In 1891 Maxwell[1] computed the electrical resistance of a narrow and short constriction (or point contact) in a metal, in the *diffusive* transport regime in which the width W of the constriction is large compared to the mean free path ℓ. In 1965 Sharvin[2] calculated the resistance in the opposite regime $\ell \gg W$ of *ballistic* transport. Subsequently, Wexler[3] studied the intermediate regime $\ell \simeq W$, where the resistance crosses over from the Maxwell to the Sharvin result.

Interestingly, the transition between the ballistic and the diffusive transport regime still has been scarcely investigated for a wire of length $L > W$. Only recently, Tarucha *et al.*[4] published measurements on the resistance of wires, of different lengths, defined in a high-mobility two-dimensional electron gas. This motivated us to study theoretically the resistance of wires from the ballistic to the diffusive regime. We assume elastic impurity scattering and specular boundary scattering, which is the relevant condition for the experiment. We restrict our investigation to semiclassical transport, and evaluate the Landauer formula exactly in this limit. The main outcome is that the resistance of a wire is the sum of the Sharvin resistance and the diffusive Drude resistance multiplied by a factor which we compute numerically, and find to be of order unity. Our exact calculation shows that — somewhat unexpectedly — the naive procedure of summing the Sharvin and Drude resistances is correct within 2.5% (3.5% for the three-dimensional case) over the range from $L \ll \ell$ to $L \gg \ell$. To illustrate the usefulness of our theory, we apply it to the experiment of Tarucha *et al.*[4] and find good agreement with no adjustable parameters. In addition, we show in the Appendix that our approach based on the Landauer formula is equivalent with the more conventional approach based on the Boltzmann equation. Finally, we would like to mention two recent, related papers. Nieuwenhuizen and Luck[5] calculate the conductance of a diffusive slab from Milne's equation, and Bauer *et al.*[6] obtain results similar to ours by concatenation of scattering matrices.

We study transport of noninteracting electrons through a two-dimensional wire, of width W and length L (see inset of Fig. 1). The electrons are scattered specularly at the wire boundaries. The wire is made of material with an ideal circular Fermi surface and a mean free path ℓ for elastic and isotropic impurity scattering. The modeling of impurity scattering by one single parameter implies that our results will be averages over the ensemble of all possible impurity configurations. We assume low temperatures, and thus neglect inelastic electron-phonon and electron-electron scattering. Our interest is in the semiclassical regime, where quantum-interference effects may be neglected, but Fermi-Dirac statistics must be retained. Beenakker and Van Houten have shown[7] that the semiclassical approximation of the Landauer formula is able to give a good description of many transport phenomena observed experimentally at temperatures on the order of 1 K. For a hard-wall wire of width W the semiclassical limit of the Landauer formula is[7]

$$G = \frac{2e^2}{h}\frac{k_F W}{\pi}\int_0^W \frac{dy}{W}\int_{-\pi/2}^{\pi/2}\frac{d\varphi}{2}\cos\varphi\, T(0,y,\varphi) \quad \text{(1a)}$$

$$= \frac{2e^2}{h}\frac{k_F W}{\pi}\langle T\rangle\,. \quad \text{(1b)}$$

The transmission probability $T(0,y,\varphi)$ is the probability that an electron which is positioned in lead 1 at $(x,y) = (0,y)$ with velocity $\mathbf{v} = v_F(\cos\varphi, \sin\varphi)$ is transmitted into lead 2 (see the inset of Fig. 1). Baranger *et al.*[8] have confirmed that Eq. (1) follows directly from the Landauer formula by using a Green function expres-

