Superconductivity phase diagram in the gauge-field description of the t-J model

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(Received 16 September 1993)

We analyze the effect that the gauge field has on the superconducting transition temperature in the t-J model. Mean-field theories of the t-J model tend to predict d-wave superconductivity at a very high temperature of the order of $T_c^0 \simeq 0.15$ J. We will show that this transition temperature is suppressed, if one takes a fluctuating gauge field into account. The underlying idea is that there is a significant reduction of free energy due to gauge-field fluctuations, which is partly lost when a superconducting gap opens up. This cost of energy prevents the system from going into a superconducting state. Superconductivity is only possible at an intermediate range of doping, when the superfluid density of holons is sufficiently large to stiffen the gauge field. These ideas are supported by a numerical analysis. We obtained a phase diagram in the doping-temperature plane, that shows that for $t/J = 3$ the optimal T_c occurs at a doping of $x \simeq 0.15$. One consequence of our analysis is that in this model the spin-gap phase is completely destroyed by gauge-field fluctuations.

I. INTRODUCTION

Due to an intense effort by many researchers the unusual properties of the high-T_c copper oxides are now quite well documented. In the normal state the copper oxides seem to be an example of a strongly correlated electronic system, which cannot be described by conventional Fermi-liquid theory. One of the unusual normal-state properties is the resistivity, which is proportional to temperature over a large range of temperature. The superconducting state is in some sense less unusual than the normal state, because in many respects it behaves like a BCS superconductor, but with an unusual pairing mechanism. The onset of superconductivity occurs at temperatures that are so high that the pairing between electrons cannot be solely due to phonons. Moreover, microwave measurements of the quasiparticle contribution to the conductivity have shown that the scattering rate decreases strongly below T_c, which is inconsistent with a scattering mechanism due to phonons.[1] It is therefore more likely that the pairing mechanism has an unconventional origin, which could be magnetism. This is further supported by experimental reports of gapless excitations,[1–5] and even evidence for a d-wave symmetry of the order parameter.[6] Another peculiar aspect of the superconducting state is that it only occurs in an intermediate rage of doping of $0.05 \lesssim x \lesssim 0.3$, but disappears when the doping is too small or too big.

Many of the microscopic models that have been proposed to describe the properties of the high-T_c copper oxides are based on the two-dimensional Hubbard model or the t-J model.[7] In the case of the t-J model, one can obtain a crude approximation of the onset of superconductivity, by means of a BCS-like mean-field decoupling of the Hamiltonian. Several mean-field phases have been suggested, and depending on the doping and temperature different phases can be energetically favored.[8–16] In general these mean-field phases predict a pairing-transition temperature of the order of $T_c^0 \simeq 0.15J$ close to half filling, which corresponds to a temperature of several hundred degrees kelvin. The reason for this overestimate of T_c is that a simple mean-field theory ignores fluctuations, which are very important in a strongly correlated system.

We will take the gauge-field formulation of the t-J model as our starting point, which goes beyond mean-field theory by including Gaussian fluctuations of a gauge field.[13,17–20] The gauge-field model has been succesful in explaining some of the normal-state properties, such as the linear resistivity, in the regime above the Bose-condensation temperature.[18–20] So far, little work has been done on what the effect is of the gauge field on the superconducting state. This article will focus on how the interaction with the gauge field can suppress the pairing-transition temperature to a temperature scale that agrees more with the experimental values of $T_c \lesssim 100$ K. The main argument for this suppression is that the gauge field introduces an additional term in the free energy,[21] which opposes the opening of a superconducting gap. Our numerical calculations show that this suppression is very significant, and that in fact superconductivity only survives at an intermediate range of doping, with a maximum T_c at a doping of $x \simeq 0.15$.

II. THE ROLE OF GAUGE-FIELD FLUCTUATIONS IN THE t-J MODEL

In this section we will first give a quick review of how the gauge-field model is derived from the t-J model. We refer to other papers for a more lengthy discussion of this derivation.[13,17,19,22] Our starting point is the t-J model on a square lattice,

$$H = -t \sum_{\langle i,j\rangle\sigma} c_{i\sigma}^\dagger c_{j\sigma} + J \sum_{\langle i,j\rangle} \left(\mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right), \quad (1)$$

where $\mathbf{S}_i = \frac{1}{2} c_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} c_{j\beta}$ and $n_i = \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}$. This Hamiltonian is under the important constraint that no site is double occupied. In order to satisfy this constraint we employ the slave-boson formalism,[23,24] in which the electron operator $c^\dagger_{i\sigma}$ is replaced by $c^\dagger_{i\sigma} = f^\dagger_{i\sigma} b_i$. The boson operator $b^\dagger_i$ keeps track of the empty sites, and the fermion operator $f^\dagger_{i\sigma}$ carries the spin.[8,9] The constraint of no double occupancy is satisfied by requiring that $b^\dagger_i b_i + \sum_\sigma f^\dagger_{i\sigma} f_{i\sigma} = 1$ at each site i. In order to get a superconducting state in the slave-boson picture it is not sufficient that the fermions form Cooper pairs, but the bosons have to be Bose condensed as well.

The gauge-field model is obtained from Eq. (1) by decoupling the hopping term and the Heisenberg term using Hubbard-Stratonovich fields, and then making the approximation of only considering fluctuations of the phase of one of these new fields. Denoting the Hubbard-Stratonovich fields by $\xi_{ij} = \xi e^{ia_{ij}}$ and $\Delta_{ij} = \pm\Delta_0$, this leads to the Hamiltonian[9,16]

$$H = \frac{1}{2}\sum_{\langle i,j\rangle}\left[\frac{3J}{4}|\xi|^2 + \frac{4}{3J}|\Delta_0|^2 - \xi e^{-ia_{ij}}\left(\frac{3J}{4} f^\dagger_{i\sigma} f_{j\sigma} + 2t b^\dagger_i b_j\right) - \text{c.c.} - \Delta^*_{ij}(f_{i\uparrow}f_{j\downarrow} - f_{i\downarrow}f_{j\uparrow}) - \text{c.c.}\right]$$
$$+\frac{8t^2}{3J}\sum_{\langle i,j\rangle} b^\dagger_i b_j b^\dagger_j b_i - \mu_0 \sum_i f^\dagger_{i\sigma} f_{i\sigma} - i\sum_i \lambda_i (f^\dagger_{i\sigma} f_{i\sigma} + b^\dagger_i b_i - 1), \qquad (2)$$

where λ_i is a Lagrange-multiplier field that enforces the local constraint $b^\dagger_i b_i + f^\dagger_{i\sigma} f_{i\sigma} = 1$. The role of the field a_{ij} will be discussed later on.

We will first consider the mean-field solution of this Hamiltonian, which corresponds to $a_{ij} = 0$. At the mean-field level $i\lambda_i = \mu_B$ plays the role of a chemical potential for the bosons, and μ_B is chosen such that the average boson density is equal to the doping concentration x. The mean-field phase diagram is schematically shown in Fig. 1.[19,25,26] Below the dashed line the uniform resonating-valence-bond (RVB) order parameter ξ is nonzero. At a lower temperature, denoted by the dotted line, d-wave pairing between fermions occurs, i.e., $\Delta_{i,i+\hat{x}} = -\Delta_{i,i+\hat{y}} = \Delta_0$. Below the solid line the bosons condense into a superfluid state. According to mean-field theory Bose condensation occurs at a temperature scale given by $T^0_{\rm BE} \simeq 2\pi x/m_B$, where $1/m_B = 2t\xi$. The mean-field phase diagram divides naturally into four regions. Region I with $\langle b\rangle \neq 0$ is a Fermi-liquid phase. Region II with $\Delta_0 \neq 0$ but $\langle b\rangle = 0$ is called the *spin-gap* phase, because an anisotropic gap appears in the fermion spectrum which represents the spin degrees of freedom. In region III both Δ_0 and $\langle b\rangle$ are nonzero, so that d-wave pairing between physical electrons occurs, resulting in a superconducting phase. Region IV has been called the *strange metal* phase, because it exhibits some of the unusual properties of the normal state of the high-T_c copper oxides.[19,25]

Since $f_{i\sigma}$ and b_i are fictitious entities, the only true phase boundary in Fig. 1 is the transition to the superconducting state in region III. Nevertheless it is possible that the other transition lines broaden to crossover lines, such that one can still identify the regions I, II, and IV in the phase diagram, characterized by the physical properties described above. In particular much attention has been paid to the spin-gap phase, because NMR and susceptibility experiments indicate the appearance of a gap in the spin excitation spectrum in a temperature range above the superconducting T_c in underdoped materials. On the other hand, a recent analysis of the data by Millis and Monien indicated that the spin-gap phase may be absent in single-layer materials such as $La_{2-x}Sr_xCuO_4$, and present in double-layer materials such as the Y-Ba-Cu-O (YBCO) compounds.[27] Thus the identification of the spin-gap phase with region II is quite uncertain at this point.

