Critical exponents for the sinh-cosh interaction model in the zero sector

Rudolf A. Römer and Bill Sutherland
Physics Department, University of Utah, Salt Lake City, Utah 84112
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In this paper, we continue our investigation of a one-dimensional, two-component, quantum many-body system in which like particles interact with a pair potential $s(s+1)/\sinh^2(r)$, while unlike particles interact with a pair potential $-s(s+1)/\cosh^2(r)$. For an equal number of particles of the two components, the ground state for $s > 0$ corresponds to an antiferromagnet/insulator. Excitations consist of a gapless pair-hole–pair continuum, a two-particle continuum with gap and excitons with gap. For $-1 < s < 0$, the system has two gapless excitations — a particle-hole continuum and a two-spin-wave continuum. Using finite-size scaling methods of conformal field theory, we calculate the asymptotic expressions and critical exponents for correlation functions of these gapless excitations at zero temperature. The conformal structure is closely related to the Hubbard model with repulsive on-site interaction.

I. INTRODUCTION

We recently presented the exact solution to a one-dimensional, two-component, quantum many-body system of considerable complexity in the form of an asymptotic Bethe-ansatz calculation.[1] The two kinds of particles are distinguished by a quantum number $\sigma = \pm 1$, which may be thought of as either spin or charge. The system is defined by the Hamiltonian

$$H = -\sum_{1\le j\le N} \frac{1}{2}\frac{\partial^2}{\partial x_j^2} + \sum_{1\le j<k\le N} v_{jk}(x_j - x_k), \qquad (1)$$

where the pair potential is

$$v_{jk}(x) = s(s+1)\left[\frac{1+\sigma_j\sigma_k}{2\sinh^2(x)} - \frac{1-\sigma_j\sigma_k}{2\cosh^2(x)}\right]. \qquad (2)$$

We assume $s \geq -1$. We call this the SC model, for the sinh-cosh interaction. Thus for $s > 0$, like particles repel, while unlike particles attract. When like particles are near, the repulsive potential increases as $1/r^2$, while for large separations, both potentials decay exponentially with a decay length which we take as our length scale, and hence unity. The potentials might usefully be thought of as a screened $1/r^2$ potential. This system was first introduced by Calogero *et al.*,[2] who showed it to be integrable. Sutherland[3] soon afterward showed that the system could be exactly solved, and gave the solution for a single-component system. In the present paper, we continue our study of the SC model with an investigation of the correlation functions and their critical exponents, at zero temperature, by methods of conformal field theory.

The concept of conformal invariance in one-dimensional (1D) quantum systems at criticality constrains the possible asymptotic behavior of correlation functions and allows a classification into universality classes, distinguished by the value of the central charge c of the underlying Virasoro algebra.[4] For models with short-range interactions and a gapless excitation spectrum with a single Fermi velocity, we can determine both c and the critical exponents of correlators directly from finite-size corrections to the ground state energy and the low-lying exited states. In most cases, such models have been found to belong to the universality class of the 1D Luttinger model,[5] i.e., $c = 1$, and the critical exponents to vary as functions of the coupling constant of the corresponding conformal theory.

Recently, various authors have extended these concepts to include multicomponent systems with different excitation velocities, such as the Hubbard model.[6,7] In general, one finds a $c = 1$ Virasoro algebra for each critical degree of freedom, i.e., each gapless excitation with a unique velocity. It is then possible to construct the full theory as a semidirect product of these independent algebras. Again, critical exponents may be calculated from finite-size corrections but now they are functions of a matrix of coupling constants.

In another recent development, the ideas of conformal field theory have been applied to models with long-range interactions such as the $1/r^2$ system.[8,9] It turns out that one can no longer simply read off the value of the central charge from the finite-size corrections to the ground state energy. However, one may still calculate the correct critical exponents of the asymptotics of the correlations functions from the finite-size scaling behavior of the low-lying excitations.[10]

Our paper is organized as follows: In Sec. II we briefly review the asymptotic Bethe-ansatz solution of the SC model in the zero sector as obtained in Ref. 1. Section III outlines the Luttinger liquid approach for long-ranged models. We give arguments why the standard evaluation of conformal field theory for c may fail for long-ranged models. For $s > 0$, there is only one gapless excitation corresponding to a single $c = 1$ conformal theory. For $-1 < s < 0$, however, there are two gapless excitations with different velocities, so that we briefly review the main formulas for a semidirect product of two $c = 1$ Virasoro algebras. In Secs. IV and V, we give expressions

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for the correlation functions and calculate their critical exponents from the Bethe-ansatz equations for the $-1 < s < 0$ and the $s > 0$ cases, respectively. For simplicity, the two types of particles are assumed to be either both bosons or both fermions, although a mixed Bose-Fermi system can be studied along similar lines. We close our paper with Sec. VI, where we briefly show how both the $-1 < s < 0$ and the $s > 0$ cases fit together as $s \to 0^{\mp}$.

II. BETHE-ANSATZ SOLUTION IN THE ZERO SECTOR

Let us recall the results of Ref. 1: We have $N_\downarrow$ particles with $\sigma = -1$ and $N_\uparrow$ with $\sigma = +1$, for a total of $N = N_\downarrow + N_\uparrow$ and $N_\uparrow \geq N_\downarrow$. The zero sector corresponds to an equal number of up and down spins, i.e., $N_\downarrow = N_\uparrow$. For $s > 0$, pairs of up-down spins bind into a variety of bound states, or pairs, which we will label by $m = 1, \ldots, M(s)$, where $M(s)$ is the smallest integer larger than s. Let there be N_m of each type. Unbound particles correspond to spinons and/or ions in the spin-charge picture and there are $N_0 = N - 2\sum_{1\leq m\leq M} N_m$ of these. Let us call particles with spin down spin waves; there are $N_{-1} = N_\downarrow - \sum_{1\leq m\leq M} N_m$ of these.