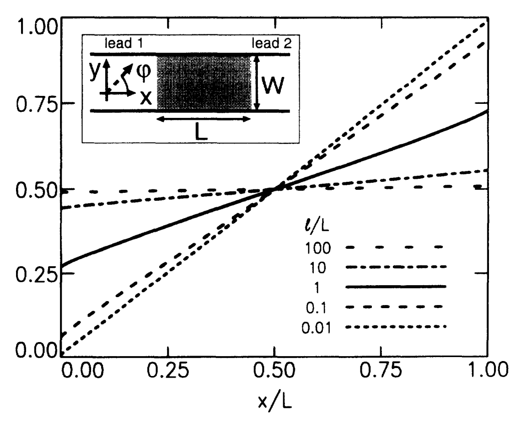

FIG. 1. The conditional transmission probability $\bar{T}(x)$ as a function of the position along the wire, for various ℓ/L. The inset shows schematically the wire and its coordinates.

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sion for the transmission amplitudes and then taking the semiclassical limit via a stationary-phase approximation.

Equation (1) is easily evaluated in two opposite regimes. First, the ballistic regime $\ell \gg L$. Then $T(0,y,\varphi) = \langle T\rangle = 1$ and hence Eq. (1) reduces to the familiar two-dimensional Sharvin conductance

$$G_S = \frac{2e^2}{h}\frac{k_F W}{\pi} . \tag{2}$$

Second, the diffusive regime $\ell \ll L$. Then[9] $\langle T\rangle = \pi\ell/2L$, so that Eq. (1b) becomes the Drude conductance

$$G_D = \frac{2e^2}{h}\frac{k_F l}{2}\frac{W}{L} . \tag{3}$$

The purpose of this paper is to derive a formula which describes the transition from G_S to G_D when $l \simeq L$.

We first note that, because boundary scattering is specular, electrons starting with equal angle of incidence, but different transverse coordinates, have the same transmission probability: $T(0,y,\varphi) = T(0,y',\varphi)$ for all y, y', φ. Furthermore, symmetry requires that $T(0,y,\varphi) = T(0,y,-\varphi)$. The average transmission probability $\langle T\rangle$ thus simplifies to

$$\langle T\rangle = \int_0^{\pi/2} d\varphi\, \cos\varphi\, T(0,\varphi) , \tag{4}$$

where the irrelevant y-coordinate has been dropped. By integrating over all possible electron trajectories starting with incoming angle φ, $T(0,\varphi)$ can be determined. We now introduce $T(x,\varphi)$ with $x \in [0,L]$ and $\varphi \in [0,\pi]$ as the probability that an electron at position x in the wire with direction φ reaches lead 2. By definition, a mean free path ℓ implies that an electron traversing an infinitesimal distance $\Delta s = \Delta x/\cos\varphi$ has a scattering probability of $\Delta s/\ell$. If an electron is scattered at position x, it has a probability $\overline{T}(x)$ to reach lead 2. Since the scattering is isotropic, this conditional transmission probability is given by

$$\overline{T}(x) = \frac{1}{\pi}\int_0^{\pi} d\varphi\, T(x,\varphi) . \tag{5}$$

An electron at position x with direction φ can be transmitted into lead 2 either after being scattered at x or by continuing in the direction φ. This leads to the integrodifferential equation

$$\ell\cos\varphi\,\frac{\partial T(x,\varphi)}{\partial x} = T(x,\varphi) - \overline{T}(x) . \tag{6}$$

At $x = 0$ and $x = L$ we have the boundary conditions

$$\begin{aligned} T(0,\varphi) &= 0 \qquad \text{if } \varphi \in [\pi/2,\pi] , \\ T(L,\varphi) &= 1 \qquad \text{if } \varphi \in [0,\pi/2] . \end{aligned} \tag{7}$$

Furthermore, from symmetry and from the fact that an electron exits the wire either through lead 1 or through lead 2, we deduce the sum rules

$$T(x,\varphi) + T(L-x,\pi-\varphi) = 1 , \tag{8a}$$

$$\overline{T}(x) + \overline{T}(L-x) = 1 . \tag{8b}$$

The integrodifferential equation (6) forms the basis of our calculation of the crossover from the ballistic to the diffusive regime. It is exact for isotropic impurity scattering and specular boundary scattering and can easily be solved numerically. In the Appendix an exact relation between the transmission probability and the solution of the Boltzmann transport equation is derived.