A serious difficulty with the schematic mean-field diagram shown in Fig. 1 is that the temperature scale for Bose condensation is much too high, if one uses the mean-field expression $T^0_{\rm BE} \simeq 4\pi t\xi x$. Furthermore, close to half filling ($x \lesssim 0.04$) the d-wave pairing state is unstable to more complicated phases, such as dimerized phases,[12,13] incommensurate flux phases,[14] and staggered flux phases.[15,16] We restrict our attention to $x \gtrsim 0.04$, which is indicated by the shaded region in Figs. 3 and 4. The dotted line in this phase diagram is the mean-field transition to a d-wave pairing state. On the same plot the $T^0_{\rm BE}$ line would lie entirely *inside* the shaded area.

For noninteracting bosons Bose condensation does not really exist in two dimensions, but one can still consider $T^0_{\rm BE}$ as a crossover temperature below which the boson susceptibility diverges exponentially:

$$\chi^0_B = \frac{1}{24\pi m_B}\left(e^{T^0_{\rm BE}/T} - 1\right). \qquad (3)$$

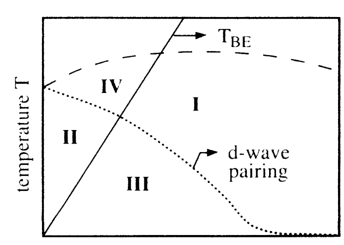

FIG. 1. Schematic mean-field phase diagram of the t-J model. Below the dashed line the uniform RVB order parameter ξ is nonzero. The mean-field pairing line (dotted) and the Bose-condensation line (solid) divide the phase diagram into four regions. Region I is a Fermi-liquid phase, region II is the spin-gap phase, region III is the superconducting phase, and region IV is the strange metal phase. In this paper we argue that the spin-gap phase is destroyed by gauge-field fluctuations.

It has been argued that the characteristic temperature scale for the increase in χ_B will be strongly suppressed by fluctuations around the mean-field solution.[18,28–31]

We will now discuss how fluctuations alter the mean-field results mentioned above. We will restrict our analysis to Gaussian fluctuations of the phase a_{ij} of the RVB order parameter ξ_{ij}, which is a massless Goldstone mode. The field a_{ij} is called a gauge field, because the Hamiltonian (2) is invariant under the local gauge transformation[13]

$$\begin{aligned} f_{i\sigma} &\longrightarrow f_{i\sigma}e^{i\theta_i} , \\ b_i &\longrightarrow b_i e^{i\theta_i} , \\ a_{ij} &\longrightarrow a_{ij} + \theta_i - \theta_j . \end{aligned} \tag{4}$$

We will choose to work in the Coulomb gauge $\nabla \cdot \mathbf{a} = 0$, in which case the gauge-field propagator $D_{\mu\nu}(q) = (\delta_{\mu\nu} - p_\mu p_\nu/p^2)/\Pi(q)$ is purely transverse. $\Pi(q) = \Pi_F(q) + \Pi_B(q)$ is the sum of the fermion and boson transverse current-current polarization functions.[19] Here we defined $p_\mu = 2\sin q_\mu/2$ to take the lattice structure into account. The Lagrange-multiplier field λ_i can be considered as the time component of the gauge field.[13,19] In this paper we will simply replace $i\lambda_i$ by its saddle-point value μ_B, which will serve as the chemical potential of the bosons. The gauge field couples to both the fermions and the bosons, so one expects that both the d-wave pairing line and the Bose-condensation line in Fig. 1 will be affected by the fluctuating gauge field.[19] While the main topic of this paper concerns the coupling of the fermions to the gauge field, this problem cannot be addressed without considering the coupling of the bosons to the gauge field as well.

The coupling of the bosons to the gauge field is a strong-coupling problem in the physically interesting case of $t/J > 1$, and is therefore difficult to analyze. This was illustrated by a diagrammatic analysis of Ioffe and Kalmeyer,[28] who calculated the lowest-order gauge-field correction to the diamagnetic susceptibility χ_B. They showed that this correction becomes very large if one approaches the Bose-condensation temperature $T^0_{\rm BE}$, at which point the perturbative analysis becomes unreliable. The problem has also been treated by a renormalization-group analysis[29] and by path-integration methods. It was pointed out that the fluctuating gauge field tends to reduce the projected area of Feynman paths, so that the path integral is dominated by almost retracing paths.[32] Wheatley and co-workers[30,31] did a further analysis of the path-integral formulation, by making the rather drastic approximation of relating the problem to one where the bosons couple to a dissipative bath. This problem of noninteracting bosons coupled to a heat bath with a damping time τ_0 can be solved exactly. For strong coupling, i.e., for $\tau_0^{-1} \gg k_BT$, the boson susceptibility χ_B is given by[31]

$$\chi_B = \frac{\tau_0 x}{\pi m_B^2}. \tag{5}$$

It is tempting to identify τ_0^{-1} with the transport scattering rate of the bosons by the gauge field, which is given in Born approximation by $\tau_0^{-1} \simeq k_BT/4m_B\chi$. If we further follow Wheatley *et al.* and replace χ by the free-fermion expression $\chi_F^0 = 1/(12\pi m_F)$, we obtain

$$\chi_B^{\rm diss} \simeq \frac{2\chi_F^0 T^0_{\rm BE}}{\pi^2 T}. \tag{6}$$

We will refer to this result as the *dissipative* model. Note that $\chi_B^{\rm diss}$ diverges only as T^{-1}, as opposed to the exponential growth of χ_B^0 given by Eq. (3).

We believe that Eq. (6) grossly overestimates the effect of the gauge-field fluctuations for at least the following reason. The susceptibility χ that controls the strength of the gauge-field fluctuations is the sum $\chi_F + \chi_B$, where χ_B should be treated self-consistently. Note that this self-consistency is missing in Eq. (6), because χ was simply replaced by χ_F. As χ_B grows τ_0^{-1} is reduced so that the dissipation crosses over to the weak-coupling limit, and Eq. (5) no longer applies. We have carried out a self-consistent calculation of χ_B, where we used the full solution of the susceptibility valid for arbitrary τ_0. The resulting self-consistent $\chi_B^{\rm SC}$ is shown in Fig. 2. A recent analysis based on self-retracing Feynman paths yields qualitatively the same results.[33]

The value of $\chi_B^{\rm SC}$ (solid line) lies between χ_B^0 (dotted line) and $\chi_B^{\rm diss}$ (dashed line), and diverges exponentially below a temperature which is a fraction of $T^0_{\rm BE}$. In the absence of a full theory, we believe that $\chi_B^{\rm SC}$ is a reasonable guess of the behavior of the boson susceptibility, which we can use in the interim. We have to keep in mind however that at low doping $\chi_B^{\rm SC}$ is probably too large, because the self-consistent dissipative model assumes that $\Pi_B(q) = \chi_B q^2$,[31] while in reality $\Pi_B(q)$ levels off to $\rho_S = x/m_B$ for large q. This means that when ρ_S is small the self-consistent dissipative model underestimates the gauge-field fluctuations for large regions of $\mathbf{q}$ space, resulting in a susceptibility $\chi_B^{\rm SC}$ that is too large.