Imposing periodic boundary conditions on the wave function and taking any particle, pair, or spin wave around a ring of large circumference yields the following set of Bethe-ansatz equations:

$$L\eta_m \mathbf{k}_m = 2\pi \mathbf{I}_m(\mathbf{k}_m) + \sum_{-1\leq m'\leq M} \sum_{\mathbf{k}'_{m'}} \theta_{m,m'}(\mathbf{k}_m - \mathbf{k}_{m'}),$$

$$m = -1, 0, 1, \ldots, M. \tag{3}$$

Here the $\mathbf{I}_m(\mathbf{k}_m)$ denote the set of quantum numbers for each type of particle. Depending on the parities of N_m and the particle statistics, the quantum numbers will be restricted to integer or half-odd integer values. Note that for the spin waves, $\mathbf{I}_{-1}$ ranges only over $1, \ldots, N_0$. $\theta_{m,m'}(\mathbf{k}_m - \mathbf{k}_{m'})$ is the phase shift for the scattering of particles of type m with type m' and has been calculated previously.[1] Note that $\theta_{mm'}(k) = -\theta_{mm'}(-k) = \theta_{m'm}(k)$. Furthermore, we define

$$\eta_m = \begin{cases} 0, & m = -1, \\ 1, & m = 0, \\ 2, & m = 1, 2, \ldots, M(s)\ . \end{cases} \tag{4}$$

We can write the momentum and energy for a solution of (3) as

$$P = \sum_{-1\leq m\leq M} \eta_m \sum_{\mathbf{k}_m} \mathbf{k}_m, \tag{5}$$

$$E = \frac{1}{2} \sum_{-1\leq m\leq M} \eta_m \sum_{\mathbf{k}_m} \mathbf{k}_m^2 - \sum_{1\leq m\leq M} N_m \kappa_m^2. \tag{6}$$

Here $\kappa_m = s + 1 - m$ denotes the poles in the transmission and reflection coefficients that give rise to bound states of type m.

For $0 > s > -1$, there are no bound states and we will call this the *unbound case* in the sequel. We therefore have only two coupled equations for N_0 particles with pseudomomenta $\mathbf{k}_0 = (k_1, \ldots, k_{N_0})$ and N_{-1} spin waves with rapidities $\mathbf{k}_{-1} = (\lambda_1, \ldots, \lambda_{N_{-1}})$:

$$\begin{aligned} Lk_j &= 2\pi I_j(k_j) + \sum_{\alpha=1}^{N_{-1}} \theta_{0,-1}(k_j - \lambda_\alpha) + \sum_{l=1}^{N_0} \theta_{0,0}(k_j - k_l), \\ 0 &= 2\pi J_\alpha(\lambda_\alpha) + \sum_{\beta=1}^{N_{-1}} \theta_{-1,-1}(\lambda_\alpha - \lambda_\beta) + \sum_{j=1}^{N_0} \theta_{-1,0}(\lambda_\alpha - k_j). \end{aligned} \tag{7}$$

The particle quantum numbers I_j and the spin-wave quantum numbers J_α are restricted by the parities of N_0, N_{-1} and the statistics of the particles to the following combination of integers and half-odd integers: If both spin-up and spin-down particles are bosons,

$$\begin{aligned} I_j &= (N_0 - 1)/2 \pmod 1), \\ J_\alpha &= (N_{-1} - 1)/2 \pmod 1), \end{aligned} \tag{8}$$

whereas for fermions,

$$\begin{aligned} I_j &= N_{-1}/2 \pmod 1), \\ J_\alpha &= (N_0 + N_{-1} - 1)/2 \pmod 1). \end{aligned} \tag{9}$$

In the thermodynamic limit, i.e., $L \to \infty$ with fixed $d_0 \equiv N_0/L$, $d_{-1} \equiv N_{-1}/L$, the ground state is a filled Fermi sea characterized by the distribution function $\rho(k)$ of particles and $\sigma(\lambda)$ of down spins:

$$\begin{aligned} \rho(k) &= \frac{1}{2\pi} + \frac{1}{2\pi}\int_{-C}^{C} \theta'_{0,-1}(k-\mu)\sigma(\mu)d\mu + \frac{1}{2\pi}\int_{-B}^{B} \theta'_{0,0}(k-h)\rho(h)dh, \\ \sigma(\lambda) &= 0 + \frac{1}{2\pi}\int_{-C}^{C} \theta'_{-1,-1}(\lambda-\mu)\sigma(\mu)d\mu + \frac{1}{2\pi}\int_{-B}^{B} \theta'_{-1,0}(\lambda-h)\rho(h)dh. \end{aligned} \tag{10}$$

Here the prime denotes the first derivative. The values of B and C are fixed by the following equations:

$$\int_{-B}^{B} \rho(k)dk = d_0, \tag{11}$$

$$\int_{-C}^{C} \sigma(\lambda)d\lambda = d_{-1} = d_0/2 - \mathcal{M}, \tag{12}$$

where $\mathcal{M} = (N_\uparrow - N_\downarrow)/2L$ is the magnetization per unit length. Let us now restrict our discussion to the zero sector, when $N_0 = N$, $N_{-1} = N/2$, $\mathcal{M} = 0$, and the limit C of the spin-wave distribution is ∞. Then we can solve for the spin-wave distribution by Fourier transform in terms of the particle distribution, which we then substitute into the particle equation, giving a single integral equation for the distribution of particles $\rho(k)$:

$$\frac{1}{2\pi} = \rho(k) + \frac{1}{2\pi}\int_{-B}^{B} \theta'(k-h)\rho(h)dh. \qquad (13)$$

Here the kernel $\theta'(k)$ is given as

$$\theta'(k) = \theta'_{00}(k) - 1/2\int_{-\infty}^{\infty} dte^{ikt}\frac{\sinh t(1+s)}{\sinh t \cosh ts}. \qquad (14)$$