A closed expression for $\overline{T}(x)$ can be found by transforming Eq. (6) into an integral equation and integrating over ϕ. This leads to

$$\overline{T}(x) = \int_0^L dx'\, G(x,x')\,\overline{T}(x') + T_0(x) , \tag{9}$$

$$G(x,x') = \frac{1}{\pi}\int_0^{\pi/2}\frac{d\varphi}{\ell\cos\varphi}\, e^{-|x-x'|/\ell\cos\varphi} , \tag{10}$$

$$T_0(x) = \frac{1}{\pi}\int_0^{\pi/2} d\varphi\, e^{-(L-x)/\ell\cos\varphi} . \tag{11}$$

Equation (9) is known as Milne's equation describing scattering of light through a diffusive medium.[10] It is interesting to note that this similarity between electron and photon transport is due to the fact that in linear response the conductance is independent of the screening properties of the electron gas, as can be found in the Appendix. This justifies the single-particle transmission approach.

An exact analytical solution of Eq. (9) is known for an infinite system ($L = \infty$).[11] To obtain solutions for all possible ratios of ℓ/L we have computed $\overline{T}(x)$ by discretizing the x-values and numerically integrating Eqs. (10) and (11). Equation (9) then becomes a matrix equation which is easily solved. Once $\overline{T}(x)$ is known, the transmission probability $T(x,\varphi)$ can be found. Through Eq. (4) the average $\langle T\rangle$ and hence the conductance from Eq. (1b) are derived. Results for $\overline{T}(x)$ and G are plotted in Figs. 1 and 2(a). Note that $\overline{T}(x)$ satisfies the sum rule (8b). Figure 2(a) also shows the interpolation formula

$$G_{\rm IP}^{-1} = G_D^{-1} + G_S^{-1} , \tag{12}$$

which approximates the resistance of the wire by the sum of the Sharvin and Drude resistances. It is remarkable how well the interpolation formula $G_{\rm IP}$ compares with the exact result G. In Fig. 2(a) the relative error $(G_{\rm IP} - G)/G$ is also given. It is at most 2.5% when $\ell \simeq L$ and goes to zero for the ballistic as well as the diffusive limit.

By analogy with Wexler's result for a point contact,[3] we write the exact solution in the form

$$G = \frac{2e^2}{h}\frac{k_F W}{\pi}\left[1 + \gamma_{\rm 2D}\frac{2L}{\pi\ell}\right]^{-1} , \tag{13}$$

where the dimensionless parameter $\gamma_{\rm 2D}$ depends on the ratio ℓ/L as plotted in Fig. 2(b). Its limiting values are

$$\lim_{\ell/L\to 0}\gamma_{\rm 2D} = 1 , \qquad \lim_{\ell/L\to\infty}\gamma_{\rm 2D} = \frac{\pi^2}{8} . \tag{14}$$

The diffusive limit follows from the condition $G \to G_D$ if $\ell/L \to 0$, with G_D given by Eq. (3). The ballistic limit is obtained by solving the integral equation (9) to first