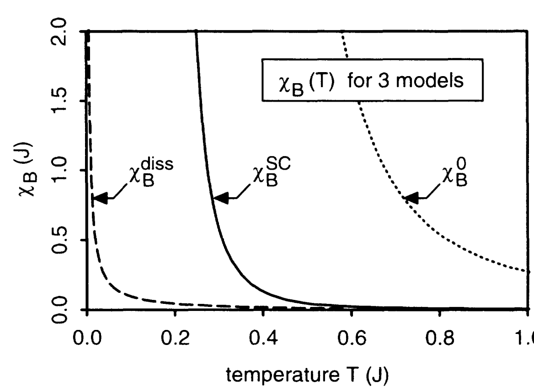

FIG. 2. Boson susceptibility $\chi_B(T)$ for three different models for a doping $x = 0.07$. The fact that $\chi_B(T)$ increases rapidly at low temperatures indicates that the bosons effectively condense into a superfluid state below a certain crossover temperature. The dotted line is the mean-field value χ_B^0, given by Eq. (3). The dashed line represents the dissipative model for $\chi_B^{\rm diss}$, given by Eq. (6), which we believe grossly overestimates the effectiveness of the gauge field to suppress Bose condensation. The solid line is the self-consistent dissipative $\chi_B^{\rm SC}$, which takes into account that a large χ_B stiffens the gauge field. In the absence of a full theory $\chi_B^{\rm SC}$ is a reasonable guess for the behavior of the susceptibility $\chi_B(T)$. We believe, however, that at low doping $\chi_B^{\rm SC}$ underestimates the suppression of Bose condensation.

In this paper we have carried out the calculation of the phase diagram using all three different choices for χ_B. For reasons to be explained below, it turns out that χ_B^0 and χ_B^{SC} yield practically indistinguishable phase boundaries for the onset of superconductivity. That result is shown in Fig. 3. For completeness the phase boundary using χ_B^{diss} is shown in Fig. 4.

The main result of this paper is that *quantum* fluctuations of the gauge field are very effective in suppressing the pairing between fermions. In fact, the suppression is so effective that the spin-gap phase (region II in Fig. 1) is destroyed completely, and only a direct transition to a d-wave pairing state remains. The transition temperature T_c is reduced compared to the mean-field value T_c^0, and T_c vanishes completely at low doping. The result of our numerical calculation of T_c is shown in Figs. 3 and 4. Before going into any technical details, we give a qualitative discussion of the physics behind this suppression. The coupling of the fermions to the gauge field is very much analogous to the coupling of electrons to an electromagnetic field, except that the magnitude of the dimensionless coupling constant is very different. The dimensionless coupling constant is very small for an electromagnetic gauge field, but of order unity in the t-J model. It is known that in metals the low-lying excitations associated with a fluctuating gauge field give rise to a large negative contribution to the free energy, so that the specific heat in three dimensions varies as $T\ln T$.[34] In ordinary metals this is a small effect because it is proportional to v_F^2/c^2,

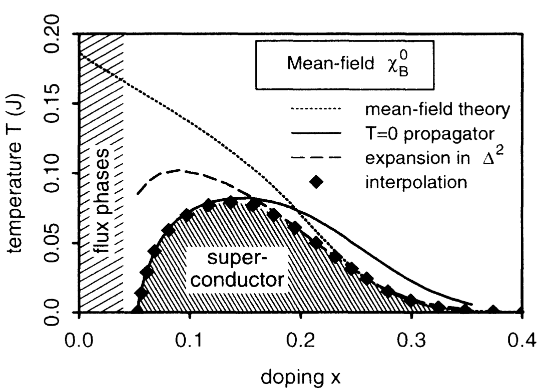

FIG. 3. Phase diagram of the t-J model for $t/J = 3$ using a mean-field expression for the susceptibility χ_B^0, given by Eq. (3). The self-consistent dissipative model χ_B^{SC} produces a phase diagram that is essentially indistinguishable. The solid line for $T_c(x)$ uses $T = 0$ expressions for $\Pi_F(\nu)$ (see Sec. III A), while the dashed line is obtained by expanding $\text{Im}\Pi_F(\nu)$ and $\text{Re}\Pi_F(\nu)$ in Δ_0^2 (see Sec. III B). The first-order jump in Δ_0 at the transition is quite small for $x \gtrsim 0.1$, and hence the expansion in small Δ_0^2 (dashed line) is a good approximation in this case. For $x \lesssim 0.1$ the first-order jump in Δ_0 becomes so large that the solid line is more appropriate. The line denoted by black diamonds is our best guess of the correct phase boundary within this model. For $x \lesssim 0.05$ superconductivity vanishes completely, which is directly related to the fact that the gauge field becomes unstable against flux phases for $x \lesssim 0.04$. For $x \gtrsim 0.2$ the gauge field becomes so stiff that the transition line $T_c(x)$ approaches the d-wave BCS value $T_c^0(x)$ (dotted line).

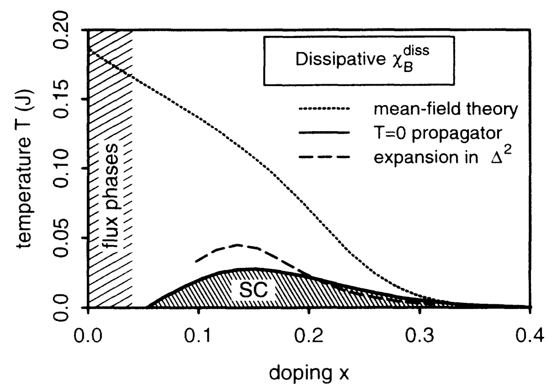

FIG. 4. Phase diagram of the t-J model in the doping-temperature plane for $t/J = 3$, using the dissipative model for the susceptibility χ_B^{diss}. The solid line for $T_c(x)$ uses $T = 0$ expressions for $\Pi_F(\nu)$, while the dashed line is obtained by expanding $\text{Im}\Pi_F(\nu)$ and $\text{Re}\Pi_F(\nu)$ in Δ_0^2. The $T = 0$ approximation for the propagator (solid line) is more appropriate, because of the large first-order jump in Δ_0 at the transition. In the underdoped case the susceptibility χ_B^{diss} is relatively small at the transition, so this model predicts a direct transition from a strange metal to a superconducting phase. Note that in this phase diagram the transition temperature $T_c(x)$ is much lower than in Fig. 3. This is directly related to the fact that in the dissipative model Bose condensation occurs at a much lower temperature than if one uses χ_B^0 or χ_B^{SC} (see Fig. 2). For large doping $x \gtrsim 0.35$ the gauge field becomes so stiff that the transition line $T_c(x)$ approaches $T_c^0(x)$ (dotted line).

and has not yet been observed. This small factor is absent in the t-J model, which makes these fluctuations very important. In two dimensions the specific heat varies as $T^{2/3}$, implying a free-energy term F_{gauge} proportional to $T^{5/3}$. The importance of the gauge-field contribution to the free energy has been pointed out by Hlubina *et al.*,[21] who showed that the contribution from the transverse gauge-field fluctuations, together with the longitudinal gauge-field fluctuations, brings the mean-field free energy much closer to that given by high-temperature expansions. Unlike the transverse mode, the longitudinal mode does not give rise to singular corrections at low temperatures, because it is screened. We will therefore ignore the longitudinal contribution to the free energy in what follows below.

In a pairing state a gap Δ opens up in the fermion spectrum. This introduces a gap in the gauge-field excitation spectrum as well, so that gauge-field modes with frequencies less than 2Δ do not contribute to the free energy, resulting in a net free-energy cost. We can estimate the free-energy cost δF_{gauge} by replacing the temperature cutoff in F_{gauge} by Δ, resulting in $\delta F_{\text{gauge}} \propto \Delta^{5/3}$. On the other hand the BCS-like free-energy gain from pairing is proportional to Δ^2, so that δF_{gauge} always dominates, at least for small enough Δ. This situation will change when the boson susceptibility becomes so large that the bosons effectively condense into a superfluid state. In that case the gauge field becomes effectively massive, with a stiffness equal to the superfluid density $\rho_S \simeq x$ of the bosons, due to the Anderson-Higgs mechanism. The gauge field is then so stiff that $F_{\rm gauge}$ is no longer dominating, and its role in suppressing fermion pairing disappears. We expect then a direct transition to a superconducting phase with d-wave pairing between physical electrons.