The excited states in the zero sector are given by the following: (i) Remove a particle from the ground state distribution, and place it outside the limits; we call this creating a hole and a particle, and it gives a two-parameter continuum. (ii) Remove a spin wave from the ground state distribution, and place it on the line with imaginary part equal to i; we call this creating two spin waves, one with spin up and the other with spin down. It gives a two-parameter continuum of the type familiar from the Heisenberg-Ising model.[11] Each of these two types of two-particle continua has a single Fermi velocity. Let us denote by v_0 the Fermi velocity of the first excitation and by v_{-1} the Fermi velocity of the second. As has been pointed out in Ref. 1, the two velocities are in general not identical. The same is true of the Hubbard model with repulsive on-site interaction, and we will later make extensive use of the conformal results obtained for this model.[6,7]

For $s > 0$, which we call the *bound case*, the ground state in the zero sector consists of a spin fluid of type $m = 1$, and thus spin 0. In the ground state, the k's for the pairs distribute themselves densely with a density $\tau(k)$, between limits $\pm D$, normalized so that

$$d_1 \equiv N_1/L = \int_{-D}^{D} \tau(k)dk = N/2L. \qquad (15)$$

The energy and momentum are given by

$$P/L = 2\int_{-D}^{D} \tau(k)kdk = 0, \qquad (16)$$

$$E/L = \int_{-D}^{D} \tau(k)k^2dk - s^2N_1/L. \qquad (17)$$

The integral equation which determines $\tau(k)$ is

$$1/\pi = \tau(k) + \frac{1}{2\pi}\int_{-D}^{D} \theta'_{11}(k-h)\tau(h)dh. \qquad (18)$$

The kernel of the equation, $\theta'_{11}(k)$, is the derivative of the phase shift for pair-pair scattering.

The low-energy excited states are given by the following: (i) Remove a pair from the ground state distribution, and place it outside the limits; we call this creating a pair-hole and a pair, and it gives a gapless two-parameter continuum. (ii) Break a pair, to give two particles, one spin up and the other spin down; this also gives a two-parameter continuum. However, there is a finite energy gap for breaking a pair. These are the spinons or ions. (iii) Excite a pair into a higher-energy bound state, if allowed; these we call excitons, and they have simple single-parameter dispersion relations.

Let us denote the unique velocity of the excitations of type (i) by v_1. The Bethe-ansatz equations that describe these excitations may be written as

$$2Lk_j = 2\pi H_j(k_j) + \sum_{l=1}^{N/2} \theta_{1,1}(k_j - k_l). \qquad (19)$$

Note here that k_j is the pseudomomentum of a pair, and is not the pseudomomentum of an individual particle, which would be complex and of the form $k_j/2 \pm is$. The pair quantum numbers H_j are restricted by the parity of N_1, and Bose and Fermi statistics are given as

$$H_j = (N_1+1)/2 \text{ (mod 1)}, \qquad (20)$$

since pair-pair scattering is symmetric for pairs of bosons and pairs of fermions. The pairs will be singlets.

III. CONFORMAL APPROACH FOR CORRELATION FUNCTIONS

A. Finite-size scaling in conformal theories of Luttinger liquids

The behavior of the correlation functions for a given one-dimensional model at large distances and low temperatures is determined by the gapless excitations.[12] These gapless excitations are due to hydrodynamic fluctuations and it has been argued[5] that the low-energy physics of such a system may be described by the exactly solvable Luttinger model,[13] the 1D quantum version of the classical 2D Gaussian model. The Luttinger model is a critical system with continuously varying exponents and corresponds to the universality class of $c = 1$ conformal field theories.[14] Application of conformal theory allows the calculation of these critical exponents purely from finite-size scaling arguments.[15]

The value of the central charge c may be read off from the following finite-size scaling formula:

$$E_0 \sim \epsilon_0 L - \frac{\pi v}{6L}c, \qquad (21)$$

thus enabling an independent check of the above arguments. Here, E_0 is the ground state energy of the finite system, ϵ_0 is the ground state energy density in the thermodynamic limit, and v is the Fermi velocity in the system. In short-ranged 1D quantum models, including Bethe-ansatz solvable models, the above universal picture is confirmed.[16] However, for long-ranged models, straightforward application of this equation may lead to unphysical results.[9] (We include the SC model in this class, although its pair potential decays exponentially, since it can only be solved by means of the *asymptotic* Bethe ansatz.) For instance, in the $1/r^2$ models c is predicted to be equal to the interaction strength, although independent calculations show that the critical exponents are those of the $c = 1$ universality class.[10] However, if one instead estimates c from the low-temperature expansion of the free energy,[17] one does get the correct answer $c = 1$.

Let us give an argument that may explain the failure of (21) in long-ranged models. The crucial point is that, due to the long-range character of the interactions, finite systems will always "feel" the particular boundary conditions chosen, so that (21) includes an additional correction term E_{bc}, representing the boundary energy, and so

$$E_0 \sim \epsilon_0 L - \frac{\pi v}{6L} c + \frac{E_{bc}}{L}. \tag{22}$$

The low-temperature expansion, however, uses boundary conditions instead for the time axis of the model and we thus have no such corrections. We may therefore write the free energy of a long-ranged $c = 1$ Luttinger liquid as

$$F(T) \simeq F(T=0) - \frac{\pi T^2}{6v}. \tag{23}$$

Let us recall the main formulas for calculating the correlation functions and their critical exponents.[4] Every primary field $\phi_\pm$ in a conformal field theory on an infinite strip of width L in the space direction gives rise to a tower of exited states. Let $x = \Delta^+ + \Delta^-$ denote the scaling dimension and $\sigma = \Delta^+ - \Delta^-$ the spin of $\phi_\pm$. Then the energies and momenta of these exited states scale as

$$E(\Delta^\pm, N^\pm) - E_0 \sim \frac{2\pi v}{L}(x + N^+ + N^-), \tag{24}$$

$$P(\Delta^\pm, N^\pm) - P_0 \sim \frac{2\pi}{L}(\sigma + N^+ - N^-) + 2Dk_f. \tag{25}$$