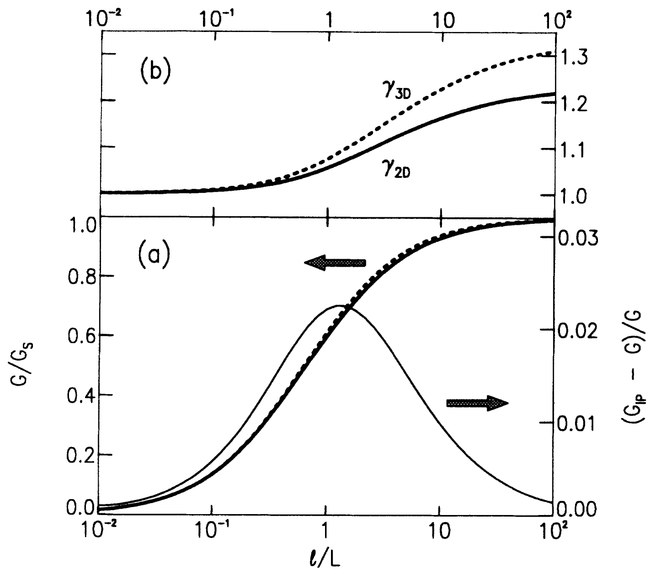

FIG. 2. (a) The conductance [normalized by the Sharvin conductance (2)] plotted against the ratio ℓ/L. The solid line is from the numerical solution of Eq. (9), the dotted line is G_{IP} according to the interpolation formula (12). The thin solid line shows the relative error of the interpolation formula. It remains below 2.5%. (b) The dependence on ℓ/L of the factor γ_{2D} in Eq. (13) (solid line) and γ_{3D} in Eq. (16) (dotted line).

order in L/ℓ, which can be done analytically.

The calculation for a three-dimensional wire goes along the same lines. In the semiclassical regime, the Landauer formula reads

$$G = \frac{2e^2}{h}\frac{k_F^2 S}{4\pi}\langle T\rangle \,, \tag{15}$$

where S is the cross-sectional area of the wire. (We assume that the wire has a constant cross section along the x axis.) The Sharvin conductance G_S is given by Eq. (15) with $\langle T\rangle = 1$. From comparison with the Drude conductance it follows that in the diffusive regime $\langle T\rangle = 4\ell/3L$.

As in the two-dimensional case, we assume specular boundary scattering. Therefore, transverse coordinates are irrelevant. The transmission probability $T(x,\varphi)$ with $x \in [0,L]$ and $\varphi \in [0,\pi]$ gives the probability that an electron at position x, and whose velocity is directed at an angle φ with respect to the x axis, reaches lead 2. Now, Eqs. (6), (7), and (9) still apply, whereas the integrations in Eqs. (4), (5), (10), and (11) are modified because of the different dimensionality.

The results are similar to the two-dimensional case. A simple interpolation formula which adds the Sharvin and Drude resistances is within 3.5% different from the exact G. We write the exact solution as

$$G = \frac{2e^2}{h}\frac{k_F^2 S}{4\pi}\left[1 + \gamma_{3D}\frac{3L}{4\ell}\right]^{-1} . \tag{16}$$

In Fig. 2(b) γ_{3D} is plotted. Its limiting values are

$$\lim_{\ell/L\to 0}\gamma_{3D} = 1 \,, \qquad \lim_{\ell/L\to\infty}\gamma_{3D} = \tfrac{4}{3} \,. \tag{17}$$

We conclude this paper by comparing our theoretical result for a two-dimensional wire (13) with the experiment by Tarucha *et al.*,[4] in which the resistances of wires with three different widths are measured for several lengths. The wires are defined in the two-dimensional electron gas in a high-mobility (Al,Ga)As heterostructure using wet etching. Their widths are $W_1 = 1.5\,\mu$m, $W_2 = 3.5\,\mu$m, and $W_3 = 7.5\,\mu$m. The lengths vary between $L = 4.0\,\mu$m and $L = 60\,\mu$m. The electron density $n = k_F^2/2\pi = 2.6 \times 10^{11}\,\text{cm}^{-2}$ and the bulk mean free path $\ell = 67\ \mu$m. In Fig. 3 the measurements are compared with the theoretical curves. We note that our theory — in which simply the experimental parameters without any fitting are used — provides quite a reasonable agreement with the measurements. This agreement indicates that reflection at the boundaries of the wet etched channels is indeed predominantly specular. Our results do not support the surmise of Ref. 4 that the mean free path in the narrowest wire is substantially enhanced above the bulk value due to lateral restriction (an effect attributed to the presence of one-dimensional subbands in the wire). It would be of interest to compare our theory with measurements further into the diffusive regime.