This qualitatively explains the phase diagrams shown in Figs. 3 and 4, which were obtained by a detailed numerical calculation to be described in the rest of the paper. We see in Figs. 3 and 4 that, in contrast to the mean-field solution, T_c now vanishes for sufficiently low doping $x \lesssim 0.05$. For larger doping the stiffness of the gauge field increases and it becomes less effective in suppressing pairing. Therefore for large doping $x \gtrsim 0.2$ the transition line $T_c(x)$ is close to the mean-field line $T_c^0(x)$, as one can see in Figs. 3 and 4. At low doping the phase diagram now describes a direct transition from a metallic phase to a superconducting state, bypassing the spin-gap phase. This implies that we have to look beyond the single-layer t-J model for an explanation of the spin-gap phase in bilayer materials, in agreement with the analysis of Millis and Monien mentioned earlier.[27] In the overdoped region $x \gtrsim 0.2$ we expect a direct transition from a Fermi-liquid phase, i.e., a phase in which the bosons are Bose condensed, to a superconducting state.

Another consequence of our numerical analysis is that we expect the transition to be first order, with a relatively large jump of Δ. Of course, in practice the transition will be rounded off by phase fluctuations of the pairing field Δ_{ij}, which are not considered in this analysis. Therefore the calculation presented here should be considered as a calculation of the mean-field T_c as far as two-dimensional fluctuations in the pairing field Δ_{ij} are concerned.

In the remainder of this paper we will describe the details of our calculation of $F_{\rm gauge}$ and its dependence on Δ_0. In Sec. III we derive a formal expression for the free energy $F_{\rm gauge}(\Delta_0)$, which will be analyzed in two limiting cases in Secs. III A and III B. We present our numerical analysis in Sec. IV. This numerical analysis takes several important aspects of the t-J model into account, that we did not discuss so far. For instance, an important role is played by the d-wave symmetry of the gap function $\Delta(\mathbf{k})$ and the non-spherical Fermi surface. These anisotropies have a large effect on the free energy $F_{\rm gauge}$, and are taken fully into account in our numerical analysis described in Sec. IV.

III. THE GAUGE-FIELD CONTRIBUTION TO $F(\Delta_0)$ IN THE PRESENCE OF A GAP

We now present a detailed calculation of the gauge-field contribution to the free energy in the presence of pairing between the fermions. In the presence of a gauge field the onset of pairing is still determined by minimizing the total free energy $F_{\rm tot}(\Delta_0)$, analogous to BCS theory. However, the total free energy will now have a contribution from the gauge field as well.[21] We note that the free energy is a gauge-invariant quantity, and is free of the singularities that plague quantities that are not gauge invariant, such as the fermion Green function.[35] This is why we choose to analyze $F_{\rm tot}(\Delta_0)$, rather than a diagrammatic study of the pairing amplitude in the presence of a gauge field.

We will calculate $F_{\rm tot}(\Delta_0)$ using the following procedure. First we integrate over the matter fields $f_{i\sigma}$ and b_i, which leads to an effective action $S_{\rm eff} = \beta F_{\rm MF}(\Delta_0) + S_{\rm gauge}[a]$, where

$$F_{\rm MF}(\Delta_0) = \frac{3J}{4}\xi^2 + \frac{4}{3J}\Delta_0^2 - 2T\int \frac{d^2k}{(2\pi)^2}\ln[\cosh(E_k/2T)] + T\int \frac{d^2k}{(2\pi)^2}\ln(1-e^{-\Omega_k/T}) , \qquad (7)$$

$$S_{\rm gauge}[a] = \frac{T}{2}\sum_{i\nu_n}\int \frac{d^2q}{(2\pi)^2}\Pi_{\mu\nu}(\mathbf{q},i\nu_n)a_q^\mu a_{-q}^\nu .$$

The dispersion relations $E_k = \sqrt{\epsilon_k^2+\Delta_k^2}$ for the fermions and Ω_k for the bosons are given by

$$\begin{aligned} \Omega_k &= 2t\xi(\cos k_x + \cos k_y) - \mu_B; \\ \epsilon_k &= \frac{3J\xi}{4}(\cos k_x + \cos k_y) - \mu_F; \\ \Delta_k &= \Delta_0(\cos k_x - \cos k_y). \end{aligned} \qquad (8)$$

Finally we obtain the total free energy $F_{\rm tot}(\Delta_0) = F_{\rm MF}(\Delta_0) + F_{\rm gauge}(\Delta_0)$, by integrating over the gauge field in the action $S_{\rm gauge}[a]$. By distorting the contour integral and noting that the analytical continuation of $\log\Pi(\mathbf{q},i\nu_n)$ has a branch cut along the real axis, one finds that the contribution of the gauge field to the free energy can be written as

$$F_{\rm gauge}(\Delta_0) = \int \frac{d^2q}{(2\pi)^2}\int_0^\infty \frac{d\nu}{2\pi}[2n_B(\nu)+1] \times \arctan\left(\frac{{\rm Im}\Pi(q,\nu+i\delta,\Delta_0)}{{\rm Re}\Pi(q,\nu+i\delta,\Delta_0)}\right), \qquad (9)$$

where $n_B(\nu) = (e^{\nu/k_BT}-1)^{-1}$ is the Bose occupation number. For $\Delta_0 = 0$, Eq. (9) is equivalent to the expression written down by Hlubina *et al.*[21] Notice that while Hlubina *et al.* needed a regularization scheme to keep $F_{\rm gauge}$ finite, we avoided the infinite constant by taking the analytical continuation $i\nu_n \to \nu + i\delta$.

The opening of a gap Δ_0 will mostly affect the fermionic contribution to the inverse of the total gauge-field propagator $\Pi = \Pi_F + \Pi_B$. In the normal state, Π_F has the form $\Pi_F(\mathbf{q},\nu) = \chi_F p^2 - i\gamma_F(\mathbf{q})\nu/p$, where we defined $p_\mu = 2\sin q_\mu/2$ to take the lattice structure of the t-J model into account. The fermion susceptibility χ_F is given by

$$\chi_F = -\frac{1}{12}\int \frac{d^2k}{(2\pi)^2}\frac{\partial^2\epsilon_k}{\partial k_x^2}\frac{\partial^2\epsilon_k}{\partial k_y^2}f'(\epsilon_k). \qquad (10)$$

Notice that χ_F is negative for the t-J model for any doping $x \lesssim 0.5$, which indicates that the uniform phase $\xi_{ij} = \xi$ is unstable towards flux phases close to half filling.[16] Away from half filling the uniform phase regains its stability, because the sum $\chi = \chi_F + \chi_B$ becomes positive as soon as the density of bosons is sufficiently large. The damping parameter $\gamma_F(\mathbf{q})$ is a finite number which depends on the direction of $\mathbf{q}$, and for the t-J model close to half filling $\gamma_F(\mathbf{q})$ is small in the $(1,0)$ direction, but strongly enhanced in the $(1,1)$ direction.

When a gap Δ_0 opens up the polarization function $\Pi_F(\mathbf{q},\nu)$ will be modified, which will change the free energy $F_{\rm gauge}$ in Eq. (9). The most obvious change is that $\Pi_F(q=0)$ becomes massive with a mass proportional to Δ_0^2. This is responsible for the Meissner effect in ordinary superconductors. The effect of this mass term on the free energy has been discussed earlier by Halperin *et al.*[36] in the context of a fluctuating electromagnetic field in a BCS superconductor. Their work was a classical calculation, based on a Ginzburg-Landau free energy. They argued that this mass term in $\mathrm{Re}\Pi_F$ gives rise to an additional term in the free energy, that is nonanalytic in Δ^2. This implies that the superconducting transition must be a first-order transition.