Here N^+ and N^- are positive integers, v is the common Fermi velocity of the excitations, and $2D$ is the momentum of the state in units of the Fermi momentum k_f. Note that the quantities on the left hand side of these equations are evaluated with respect to the same boundary condition and therefore the above mentioned correction terms cancel. We may write the correlation functions of the primary fields at zero temperature (expressions for low but finite temperature may also be given) as

$$\langle \phi_{\Delta^\pm}(x,t)\phi_{\Delta^\pm}(0,0)\rangle = \frac{\exp(-2iDk_f)}{(x - ivt)^{2\Delta^+}(x + ivt)^{2\Delta^-}}. \tag{26}$$

However, the excitation spectrum of the SC model is quite different for the bound ($s > 0$) and the unbound ($-1 < s < 0$) cases as we have argued in the previous section. Most importantly, the unbound case does not have a common velocity for all excitations anymore and so the formulas given above for a Lorentz-invariant conformal field theory can no longer hold.

B. Conformal weights and the dressed charge in the bound case

For the bound case in the zero sector, only the pair–pair-hole excitation branch is gapless. Thus there is only one excitation velocity, and from the above arguments, we expect the dimensions of the primary operators to obey the formulas for a single $c = 1$ Gaussian model, i.e.,

$$\Delta^\pm(\Delta N_1, D_1) = \frac{1}{2}\left(D_1\xi_1 \pm \frac{\Delta N_1}{2\xi_1}\right)^2. \tag{27}$$

The coupling constant ξ_1 of this Gaussian model depends on the system parameters. It is sometimes called the dressed charge and may be calculated from the Bethe-ansatz equations by means of an integral equation[6]

$$\xi_1(k) = 2 + \frac{1}{2\pi}\int_{-D}^{D} \xi_1(h)\theta'_{1,1}(h-k)dh, \tag{28}$$

where the constant is 2 because this excitation is a pair. However, we can also calculate $\xi_1 \equiv \xi_1(D)$ by purely thermodynamical arguments as follows: Let us change a given ground state configuration by adding pairs while keeping the Fermi sea at zero momentum, so that the excitation can be described by the pair $(\Delta N_1, D_1 = 0)$. Then a second order expansion gives

$$\Delta E = -\mu_1(\Delta N_1) + \frac{1}{2}\frac{1}{L\kappa_1 d_1^2}(\Delta N_1)^2, \tag{29}$$

where $\mu_1 = -\frac{\partial E}{\partial N_1}$ is the chemical potential for adding pairs and κ_1 is the pair compressibility. Comparison with (24) and (27) yields

$$\xi_1^2 = \pi v_1 \kappa_1 d_1^2 = \pi d_1 / v_1. \tag{30}$$

In the last equation, we have used the well known relation $v_1^2 = 1/(\kappa_1 d_1)$. Therefore, by knowing the Fermi velocity of the pair–pair-hole excitations, we can calculate the scaling dimensions.

C. Finite-size scaling and the dressed charge matrix in the unbound case

For the unbound case, two excitation branches are gapless, giving rise to a particle-hole continuum and to a spin-wave continuum, with Fermi velocities v_0 and v_{-1}, respectively. Thus, the finite-size corrections of Eqs. (24) and (25) now become

$$E(\Delta\mathbf{N}, \mathbf{D}) - E_0 \sim \frac{2\pi}{L}\left[\frac{1}{4}\Delta\mathbf{N}^T(\Xi^{-1})^T V(\Xi^{-1})\Delta\mathbf{N} + \mathbf{D}^T\Xi V \Xi^T\mathbf{D} + v_0(N_0^+ + N_0^-) + v_{-1}(N_{-1}^+ + N_{-1}^-)\right], \tag{31}$$

$$P(\Delta\mathbf{N}, \mathbf{D}) - P_0 \sim \frac{2\pi}{L}\left[\Delta\mathbf{N}^T\,\mathbf{D} + N_0^+ - N_0^- + N_{-1}^+ - N_{-1}^-\right] + 2D_0 k_{f,\uparrow} + 2(D_0 + D_{-1})k_{f,\downarrow}\,. \tag{32}$$

Here, the matrix $V \equiv \text{diag}(v_0, v_{-1})$ and the excited state is characterized by the pairs $\mathbf{\Delta N} = (\Delta N_0, \Delta N_{-1})$ and $\mathbf{D} = (D_0, D_{-1})$. As before, $N_0^{\pm}$ and $N_{-1}^{\pm}$ are positive integers that label the descendant fields. The 2×2 matrix Ξ is the generalization of the dressed charge ξ and may be calculated by means of coupled integral equations. Thus if we denote the components of Ξ by

$$\Xi = \begin{pmatrix} \xi_{0,0}(B) & \xi_{0,-1}(C) \\ \xi_{-1,0}(B) & \xi_{-1,-1}(C) \end{pmatrix}, \tag{33}$$

then

$$\begin{aligned}
\xi_{0,0}(k) &= 1 + \frac{1}{2\pi}\int_{-B}^{B}\xi_{0,0}(h)\theta'_{0,0}(h-k)dh \\
&\quad + \frac{1}{2\pi}\int_{-C}^{C}\xi_{0,-1}(\mu)\theta'_{-1,0}(\mu-k)d\mu, \\
\xi_{0,-1}(\lambda) &= 0 + \frac{1}{2\pi}\int_{-B}^{B}\xi_{0,0}(h)\theta'_{0,-1}(h-\lambda)dh \\
&\quad + \frac{1}{2\pi}\int_{-C}^{C}\xi_{0,-1}(\mu)\theta'_{-1,-1}(\mu-\lambda)d\mu, \\
\xi_{-1,0}(k) &= 0 + \frac{1}{2\pi}\int_{-B}^{B}\xi_{-1,0}(h)\theta'_{0,0}(h-k)dh \\
&\quad + \frac{1}{2\pi}\int_{-C}^{C}\xi_{-1,-1}(\mu)\theta'_{-1,0}(\mu-k)d\mu, \\
\xi_{-1,-1}(\lambda) &= 1 + \frac{1}{2\pi}\int_{-B}^{B}\xi_{-1,0}(h)\theta'_{0,-1}(h-\lambda)dh \\
&\quad + \frac{1}{2\pi}\int_{-C}^{C}\xi_{-1,-1}(\mu)\theta'_{-1,-1}(\mu-\lambda)d\mu.
\end{aligned} \tag{34}$$