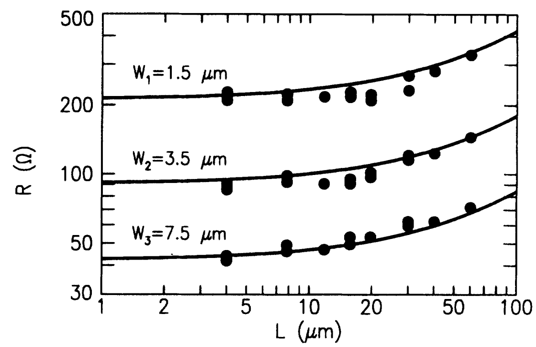

FIG. 3. Comparison between the experimental resistance data of Tarucha *et al.* (Ref. 4) (dots) and the results of our theory (curves). There are no adjustable parameters.

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APPENDIX: BOLTZMANN APPROACH

We show that the semiclassical transmission approach as described in the main text is equivalent with Boltzmann transport theory. The wire is connected via perfect leads to two electron reservoirs, with electrochemical potentials $\mu_1 = E_F + eV$ and $\mu_2 = E_F$. The electrons inside the wire at the position $\mathbf{r} = (x,y)$ and with wave vector $\mathbf{k} = k(\cos\varphi, \sin\varphi)$ have the Boltzmann distribution function $f_{\mathbf{k}}(\mathbf{r})$. In the presence of an electric field $\mathbf{E}(\mathbf{r})$ the stationary Boltzmann equation reads

$$e\mathbf{E}\cdot\frac{\partial f_{\mathbf{k}}(\mathbf{r})}{\hbar\partial\mathbf{k}} + \mathbf{v}\cdot\frac{\partial f_{\mathbf{k}}(\mathbf{r})}{\partial\mathbf{r}} = \int_0^{2\pi}\frac{d\varphi'}{2\pi\tau}f_{\mathbf{k}'}(\mathbf{r}) - \frac{f_{\mathbf{k}}(\mathbf{r})}{\tau} \,, \tag{A1}$$

where the right-hand side is the impurity-scattering term (with scattering time τ). We assume translational invariance along the x-axis and retain only the relevant x, φ coordinates. Following Wexler,[3] we introduce a function u by

$$f_{\mathbf{k}}(\mathbf{r}) = f_0(\varepsilon) + e\left(\frac{\partial f_0}{\partial \varepsilon}\right)[\phi(x) - Vu(x,\varphi)] , \tag{A2}$$

where $f_0(\varepsilon) = \Theta(E_F - \varepsilon)$ is the Fermi-Dirac distribution function at energy $\varepsilon = \hbar^2 k^2/2m$ and $\phi(x)$ is the electrostatic potential. Substitution of Eq. (A2) into Eq. (A1) yields in linear response and at zero temperature

$$\ell \cos\varphi \, \frac{\partial u(x,\varphi)}{\partial x} = \overline{u}(x) - u(x,\varphi) , \tag{A3}$$

$$\overline{u}(x) = \int_0^\pi \frac{d\varphi}{\pi} \, u(x,\varphi) . \tag{A4}$$

The physical meaning of $\overline{u}$ is that $E_F + eV\,\overline{u}(x)$ is the local electrochemical potential at x. The boundary conditions on $u(x,\varphi)$ follow from the requirement that the incoming electrons in the leads must have the same electrochemical potential as the attached reservoirs:

$$\begin{aligned} u(0,\varphi) &= 1 \qquad \text{if } \varphi \in [0,\pi/2] , \\ u(L,\varphi) &= 0 \qquad \text{if } \varphi \in [\pi/2,\pi] . \end{aligned} \tag{A5}$$

The conductance can be expressed in terms of u,

$$G = \frac{2e^2}{h} \frac{k_F W}{\pi} \int_0^\pi d\varphi \, \cos\varphi \, u(x,\varphi) , \tag{A6}$$

which is independent of x because of Eq. (A3).