We are more interested in the question how the total free energy is influenced by *quantum* fluctuations of the gauge field. We therefore need to know how a nonzero gap modifies $\Pi_F(\nu)$ at *finite* frequencies. We first consider $\Pi_F(\nu)$ in the normal state, i.e., without a gap. To get a simple estimate of $F_{\rm gauge}(\Delta_0=0)$ we consider first for simplicity an isotropic band structure, in which case $\Pi_F(q,\nu+i\delta)=\chi_F q^2 - ik_F\nu/q$ for $\nu<k_Fq/m_F$. We further concentrate on the *quantum fluctuations* in Eq. (9), i.e., $\nu>k_BT$, so that $2n_B+1$ may be replaced by unity. In the normal state we approximate $F_{\rm gauge}$ by

$$F_{\rm gauge}(\Delta_0=0)\simeq -\int\frac{d^2q}{(2\pi)^2}\int_{k_BT}^{k_Fq/m_F}\frac{d\nu}{2\pi}\times\arctan\left(\frac{k_F\nu}{\chi q^3}\right). \tag{11}$$

This is a large negative quantity, which is finite because $\mathbf{q}$ is restricted to the first Brillouin zone. We are interested in how this quantity depends on the lower cutoff k_BT in the frequency integral. For small frequencies $\nu\ll\chi k_F^2$ the q integration can be done first, which yields a factor proportional to $(k_F\nu/\chi)^{2/3}$. The ν integration then yields a large negative term plus a $T^{5/3}$ correction due to the lower cutoff at k_BT.

When a gap Δ opens up both the real and imaginary part of the polarization function are modified, and in general $\mathrm{Im}\Pi_F(\nu)$ and $\mathrm{Re}\Pi_F(\nu)$ become very complicated functions of the frequency ν, especially when the gap $\Delta(\mathbf{k})$ is anisotropic. Before studying the general case, we will first consider the simpler case of an isotropic gap at $T=0$. In that case the propagator is modified by the gap as follows.

(i) Low-lying gauge-field modes with $\nu<2\Delta$ are undamped, i.e., $\mathrm{Im}\Pi_F=0$ for $\nu<2\Delta$ (at $T=0$). In ordinary superconductors this is responsible for the anomalous skin effect.[37,38]

(ii) The real part of $\Pi_F(q,\nu)$ is enhanced, which stiffens the gauge field.

We now see that this change in $\mathrm{Im}\Pi_F(\nu)$ and $\mathrm{Re}\Pi_F(\nu)$ gives rise to a huge cost in the free energy. From the discussion following Eq. (11) we learned that in the normal state all gauge-field modes give a negative contribution to the free energy $F_{\rm gauge}(\Delta=0)$. When an isotropic gap $\Delta\gg T$ opens up, the gauge-field modes with frequencies $\nu<2\Delta$ do not contribute to the free energy anymore. The ν integral in Eq. (11) is now cut off by Δ instead of k_BT, so that a free-energy cost of $\Delta^{5/3}$ results. This free-energy cost is in general much larger than the free-energy gain coming from $F_{\rm MF}(\Delta)$, which is proportional to Δ^2. This implies that due to low-lying gauge-field fluctuations it is not favorable anymore to open up a gap Δ, and hence the superconducting transition temperature will be suppressed.

We now turn to the the calculation of $\mathrm{Im}\Pi_F(\nu)$ and $\mathrm{Re}\Pi_F(\nu)$ in the general case of an anisotropic gap $\Delta(\mathbf{k})$ at finite temperatures. The calculation is analogous to the calculation of the complex conductivity in a BCS superconductor, which was first performed by Mattis and Bardeen,[37,38] and by Abrikosov and Gorkov.[39] Their results were originally meant for an s-wave gap $\Delta(\mathbf{k})=\Delta$, but it is a straightforward exercise to generalize these expressions to the anisotropic case of a d-wave gap $\Delta(\mathbf{k})$.[40] We will do our calculations in the extreme anomalous limit (i.e., $1/q$ much larger than the coherence length), which is the appropriate limit in our case. In this limit one obtains

$$\mathrm{Im}\Pi_F(\mathbf{q},\nu)=-\frac{\gamma_F(\mathbf{q})}{q}\left[\int_{|\Delta|,|\Delta'|-\nu}^{\infty}+\int_{-\infty}^{-|\Delta|,-|\Delta'|-\nu}+\theta(\nu-|\Delta|-|\Delta'|)\int_{|\Delta'|-\nu}^{-|\Delta|}\right]\times[f(E)-f(E+\nu)]\left|\frac{E(E+\nu)+\Delta\Delta'}{[E^2-\Delta^2]^{\frac12}[(E+\nu)^2-\Delta'^2]^{\frac12}}\right|dE \tag{12}$$

$$\mathrm{Re}\Pi_F(\mathbf{q},\nu)=\frac{\gamma_F(\mathbf{q})}{q}\int dE\left[-\frac{E(E+\nu)+\Delta\Delta'}{[\Delta^2-E^2]^{\frac12}[(E+\nu)^2-\Delta'^2]^{\frac12}}f(E+\nu)+\frac{E(E+\nu)+\Delta\Delta'}{[\Delta'^2-E^2]^{\frac12}[(E+\nu)^2-\Delta^2]^{\frac12}}f(-E-\nu)\right]\mathrm{sgn}(E+\nu), \tag{13}$$

where $\Delta=\Delta_0\varphi_{k+q/2}$ and $\Delta'=\Delta_0\varphi_{k-q/2}$ ($\varphi_{\mathbf{k}}=\cos k_x-\cos k_y$). Here we must choose $\mathbf{k}$ such that $\mathbf{k}\pm\mathbf{q}/2$ are on the Fermi surface, so that Δ and Δ' are completely determined by $\mathbf{q}$. The lower limit in the first integral in Eq. (12) is the maximum of $|\Delta|$ and $|\Delta'|-\nu$. Similarly, the integrals in Eq. (13) are restricted to those values of E for which the arguments of the square roots are positive. For $\Delta=\Delta'$ the Eqs. (12) and (13) reduce to

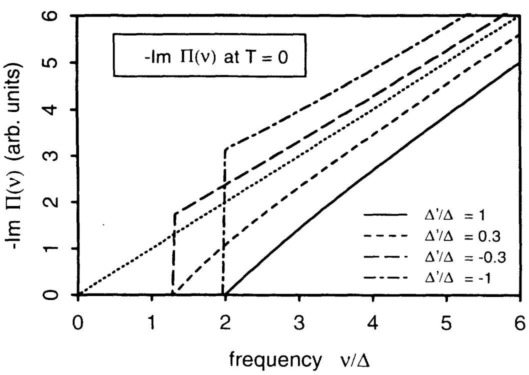

FIG. 5. This figure shows $\mathrm{Im}\Pi_F(\nu)$ at $T=0$ for $\Delta'/\Delta = 1$, 0.3, -0.3, and -1. Notice that there is no absorption for $\nu < |\Delta| + |\Delta'|$. For $\nu > |\Delta| + |\Delta'|$ it depends on the relative sign of Δ and Δ' whether $\mathrm{Im}\Pi_F(\nu)$ is enhanced or suppressed by the gap. The dotted line is the normal-state ($\Delta = 0$) value.

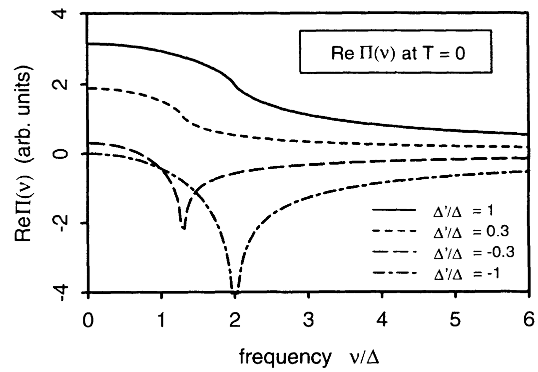

FIG. 6. This figure shows $\mathrm{Re}\Pi_F(\nu)$ at $T=0$ for $\Delta'/\Delta = 1$, 0.3, -0.3, and -1. Similar to the case of $\mathrm{Im}\Pi_F(\nu)$, it depends on the relative sign of Δ and Δ' whether $\mathrm{Re}\Pi_F(\nu)$ is positive or negative.

the expressions given by Mattis and Bardeen.[37] At $T=0$ only the last integral in Eq. (12) survives, and therefore $\mathrm{Im}\Pi_F$ is only nonzero for $\nu > |\Delta| + |\Delta'|$.

In principle we would like to solve for $F_{\mathrm{gauge}}(\Delta_0)$, by combining Eqs. (9), (12), and (13). However, doing this while taking full account of the anisotropic d-wave gap and the non-spherical Fermi surface is a complex numerical problem. Instead we will study two limiting cases for the polarization function $\Pi_F(\nu, \Delta, \Delta')$. We will first consider the zero-temperature limit (i.e., $T \ll \Delta_0$), which is the simplest case to understand from a physical point of view. We will later consider the opposite limit $\Delta_0 \ll T$, to study the possibility of a second-order transition.