Thus, the situation for $-1 < s < 0$ is analogous to the situation in the repulsive Hubbard model away from half-filling[7,6] and we may interpret Eqs. (31) and (32) in terms of a semidirect product of two independent Virasoro algebras, both with $c = 1$. The scaling behavior of the energy and momentum in terms of the conformal weights $\Delta_0^{\pm}$ and $\Delta_{-1}^{\pm}$ and the formulas for these weights as functions of the components of the dressed charge matrix Ξ have been given in Ref. 7, and we will not repeat them here. The generalization of the correlation functions of the primary fields has also been given in Ref. 7. However, as before, thermodynamic arguments may be used to calculate the values of the dressed charge matrix.

For the zero sector, i.e., $\mathcal{M} = 0$, the relevant equations simplify considerably. In this case, $k_{f,\downarrow} = k_{f,\uparrow} \equiv k_f = \pi d_0/2$, and the dressed charge matrix Ξ may again be expressed in terms of a single parameter $\xi_0 \equiv \xi_0(B)$, i.e.,

$$\Xi = \begin{pmatrix} \xi_0 & 0 \\ \frac{1}{2}\xi_0 & \frac{1}{\sqrt{2(1+s)}} \end{pmatrix}. \tag{35}$$

Thus the conformal weights $\Delta_0^{\pm}$ and $\Delta_{-1}^{\pm}$ are given as

$$\begin{aligned}
\Delta_0^{\pm} &= \frac{1}{2}\xi_0^2\left(D_0 + \frac{1}{2}D_{-1}\right)^2 + \frac{1}{8\xi_0^2}(\Delta N_0)^2 \\
&\quad \pm \frac{1}{4}\Delta N_0(2D_0 + D_{-1}) + N_0^{\pm},
\end{aligned} \tag{36}$$

$$\begin{aligned}
\Delta_{-1}^{\pm} &= \frac{1}{4(1+s)}(D_{-1})^2 + \frac{(1+s)}{4}\left(\Delta N_{-1} - \frac{1}{2}\Delta N_0\right)^2 \\
&\quad \pm \frac{1}{4}(2\Delta N_{-1} - \Delta N_0)D_{-1} + N_{-1}^{\pm}.
\end{aligned} \tag{37}$$

Note that the second equation is independent of ξ_0. However, there is an explicit dependence on the interaction strength s and only for $s = 0$ do we recover the result of the Hubbard model.

This s dependence can be understood by realizing that for the zero sector and $-1 < s < 0$ the Bethe-Ansatz equations of the rapidities $\mathbf{k}_{-1} = (\lambda_1, \ldots, \lambda_{N_{-1}})$ are essentially the Bethe-Ansatz equations of the Heisenberg-Ising model. The effect of the Bethe-Ansatz equations for the pseudomomenta is simply a renormalization. Following Ref. 11 we parametrize the anisotropy in the Heisenberg-Ising model by $\Delta = -\cos(\mu)$. Then the correspondence is established by setting $\mu = -\pi s$. Thus we may say that the behavior of the spin wave excitations changes from ferromagnetic at $s \to -1^+$ ($\Delta = 1$) to antiferromagnetic at $s \to 0^-$ ($\Delta = -1$). Furthermore, we expect to see free spin waves at $s = -\frac{1}{2}$. This picture has been confirmed by a study of the transport properties of the SC model which we present in another publication. An integral equation can also be given for ξ_0,

$$\xi_0(k) = 1 + \frac{1}{2\pi}\int_{-B}^{B}\xi_0(h)\theta'(h-k)dh, \tag{38}$$

where the kernel is as in Eq. (14). Alternatively, we may simply express ξ_0 in terms of thermodynamical response functions as

$$\xi_0^2 = \pi v_0 \kappa_0 d_0^2 = \pi d_0/v_0. \tag{39}$$

D. Correlation functions and conformal expansion

Given the conformal weights, we now construct the asymptotic expressions for correlation functions. For $-1 < s < 0$, we want to consider the following set of correlators: Let $\psi_\sigma(x,t)$ denote the field operator of a particle with spin σ. Later, we will additionally restrict the statistics to be either bosonic or fermionic by restricting the possible values of the pair $\mathbf{D}$. Then the field correlator — also called the one-particle reduced density matrix — is given by

$$C_\psi(x,t) = \langle \psi_\downarrow(x,t)\psi_\downarrow^\dagger(0,0)\rangle. \tag{40}$$

Defining the number operator $n(x,t) = n_\uparrow(x,t) + n_\downarrow(x,t)$, we write the density-density correlator

$$C_n(x,t) = \langle n(x,t)n(0,0)\rangle. \tag{41}$$

The spin-spin correlation functions are

$$C_\sigma^z(x,t) = \langle S^z(x,t)S^z(0,0)\rangle, \tag{42}$$

$$C_\sigma^\perp(x,t) = \langle S^-(x,t)S^+(0,0)\rangle, \tag{43}$$

where we used $S^z = (n_\uparrow - n_\downarrow)/2$ and $S^+ = \psi_\uparrow^\dagger\psi_\downarrow$. Note