By comparing Eqs. (A3), (A5), and (A6) with Eqs. (6), (7), and (4) we conclude that the semiclassical transmission approach is equivalent to the Boltzmann approach, upon the identification

$$u(x,\varphi) = 1 - T(x, \pi - \varphi) . \tag{A7}$$

This is a simple but instructive example of the equivalence between the Landauer formula and conventional transport theory: previously[12] this equivalence has been derived for the full quantum-mechanical case, starting from the Kubo formula. (Of course, neither the derivation in Ref. 12 nor the present one fully do justice to the connection between the reservoirs and the leads.[13]) Since the electrostatic potential $\phi(x)$ does not appear in Eq. (A3), we conclude that the conductance is independent of the screening properties of the electron gas. This is a generic feature of linear response.[9]

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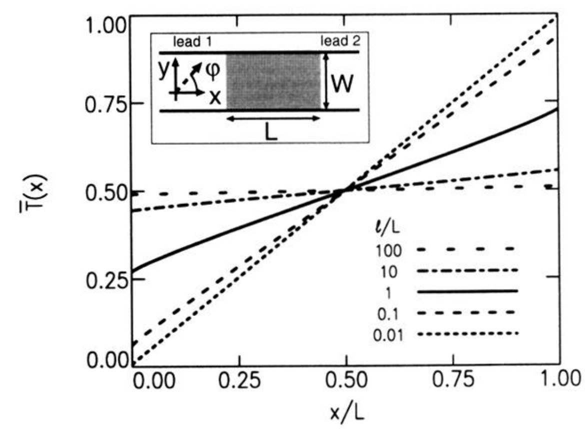

FIG. 1. The conditional transmission probability $\bar{T}(x)$ as a function of the position along the wire, for various ℓ/L. The inset shows schematically the wire and its coordinates.

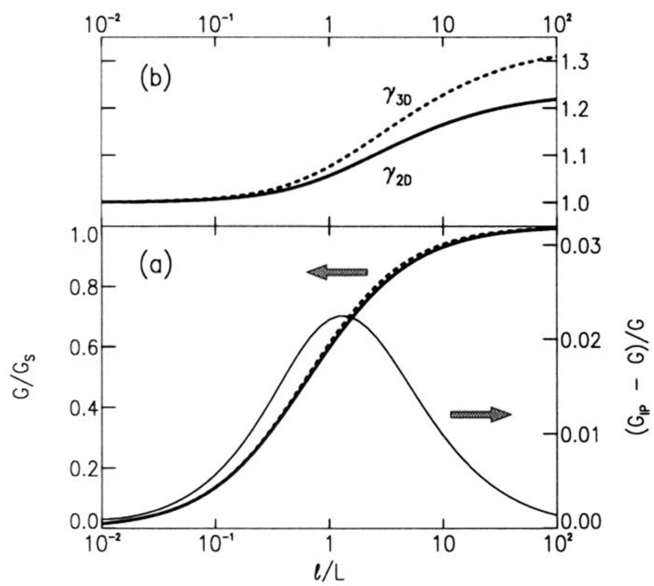

FIG. 2. (a) The conductance [normalized by the Sharvin conductance (2)] plotted against the ratio ℓ/L. The solid line is from the numerical solution of Eq. (9), the dotted line is $G_{\rm IP}$ according to the interpolation formula (12). The thin solid line shows the relative error of the interpolation formula. It remains below 2.5%. (b) The dependence on ℓ/L of the factor $\gamma_{\rm 2D}$ in Eq. (13) (solid line) and $\gamma_{\rm 3D}$ in Eq. (16) (dotted line).