A. The polarization function at $T = 0$

The objective of this section is to give an estimate of $F_{\mathrm{gauge}}(\Delta_0)$, using zero-temperature expressions for $\mathrm{Im}\,\Pi_F(\nu)$ and $\mathrm{Re}\,\Pi_F(\nu)$. The dependence of $\mathrm{Im}\Pi_F(\nu)$ and $\mathrm{Re}\,\Pi_F(\nu)$ on the gap Δ_0 at $T=0$ is shown in Figs. 5 and 6, for various values of $\Delta'/\Delta = \Delta(\mathbf{k}-\mathbf{q}/2)/\Delta(\mathbf{k}+\mathbf{q}/2)$. For $q \to 0$ one always has $\Delta'/\Delta \simeq 1$, and $-\mathrm{Im}\Pi_F(\nu)$ is suppressed for all ν, even for $\nu > |\Delta|+|\Delta'|$. However for large q it is possible that $\Delta\Delta' < 0$, in which case $-\mathrm{Im}\Pi_F(\nu)$ is actually *enhanced* for $\nu > |\Delta| + |\Delta'|$. This is an important point to make, because this means that while scattering processes that involve a small momentum transfer are pair breaking, scattering processes with a large momentum transfer can have exactly the opposite effect. The dependence of $\mathrm{Re}\Pi_F(\nu)$ on Δ_0 shows a similar behavior (see Fig. 6), in the sense that $\mathrm{Re}\Pi_F(\nu)$ in Eq. (13) is positive when $\Delta\Delta' > 0$ and negative when $\Delta\Delta' < 0$.

We will initially ignore the anisotropy of the gap, which is a valid assumption as long as the important momenta $\mathbf{q}$ are small compared to the size of the Brillouin zone. The most important feature of $\mathrm{Im}\Pi_F(\nu, \Delta_0)$ is then that $\mathrm{Im}\Pi_F(\nu) = 0$ for $\nu < 2\Delta_0$, so that gauge-field modes with $\nu < 2\Delta_0$ do not contribute to $F_{\mathrm{gauge}}(\Delta_0)$ in Eq. (9) anymore. A crude estimate of $F_{\mathrm{gauge}}(\Delta_0) - F_{\mathrm{gauge}}(0)$ is therefore given by the contribution to $F_{\mathrm{gauge}}(0)$ of the "missing" gauge-field modes with $\nu < 2\Delta_0$. For $T > T_{\mathrm{BE}}$ we can use $\mathrm{Re}\Pi_F(q) \simeq \chi q^2$, so that we can do the $\mathbf{q}$ integration in Eq. (9) by scaling:

$$
\begin{aligned}
F_{\mathrm{gauge}}(\Delta_0) - F_{\mathrm{gauge}}(0) &\simeq \int_0^{2\Delta_0} \frac{d\nu}{2\pi} \coth\left(\frac{\nu}{2T}\right) \int \frac{d^2q}{(2\pi)^2} \arctan\left(\frac{\gamma_F \nu}{\chi q^3}\right) \\
&\sim \int_0^{2\Delta_0} d\nu \coth\left(\frac{\nu}{2T}\right) \left(\frac{\gamma_F \nu}{\chi}\right)^{2/3} \\
&\sim \left(\frac{\gamma_F}{\chi}\right)^{2/3} \Delta_0^{5/3}.
\end{aligned} \tag{14}
$$

Note that $F_{\mathrm{gauge}}(\Delta_0) \sim \Delta_0^{5/3}$ is nonanalytic in Δ_0^2, so that the pairing transition must be a first-order transition in this approximation. More importantly, $F_{\mathrm{gauge}}(\Delta_0) - F_{\mathrm{gauge}}(0)$ is always much larger than $F_{\mathrm{MF}}(\Delta_0)$ for any Δ_0, as long as $\chi \sim 1/(24\pi m_B)$ remains small. This implies that the opening of a gap will never happen for $T > T_{\mathrm{BE}}$.

However, for $T < T_{\mathrm{BE}}$ the susceptibility $\chi \simeq \chi_B$ will increase rapidly, which will lower $F_{\mathrm{gauge}}(\Delta_0)$ in Eq. (14) significantly. In fact, if χ_B becomes much larger than $1/m_B = 2\xi t$, the approximation $\mathrm{Re}\Pi_B(q) \simeq \chi_B q^2$ is only valid for very small q. Instead $\mathrm{Re}\Pi_B(q)$ levels off to $\mathrm{Re}\Pi_B(q) \simeq \rho_S$ for $\chi q^2 \gtrsim \rho_S$, where $\rho_S \simeq x/m_B$ is the superfluid density of bosons. The fact that $\mathrm{Re}\Pi_B(q)$

becomes massive below $T_{\rm BE}$ implies that we cannot assume anymore that the dominant momenta q are small in the integral (9) for $F_{\rm gauge}(\Delta_0)$. That means that for $T < T_{\rm BE}$ the anisotropy of the d-wave gap $\Delta(\mathbf{k})$ will start to play an important role. This decreases the pair-breaking effects of the gauge field, because as pointed out in the beginning of this section, modes with a large momentum $\mathbf{q}$ actually *favor* pairing. The reason for this is that $\Delta(\mathbf{k}+\mathbf{q}/2)$ and $\Delta(\mathbf{k}-\mathbf{q}/2)$ can have opposite signs if $\mathbf{q}$ is large enough.

The consequences of this for the phase diagram are that the gauge field suppresses pairing very strongly at low doping, but for large doping the pair-breaking effects of the gauge field will be less pronounced, because the gauge field is in that case stiffened by a large superfluid density ρ_S.

B. The polarization function for $\Delta_0 \ll T$

In the previous section we used a zero-temperature expression for the gauge-field propagator. The fact that the free energy varied as $F_{\rm gauge}(\Delta_0) \sim \Delta_0^{5/3}$ for $T > T_{\rm BE}$ implied that the superconducting transition had to be a first-order transition in that approximation. It is an interesting question whether the transition can become second-order if we would use a finite temperature expression for Im $\Pi_F(\nu)$ and Re $\Pi_F(\nu)$. To address this question we will assume that $\Delta_0 \ll T$, and expand the polarization function in powers of Δ_0^2 by writing $\Pi(\Delta_0) \simeq \Pi(\Delta_0=0) + \Delta_0^2 \partial\Pi/\partial\Delta_0^2$. We can then use this expansion to find $F_{\rm gauge}(\Delta_0)$ for $\Delta_0 \ll T$. Using Eqs. (12) and (13) one can show that in the limit $\Delta_0 \to 0$, $\partial\Pi_F/\partial\Delta_0^2$ has the following functional form:

$$\lim_{\Delta_0\to 0} \frac{\partial}{\partial\Delta_0^2}\,\Pi_F(\mathbf{q},\nu) = \frac{\gamma_F(\mathbf{q})}{q\nu}\left[h_R\left(\frac{\nu}{T}\right) + i\,h_I\left(\frac{\nu}{T}\right)\right]; \quad (15)$$

$$h_I\left(\frac{\nu}{T}\right) = \frac{1}{2}\mathcal{P}\int_{-\infty}^{\infty} dx \left(\frac{\varphi}{x} + \frac{\varphi}{1+x}\right)^2 \times \left[\frac{1}{2} - f(\nu) - f(x\nu) + f((1+x)\nu)\right]; \quad (16)$$

$$h_R\left(\frac{\nu}{T}\right) = \varphi\varphi' \tanh\left(\frac{\nu}{2T}\right) + \frac{\nu}{8T}(\varphi^2 + \varphi'^2) \times \left[1 - \tanh^2\left(\frac{\nu}{2T}\right)\right]. \quad (17)$$

Notice that we have to use a finite-temperature expression for $\partial\Pi_F/\partial\Delta_0^2$, because $h_I(\nu/T)$ diverges logarithmically for $T \to 0$. For $\varphi\varphi' > 0$, h_I and h_R are always positive, but for $\varphi\varphi' < 0$, h_I and h_R can become negative. This shows explicitly that scattering processes with a large momentum transfer (for which $\varphi\varphi' < 0$) are not necessarily pair breaking. We will later take this into account in our numerical calculations, but we will ignore the anisotropy of the gap in the simple estimates that follow below.