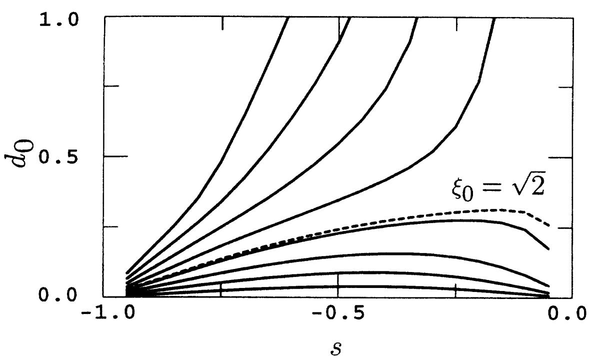

FIG. 1. Lines of constant universal behavior for the unbound case. Contours of constant value of the dressed charge ξ_0 in the (d_0, s) plane are shown. The lines represent increments of 0.2 starting from $\xi_0 = 1.0$ at $d_0 = 0$ up to $\xi_0 = 1.8$. The dashed line correspond to the value $\xi_0 = \sqrt{2}$ of a noninteracting system.

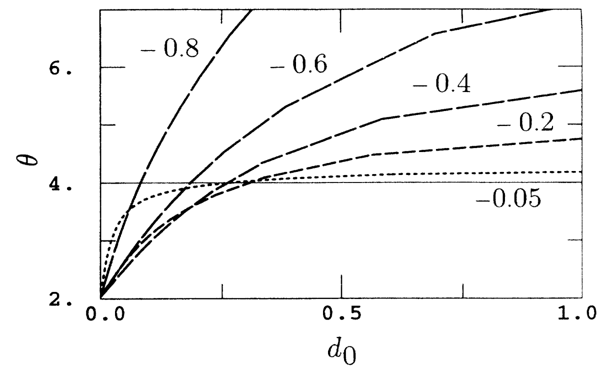

FIG. 2. Plot of θ as function of particle density d_0 for various values of interaction strength s for the unbound case.

that for systems that are rotationally invariant, such as the Hubbard model in zero magnetic field, these two spin-spin correlators are closely related, i.e., $C_\sigma^z = 2C_\sigma^\perp$.

Following Ref. 7, we also consider the correlation function for singlet pairs,

$$C_{\rm sing}(x,t) = \langle \psi_\uparrow^\dagger(x,t)\psi_\downarrow^\dagger(x,t)\psi_\uparrow(0,0)\psi_\downarrow(0,0)\rangle. \quad (44)$$

Note that all these correlators are of the form $\langle A(x,t)A^\dagger(0,0)\rangle$. By standard arguments of conformal field theory,[4] we may deduce the leading terms and the critical exponents of the long-distance behavior of these correlators by expanding A in terms of the primary fields $\phi_\pm$ while minimizing with respect to $\mathbf{D}$ at the corresponding values of $\Delta\mathbf{N}$. Here the above mentioned restrictions on $\mathbf{D}$ will become crucial. This approach, however, will leave the expansion coefficients undetermined, and at special points in the phase diagram, they may even vanish.

For $s > 0$, the model exhibits a gap for breaking of pairs and there are no spin waves. Therefore the correlators (40), (42), and (43) will decay exponentially. Let us introduce the pair field operator Ψ. The pair-density–pair-density correlator can be written in terms of the pair number operator $p = \Psi^\dagger\Psi$ as

$$C_p(x,t) = \langle p(x,t)p(0,0)\rangle \quad (45)$$

and the pair field correlator is given by

$$C_\Psi(x,t) = \langle \Psi^\dagger(x,t)\Psi(0,0)\rangle. \quad (46)$$

As before, we can construct these correlators by an expansion in primary fields, minimizing with respect to ΔN_1 and D_1.

IV. ASYMPTOTICS OF THE CORRELATION FUNCTIONS FOR THE UNBOUND CASE

Due to the restrictions (8) and (9) on the quantum numbers of a given state, the numbers $\mathbf{D} = (D_0, D_{-1})$ are integers or half-odd integers depending on the parities of the pair $\Delta\mathbf{N} = (\Delta N_0, \Delta N_{-1})$ and the statistics of $\psi^\dagger, \psi$. In particular, for fermionic particles we have

$$D_0 = \frac{\Delta N_0 + \Delta N_{-1}}{2} \pmod 1, \quad D_{-1} = \frac{\Delta N_0}{2} \pmod 1. \quad (47)$$

We can now apply the scheme for calculating the leading asymptotic behavior of the correlation function as outlined in the last section. Following our selection rules, we therefore have for a fermionic system

$$\begin{aligned}
C_\psi:&\quad \Delta N_0 = 1, && \Delta N_{-1} = 1,\\
&\quad D_0 = 0, \pm 1, \ldots, && D_{-1} = \pm\tfrac{1}{2}, \ldots;\\
C_n:&\quad \Delta N_0 = 0, && \Delta N_{-1} = 0,\\
&\quad D_0 = 0, \pm 1, \ldots, && D_{-1} = 0, \pm 1, \ldots;\\
C_\sigma^z:&\quad \Delta N_0 = 0, && \Delta N_{-1} = 0,\\
&\quad D_0 = 0, \pm 1, \ldots, && D_{-1} = 0, \pm 1, \ldots;\\
C_\sigma^\perp:&\quad \Delta N_0 = 0, && \Delta N_{-1} = 1,\\
&\quad D_0 = \pm\tfrac{1}{2}, \ldots, && D_{-1} = 0, \pm 1, \ldots;\\
C_{\rm sing}:&\quad \Delta N_0 = 2, && \Delta N_{-1} = 1,\\
&\quad D_0 = \pm\tfrac{1}{2}, \ldots, && D_{-1} = 0, \pm 1, \ldots;
\end{aligned} \quad (48)$$

This is identical to the results for the repulsive Hubbard model, and as in Ref. 7, we will write the critical exponents as functions of $\theta \equiv 2\xi_0^2$. However, there is an additional interaction strength dependence in the correlation functions due to the explicit appearance of s in Eq. (3). This is a novel feature and not true in the Hubbard model. It emphasizes the close correspondence of the Heisenberg-Ising model and the SC model for $-1 < s < 0$ in the zero sector.

Following the scheme outlined briefly in the last section, we calculate the leading asymptotics of the fermionic field correlator in the SC model to be

$$C_\psi(x,t) \sim \frac{1}{|x+iv_0t|^{1/\theta+\theta/16}|x+iv_{-1}t|^{1/2+s^2/4(s+1)}}\mathrm{Re}\left[A_0e^{-ik_fx}\left(\frac{x+iv_0t}{x-iv_0t}\right)^{1/4}\left(\frac{x+iv_{-1}t}{x-iv_{-1}t}\right)^{1/4}\right]$$
$$+\frac{1}{|x+iv_ot|^{1/\theta+9\theta/16}|x+iv_{-1}t|^{1/2+s^2/4(s+1)}}\mathrm{Re}\left[A_1e^{-i3k_fx}\left(\frac{x+iv_0t}{x-iv_0t}\right)^{3/4}\left(\frac{x+iv_{-1}t}{x-iv_{-1}t}\right)^{1/4}\right]. \quad (49)$$

The density-density correlator is given by

$$C_n(x,t) \sim n_0^2 + A_1\frac{\cos(2k_fx+\Phi_1)}{|x+iv_0t|^{\theta/4}|x+iv_{-1}t|^{1/(1+s)}} + A_2\frac{\cos(4k_fx+\Phi_2)}{|x+iv_ot|^{\theta}} + A_3\frac{x^2-(v_0t)^2}{[x^2+(v_0t)^2]^2} + A_4\frac{x^2-(v_{-1}t)^2}{[x^2+(v_{-1}t)^2]^2}, \quad (50)$$

and since the selection rules for the density-density correlator are identical to the selection rules for the longitudinal spin-spin correlator, the above calculation holds for C_σ^z with different constants and the replacement of $\mathcal{M}^2$ for n_0^2. Finally, for the transverse spin-spin and the single-particle correlator we find

$$C_\sigma^\perp(x,t) \sim A_0\frac{\cos(2k_fx+\Phi)}{|x+iv_0t|^{\theta/4}|x+iv_{-1}t|^{(1+s)}} + \frac{1}{|x+iv_{-1}t|^{2+s^2/(1+s)}}\mathrm{Re}\left[A_1\frac{x+iv_{-1}t}{x-iv_{-1}t}\right], \quad (51)$$

$$C_{\mathrm{sing}}(x,t) \sim A_0\frac{1}{|x+iv_0t|^{4/\theta}|x+iv_{-1}t|^{1/(1+s)}} + \frac{1}{|x+iv_ot|^{4/\theta+\theta/4}}\mathrm{Re}\left[A_1e^{-i2k_fx}\frac{x+iv_0t}{x-iv_0t}\right]. \quad (52)$$

Following Eq. (39), we calculate ξ_0 from the Fermi velocity v_0. In Fig. 1, we plot the lines of constant ξ_0 in the (d_0,s) plane. Note that the value of $\theta(\xi_0)$ at zero density is given by 2(1), whereas for finite densities and vanishing interaction strength $s\to 0^-$, we have $\theta\to 4$ ($\xi_0\to\sqrt{2}$). As expected, this is the same behavior as in the Hubbard model for vanishing on-site interaction strength u. However, we can not bound θ between those two values as we could for the Hubbard model. In fact, θ is larger than 4 and continues to increase for finite densities and increasing negative interaction strength $s\to -1^+$. A plot of θ as a function of the density d_0 for different values of the interaction strength s is given in Fig. 2.

For Bose statistics, D_0 and D_{-1} are restricted to integer values. The correlators of diagonal operators, i.e., the density-density correlator C_n and the longitudinal spin-spin correlator C_σ^z, are independent of statistics, and so only the correlators C_ψ, $C_\sigma^\perp$ and C_{sing} change. We find for their asymptotics

$$C_\phi(x,t) \sim A_0\frac{1}{|x+iv_0t|^{1/\theta}|x+iv_{-1}t|^{(1+s)/4}} + \frac{1}{|x+iv_ot|^{\theta/4+1/\theta}|x+iv_{-1}t|^{(s^2+4s+10)/8(1+s)}}$$
$$\times\mathrm{Re}\left[A_1e^{-i2k_fx}\left(\frac{x+iv_0t}{x-iv_0t}\right)^{1/2}\left(\frac{x+iv_{-1}t}{x-iv_{-1}t}\right)^{1/2}\right], \quad (53)$$

$$C_\sigma^\perp(x,t) \sim A_0\frac{1}{|x+iv_{-1}t|^{(1+s)}} + \frac{1}{|x+iv_0t|^{\theta/4}|x+iv_{-1}t|^{2+s^2/(1+s)}}\mathrm{Re}\left[A_1e^{-i2k_fx}\left(\frac{x+iv_{-1}t}{x-iv_{-1}t}\right)\right], \quad (54)$$

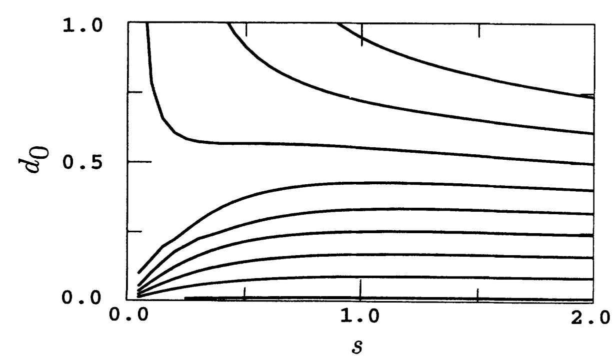

FIG. 3. Lines of constant universal behavior for the bound case. Contours of constant value of the dressed charge ξ_1 in the (d_0,s) plane are shown. The lines represent increments of 0.2 starting from $\xi_1=2.0$ at $d_0=0$ down to $\xi_1=1.2$.