We can now use the expressions for $h_I(\nu/T)$ and $h_R(\nu/T)$ to evaluate $\partial F_{\rm gauge}/\partial\Delta_0^2$, which according to Eq. (9) can be written as

$$\left.\frac{\partial F_{\rm gauge}}{\partial\Delta_0^2}\right|_{\Delta=0} = \int \frac{d^2q}{(2\pi)^2}\int_0^{\infty}\frac{d\nu}{2\pi}\,\frac{\gamma_F}{q\nu}\coth\left(\frac{\nu}{2T}\right) \times \frac{h_I(\nu/T){\rm Re}\Pi - h_R(\nu/T){\rm Im}\Pi}{({\rm Re}\Pi)^2 + ({\rm Im}\Pi)^2}. \quad (18)$$

It is instructive to estimate $\partial F_{\rm gauge}/\partial\Delta_0^2$, assuming that ${\rm Re}\Pi \simeq \chi q^2$, which is a valid assumption above the Bose-condensation temperature $T_{\rm BE}$. The calculation is similar to the calculation of $F_{\rm gauge}(\Delta_0)$ in Eq. (14). The q integration in Eq. (18) can again be done by scaling, and one obtains

$$\frac{\partial F_{\rm gauge}}{\partial\Delta_0^2} \sim \left(\frac{\gamma_F}{\chi}\right)^{2/3}\int_0^{\infty}\frac{d\nu}{\nu^{4/3}}\coth\left(\frac{\nu}{2T}\right)\tilde{h}\left(\frac{\nu}{T}\right), \quad (19)$$

where $\tilde{h}(\nu/T)$ is a linear combination of the functions $h_I(\nu/T)$ and $h_R(\nu/T)$. The frequency integral in Eq. (19) diverges at $\nu \to 0$, because $h_R(\nu/T)$ is linear in ν/T for $\nu \to 0$. This implies that a second-order transition is impossible for $T > T_{\rm BE}$.

We will now analyze Eq. (19) for a small finite gap Δ_0. One can repeat the calculations that led to Eq. (19) for a nonzero gap $\Delta_0 \ll T$, by noticing that a nonzero gap Δ_0 essentially introduces a lower cutoff in the frequency integral in Eq. (19). This cutoff is due to the fact that the expansion in Eq. (15) is only valid for $\nu \gg 2\Delta$, and the expansion clearly breaks down for $\nu < 2\Delta$. Using this cutoff one can now evaluate the frequency integral in Eq. (19), which gives

$$\frac{\partial F_{\rm gauge}}{\partial\Delta_0^2} \sim \left(\frac{\gamma_F}{\chi}\right)^{2/3}\frac{1}{\Delta_0^{1/3}}. \quad (20)$$

This is the same functional form for $F_{\rm gauge}(\Delta_0)$ as obtained in Eq. (14), where we used a zero-temperature expression for $\Pi(\nu,\Delta_0)$. The conclusion is again that the superconducting transition has to be first order for $T > T_{\rm BE}$.

As before, the arguments that led to $\partial F_{\rm gauge}/\partial\Delta_0^2 \sim \Delta_0^{-1/3}$ have to be modified if a superfluid density ρ_S develops for $T < T_{\rm BE}$. If one replaces ${\rm Re}\Pi(q) = \chi q^2$ by ${\rm Re}\Pi(\nu) = \rho_S$ one finds that the expression for $\partial F_{\rm gauge}/\partial\Delta_0^2$ in Eq. (18) converges, even for $\Delta_0 \to 0$. This makes it in principle possible to have a second-order transition for $T < T_{\rm BE}$, if ρ_S is sufficiently large. Considering that the pair-breaking effect of the gauge field diminishes as the superfluid density increases, we expect that the first-order jump of Δ_0 at the transition becomes smaller as ρ_S increases, and the transition might become second order if ρ_S is sufficiently large.

These arguments assumed that there is a true superfluid state for the bosons, i.e., ${\rm Re}\Pi_B(q) = \rho_S$ for $q \to 0$. However, we discussed in Sec. II that one approximate treatment of the interaction of the bosons with the gauge field leads to the dissipative susceptibility χ_B in Eq. (6), which increases according to a Curie law $\chi_B \sim 1/T$,[31] instead of diverging exponentially below $T_{\rm BE}^0$. In that case there is no true superfluid state anymore, which means that even at low temperatures ${\rm Re}\Pi_B(q)$ varies like χq^2 for small momenta q. This implies that $F_{\rm gauge}(\Delta_0)$ will still vary like $\Delta_0^{5/3}$ for a sufficiently small gap Δ_0. Therefore we will always find a first-order transition if we use the Curie-like expression for χ_B in Eq. (6).

IV. NUMERICAL ANALYSIS OF $F_{\mathrm{gauge}}(\Delta_0)$

We did a numerical analysis of $F_{\mathrm{tot}}(\Delta_0)$, using the expression for $F_{\mathrm{gauge}}(\Delta_0)$ in Eq. (9), and assuming a d-wave symmetry for the gap $\Delta(\mathbf{k})$. We used mean-field expressions for the RVB order parameter ξ, the susceptibility χ_F, and the damping parameter $\gamma_F(\mathbf{q})$, and we carefully took the diamondlike shape of the Fermi surface into account. As mentioned after Eq. (10), χ_F is actually negative for the t-J model (for doping $x \lesssim 0.5$), and $\gamma_F(\mathbf{q})$ is highly anisotropic due to the nonspherical shape of the Fermi surface. The value of the order parameter ξ depends on x and T, and decreases rapidly if the doping x becomes very small. We minimized the total free energy $F_{\mathrm{tot}}(\Delta_0)$ with respect to Δ_0, and the onset of superconductivity is determined by the temperature T_c at which $F_{\mathrm{tot}}(\Delta_0)$ has its global minimum at a nonzero value of Δ_0.

The most challenging part of this numerical calculation is to find expressions for $\mathrm{Im}\Pi_F(\nu,\Delta,\Delta')$ and $\mathrm{Re}\Pi_F(\nu,\Delta,\Delta')$ that take into account that the d-wave gap $\Delta = \Delta_0\varphi_{k\pm q/2}$ is anisotropic around the Fermi surface. As pointed out after Eq. (13), this anisotropy of the gap has the feature that processes with a large momentum transfer $\mathbf{q}$ tend to *favor* pair breaking, because $\Delta\Delta'$ can become negative for large $\mathbf{q}$. The anisotropy of the gap is very important for the numerical values that one obtains for the suppression of $T_c(x)$, and can definitely not be ignored. We performed the numerical calculation in the two limiting cases that we discussed in Secs. III A and III B. In the first case we used zero-temperature expressions for $\mathrm{Im}\Pi_F(\nu,\Delta,\Delta')$ and $\mathrm{Re}\Pi_F(\nu,\Delta,\Delta')$, which is a good approximation when $\Delta_0 \gg T$. In the second case we expanded $\mathrm{Im}\Pi_F$ and $\mathrm{Re}\Pi_F$ in Δ_0^2, which is a good approximation when $\Delta_0 \ll T$. For $\nu > |\Delta| + |\Delta'|$ we used the functions $h_I(\nu/T)$ and $h_R(\nu/T)$ as defined in Eqs. (16) and (17). This expansion fails for $\nu < |\Delta|+|\Delta'|$, in which case we used a high-temperature expression for the propagator instead.

To take the effect of Bose condensation into account we used a parametrization of $\mathrm{Re}\Pi_B(q)$, which interpolates between the limits $\mathrm{Re}\Pi_B(q) \simeq \chi_B q^2$ (for $q \to 0$) and $\mathrm{Re}\Pi_B(q) \simeq x/m_B$ (for large q). For χ_B we used the three possibilities discussed in Sec. II: (i) the mean-field expression χ_B^0 in Eq. (3); (ii) the dissipative expression χ_B^{diss} in Eq. (5); (iii) the self-consistent dissipative χ_B^{SC} shown in Fig. 2.