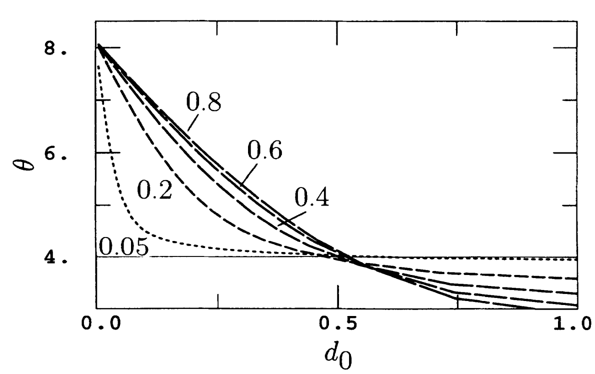

FIG. 4. Plot of θ as function of particle density d_0 for various values of interaction strength s for the bound case.

$$C_{\rm sing}(x,t) \sim A_0 \frac{1}{|x+v_0t|^{4/\theta}} + \frac{1}{|x+iv_0t|^{\theta/4+4/\theta}|x+iv_{-1}t|^{1/(1+s)}} \mathrm{Re}\left[A_1 e^{-i2k_fx}\left(\frac{x+iv_0t}{x-iv_0t}\right)\right] . \tag{55}$$

V. ASYMPTOTICS OF THE CORRELATION FUNCTIONS FOR THE BOUND CASE

Due to the restriction (20) on the quantum numbers of a given state, D_1 is an integer or half-odd integer depending on the parity of ΔN_1 for both Bose and Fermi statistics of the particles, i.e.,

$$D_1 = \frac{\Delta N_1}{2} \pmod 1 . \tag{56}$$

This selection rule is just the same as the case of one-component bosons, and so we find for the asymptotics of the pair density correlator

$$C_p(x,t) - d_1^2 \sim A_1 \frac{x^2-(v_1t)^2}{[x^2+(v_1t)^2]^2} + A_2 \cos(2k_fx+\varphi_1)\frac{1}{|x+iv_1t|^{\theta}} \tag{57}$$

and for the pair field correlator

$$C_\Psi(x,t) \sim A_1 \frac{1}{|x+iv_1t|^{1/\theta}} + \frac{1}{|x+iv_1t|^{\theta+1/\theta}} \mathrm{Re}\left[A_2 e^{-i2k_fx}\frac{x+iv_1t}{x-iv_1t}\right] . \tag{58}$$

Here we again defined an exponent $\theta = 2\xi_1^2$. Following Eq. (30), we can calculate ξ_1 from the Fermi velocity of pairs v_1. In Fig. 3, we plot the lines of constant ξ_1 in the (d_0, s) plane. Note that the value of $\theta(\xi_1)$ at zero density is given by 8(2), whereas for finite densities and vanishing interaction strength $s \to 0^+$, we have $\theta \to 4$ ($\xi_1 \to \sqrt{2}$). A plot of ξ_1 as a function of the density d_0 for different values of the interaction strength s is given in Fig. 4.

VI. NONINTERACTING TWO-COMPONENT SYSTEM

At $s = 0$, the system reduces to a noninteracting two-component gas and we may expect a certain continuity in the behavior of the correlators at this point. Indeed, as $s \to 0^-$, the two Fermi velocities v_0 and v_{-1} both approach the Fermi velocity of a noninteracting single-component model, i.e., $v_0(s \to 0^-) = v_{-1}(s \to 0^-) = \pi d_0/2$. Consequently, the correlation functions of the bosonic (fermionic) system reduce to the correlation functions of a noninteracting Bose (Fermi) system with two components, i.e., with half the one-component Fermi momentum. Using the language of conformal field theory, we can thus describe the excitations of the noninteracting two-component gas by a $c = 2$ generalized Gaussian model.[6]

From the expression of the dressed charges ξ_0 and ξ_1, we see that $\xi_1^2 = \frac{1}{2}\xi_0^2\frac{v_0}{v_1}$. As $s \to 0^+$, the Fermi velocity of the pairs goes to the Fermi velocity of a one-component free Bose gas with doubled particle mass, i.e., $v_1(s \to 0^+) = \pi d_1/2 = \frac{1}{2}v_0(s \to 0^-)$. Therefore, we expect $\theta_1 = \theta_0$ at $s = 0$ and this is indeed true as shown above. Furthermore, the free energy of the system should be uniquely specified at $s = 0$. Following (23) we may write the finite-temperature corrections for the unbound case as

$$F(T) \simeq F(T=0) - \frac{\pi T^2}{6}\left(\frac{1}{v_{-1}} + \frac{1}{v_0}\right), \tag{59}$$

whereas for the bound case we have

$$F(T) \simeq F(T=0) - \frac{\pi T^2}{6v_1}. \tag{60}$$

As predicted, these two equations are in agreement at $s = 0$ and identical to the free energy of a noninteracting $c = 2$ system.

The bound pairs for $s > 0$ are singlets. Therefore we might expect that the pair field correlator (46) becomes identical to the singlet pair correlators (44) and (55) of the unbound case as $s \to 0$. However, $\Psi^\dagger$ creates pairs with characteristic length scale $1/s$ and not just two particle wave functions. Thus, the pair wave functions include a normalization factor $\sqrt{s}$. As $s \to 0^+$, the leading terms of the conformal expansion (58) consequently vanish and higher-order terms become important. It should therefore come as no surprise that the expansions (44), (55), and (58) do not agree at $s = 0$.

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