The result of our numerical analysis is shown in the phase diagrams in Fig. 3 (using the mean-field χ_B^0) and Fig. 4 (using the dissipative χ_B^{diss}). The phase diagrams show that $T_c(x)$ is indeed strongly suppressed compared to the mean-field transition temperature $T_c^0(x)$ (dotted line), especially at low doping. The solid line represents $T_c(x)$ if one uses $T = 0$ expressions for the propagator, as discussed in Sec. III A, and the dashed line is the result for $T_c(x)$ if one assumes $\Delta_0 \ll T$, as discussed in Sec. III B. These two approximations for the propagator give results that are qualitatively similar. One has to compare the first-order jump in Δ_0 at $T_c(x)$ with $T_c(x)$ itself to decide which approximation is more appropriate.

In Fig. 3 the first-order jump in Δ_0 is small for large doping so that the dashed line is appropriate, whereas for small doping the jump in Δ_0 becomes so large that the solid line is more appropriate. The line indicated with diamonds, which interpolates between the dashed line and the solid line, is our best guess of what the correct phase boundary is. We reiterate that the need to use two different approximations for the polarization function (represented by the dashed line and the solid line) is purely technical, due to the limitation of our computational abilities. Note that the superconducting transition temperature goes to zero at a finite doping near $x = 0.05$. This is not too surprising, because the gauge field becomes unstable towards a flux phase for $x \simeq 0.04$ and hence the strength of the gauge-field fluctuations diverges in the vicinity of this point. At higher doping the transition temperature $T_c(x)$ approaches the mean-field transition line $T_c^0(x)$, because the gauge field becomes very stiff as the superfluid density $\rho_S \simeq x/m_B$ increases.

When we repeat the calculation using χ_B^{SC} the result is indistinguishable from Fig. 3. The reason is clear from Fig. 2 by observing that for $x \simeq 0.07$ the self-consistent χ_B^{SC} is exponentially large for $T \lesssim 0.2J$. Because the phase boundary is at a much lower temperature, the bosons have essentially Bose condensed at the transition, whether one uses χ_B^0 or χ_B^{SC}. This model therefore predicts a Fermi liquid in a temperature range just above T_c, even in the underdoped case. This aspect of the model may be in disagreement with experiments, and we believe that this is related to the fact that in the underdoped case $x < 0.1$ the self-consistent dissipative model underestimates the effectiveness of the gauge field to suppress Bose condensation.

For completeness we show in Fig. 4 the phase diagram using the dissipative model χ_B^{diss}. Note that the suppression of $T_c(x)$ is much larger than in Fig. 3. The first-order jump in Δ_0 is large at the transition, so that the solid line, which uses the approximation $\Delta_0 \gg T$, will be close to the correct answer for $T_c(x)$. According to Fig. 2, χ_B^{diss} is still relatively small at the transition temperature for $x = 0.07$, so that the dissipative model predicts a direct transition from a strange metal phase into a superconducting phase at low doping. The fact that in the dissipative model the gauge field is still massless at T_c is the reason why $T_c(x)$ is suppressed more strongly in Fig. 4 than in Fig. 3. As explained in Sec. II, we believe that this dissipative model grossly overestimates the effectiveness of the gauge field in suppressing T_c. We therefore think that the phase diagram in Fig. 3 is closer to the truth than the phase diagram in Fig. 4. In the underdoped case $x < 0.1$ the true answer will lie somewhere in between Fig. 3 and Fig. 4, because at low doping the self-consistent model underestimates the effectiveness of the gauge field in suppressing Bose condensation.

As pointed out earlier in Sec. II, the first-order jump in Δ_0 will be rounded off by phase fluctuations of the pairing field Δ_{ij}. Taking this into consideration the interpretation of our numerical results should be that the

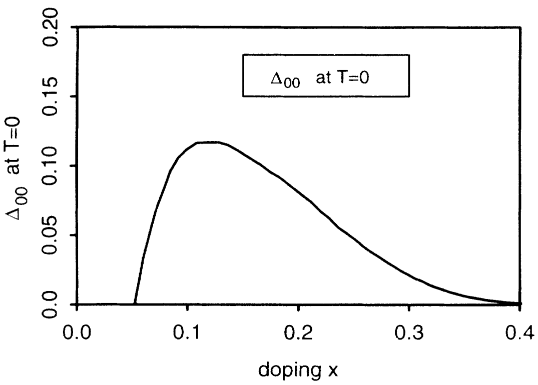

FIG. 7. This figure shows $\Delta_{00}(x)$, the gap at $T = 0$. By comparing this to $T_c(x)$ in Figs. 3 and 4 we find that the ratio $2\Delta_{00}/T_c$ is approximately 3 if one uses the mean-field χ_B^0 (or the self-consistent χ_B^{SC}), and approximately 8 if one uses the dissipative χ_B^{diss}. This should be compared to the mean-field d-wave value of $2\Delta_{00}/T_c^0 \simeq 2.6$, which does not include any gauge-field fluctuations. We believe that in the underdoped case the true value of $2\Delta_{00}/T_c$ is significantly larger than 3, because at low doping the self-consistent χ_B^{SC} underestimates the effectiveness of the gauge-field to suppress Bose condensation.

gap Δ_0 increases very rapidly below T_c, much faster than according to BCS theory. Our numerical results indicate that this rapid increase in Δ_0 will be more pronounced in the underdoped case $x \lesssim 0.1$.

In Fig. 7 we show a plot of $\Delta_{00}(x)$, where $\Delta_{00}(x)$ is the gap at $T = 0$. Note that $\Delta_{00}(x)$ is independent of the particular model for χ_B, because the bosons are condensed at $T = 0$ for all three models. As expected the function $\Delta_{00}(x)$ has essentially the same shape as $T_c(x)$. We are mostly interested in the ratio $2\Delta_{00}(x)/T_c(x)$, which is a constant according to BCS theory. For a d-wave gap without a gauge field we find this ratio to be approximately equal to 2.6. If we include the gauge field this ratio is enhanced, depending on the model that one uses to determine $T_c(x)$. If we use the dissipative χ_B^{diss} the ratio is strongly enhanced to $2\Delta_{00}/T_c \simeq 8$, but if we use the mean-field χ_B^0 or the self-consistent χ_B^{SC} this ratio is only slightly enhanced to $2\Delta_{00}/T_c \simeq 3$. Because we believe that at low doping χ_B^{SC} underestimates the importance of gauge-field fluctuations, we expect that the correct value of $2\Delta_{00}(x)/T_c(x)$ will be significantly larger than 3 in the underdoped case $x < 0.1$.

V. CONCLUSIONS

We analyzed the pair-breaking effects of the gauge field, by studying the contribution from the gauge field to the total free energy. This contribution $F_{\mathbf{gauge}}(\Delta_0)$ depends on Δ_0, because a nonzero gap Δ_0 modifies the gauge-field propagator, and therefore changes the free energy. We showed that $F_{\mathbf{gauge}}(\Delta_0) \sim \Delta_0^{5/3}$, which implies that the superconducting transition must be first order, if one ignores fluctuations in the pairing field itself.

Our numerical calculations, which took into account that the gap $\Delta(\mathbf{k})$ has a d-wave symmetry around the Fermi surface, showed that there is indeed a strong suppression of superconductivity by the gauge field. The value of the boson susceptibility χ_B, which indicates whether the holons are condensed or not, played an important role in the suppression of $T_c(x)$. We used several models for χ_B, and in all cases we found that superconductivity only survives in an intermediate range of doping $0.05 \lesssim x \lesssim 0.35$. The maximum critical temperature occurs near $x = 0.15$. These numerical results are in qualitative agreement with the well-known phase diagram of the high-T_c copper oxides.

One of our predictions is that the experimentally observed "spin-gap phase" cannot be due to pairing of fermions within the plane, because those Cooper pairs are broken by the fluctuating gauge field. We also predict that the nature of the superconducting transition is significantly altered by the gauge field, especially in the underdoped case. The signature for this is that the magnitude of the gap Δ_0 increases very rapidly below T_c. Moreover, we expect that in the underdoped case the ratio $2\Delta_{00}/T_c$, where Δ_{00} is the gap at $T = 0$, will be significantly larger than what one would obtain from BCS theory.

ACKNOWLEDGMENT

This work was supported by the NSF through the Material Research Laboratory under Grant No. DMR-90-22933.

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