Potts antiferromagnetic model on a family of fractal lattices: Exact results for an unusual phase

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(Received 18 June 1993; revised manuscript received 6 October 1993)

The three-state antiferromagnetic Potts model on a family of bipartite diamond hierarchical lattices is investigated. This exactly solvable family of models exhibits a distinctive low-temperature phase of the type predicted by Berker and Kadanoff for complex systems with a macroscopically degenerate ground state. An analytic expression for the residual entropy is derived. We prove that there are some similarities between this phase and that of Kosterlitz-Thouless, namely, the power-law decay of correlations, the vanishing of the order parameter, and the absence of divergence in the specific heat. It is also shown that the local order parameter has two distinct multifractal structures: one at the critical point and another at the unusual phase.

I. INTRODUCTION

Although the majority of macroscopic physical systems and statistical mechanics models obey the third law of thermodynamics, there are notable exceptions like, for example, the substances[1] ice, carbon monoxide, and nitrous oxide, some frustrated systems (e.g., Ising antiferromagnet on the triangular lattice,[2] spin glasses[3]), some Potts antiferromagnets,[4] and some other classical discrete spin systems.[5] The study of statistical properties of such systems is much more complex due to their infinite macroscopic ground-state degeneracy. In particular, Berker and Kadanoff[6] suggested, using a one-parameter renormalization-group (RG) argument, that systems with finite residual (i.e., at null temperature) entropy per particle may present a distinctive low-temperature phase with no obvious order parameter and with a power-law decay of correlations. In fact, they obtained, within a Migdal-Kadanoff RG approximation, such unusual phase in the q-state ($q>2$) antiferromagnetic (AF) Potts model on d-dimensional hypercubic lattices whenever $d>d_c(q)$, where $d_c(q)$ is the lower critical dimension for fixed q, below which there is no phase transition. However, Rácz and Vicsek[7] argued that the appearance of the distinctive phase could be an artifact of the one-parameter RG treatment. Since then, much work has been done looking for this phase in the AF Potts model on different Bravais (see, for example, Ref. 8 and references therein) and fractal lattices.[9-12] In particular, many approximate calculations (see Ref. 8 and references therein) for the $q=3$ case on the bipartite cubic lattice indicate a continuous transition at $T_c\neq 0$ from the paramagnetic to the long-range ordered broken sublattice symmetry (BSS) phase. The latter is characterized by a kind of staggered order that breaks the sublattice symmetry given by the predominance of one of the states in one sublattice and of the other two states (randomly distributed with equal probabilities) on the other sublattice. Excluding Ono's simulations[13] (which, according to Ref. 8, have an insufficient number of Monte Carlo steps) and some indications for the divergence of the correlation length below T_c despite the nonzero order parameter,[8] there has not been found any evidence for the unusual phase in the three-dimensional $q=3$ case or for any value of q on a number of Bravais lattices. On the other hand, the same is not true for some fractal lattices, where this phase has been established by either exact RG transformations[11,12] or by the Migdal-Kadanoff RG approximation.[9,10] It is worth mentioning that the exact results of Ref. 12 show that the distinctive phase is neither an artifact of a one-parameter RG treatment nor an artifact of the Migdal-Kadanoff bond-moving procedure and can, therefore, occur in a lattice, which is not a combination of series and/or parallel bonds. The unusual phase was detected in the RG procedures[9-12] through an attractor at a nonvanishing temperature—a feature which appears also in the RG framework of Berker and Kadanoff.[6] Furthermore, it was also proved,[12] for the Ising case, that the correlations decay algebraically with distance along the whole unusual phase. But a detailed study of this phase including the exact temperature dependence of the order parameter and other thermodynamical quantities has not, as far as we know, been reported in the literature. In particular, it would be interesting to see if a BSS ordering exists in the unusual phase despite the divergence of the correlation length below the critical temperature, like the MC data of Wang, Swendsen, and Kotecký[8] for the three-state AF Potts model on the cubic lattice indicate. Furthermore, in the case of absence of long-range ordered phase, it would be worth examining up to what extent this phase is similar to that of the bidimensional XY model.[14]

Herein we consider the three-state AF Potts model on a family of diamond hierarchical lattice (HL) types, which belongs to a larger family of HL's on which the Potts antiferromagnet presents[11] the distinctive phase. Using an exact recursive procedure,[15-18] we prove that the BSS order parameter vanishes for all temperatures and that the local magnetization distribution has a multifractal structure at the critical point different from that along the unusual phase. We also calculate exactly the average internal energy, the specific heat and the entropy

per spin as functions of temperature.

The outline of this paper is as follows. In Sec. II we define the model and the family of HL's to be considered herein. In Sec. III we prove that this family of systems has a low-temperature phase with a power-law decay of correlations. In Sec. IV we derive the recursive relations for local average quantities whose iteration leads to multifractal local magnetization profiles. In Sec. V we calculate, in an exact way, the order parameter, the internal energy, the specific heat and the entropy. Finally, the conclusions are given in Sec. VI.

II. MODEL

Qin and Yang[11] considered the q-state AF Potts model on the fractal family of diamond HL types whose generator is constituted by P branches in parallel, each of which has L bonds in series. They showed that, for an odd L and for $2<q<q_c$ (where q_c is a cutoff value of q, which depends on P and L), there appears the type of phase predicted by Berker and Kadanoff. Herein we shall consider the simplest family of HL's (with a minimum number of L) of this larger family on which the q-state (q being the minimum integer below q_c) AF Potts model presents the unusual phase. Its generator (or basic cell) contains $P\geq 10$ branches of $L=3$ bonds in series (see graph $G^{(1)}$, for $P=10$, in Fig. 1), and the HL is constructed as follows. We start with a bond between the roots R_A and R_B (see $G^{(0)}$) and, in the next level, we replace it by the generator and continue successively substituting each bond of a level by the basic cell. In the thermodynamic limit ($n\rightarrow\infty$) this family of HL's has a fractal dimension[19] $d_f(P)$ given by

$$d_f(P)=\frac{\ln(3P)}{\ln 3} \quad (P\geq 10)\ , \tag{1}$$

in particular, $d_f(10)=3.09\ldots$.

Notice that the considered HL family is bipartite, i.e., each HL can be divided into two interpenetrating sublattices $\mathcal{A}$ (represented by points in Fig. 1) and $\mathcal{B}$ (represented by squares) such that any site of one sublattice has as nearest neighbors only sites of the other sublattice at any level. Furthermore, contrarily to what happens in the case of $L=$even, each point belongs always to a given sublattice independently of the level n.

At each site i of the HL with P branches ($P\geq 10$), we associated a Potts variable $\sigma_i=0,1,2$ and consider the three-state AF Potts model described by the following dimensionless Hamiltonian at the n level:

$$\beta\mathcal{H}_n^{(P)}=-3K_n^{(P)}\sum_{\langle ij\rangle}\delta(\sigma_i,\sigma_j)$$

$$(\beta\equiv 1/k_BT\ ,\quad K_n^{(P)}\equiv\beta J_n^{(P)})\ , \tag{2}$$

where $J_n^{(P)}<0$ is the AF coupling constant between nearest-neighbor spins at the n level of the HL with P branches, the sum is over all nearest-neighbor pairs $\langle i,j\rangle$ of spins, $\delta(\sigma_i,\sigma_j)$ is the Kronecker δ function, and T is the absolute temperature.

III. THE DISTINCTIVE LOW-TEMPERATURE PHASE

Let us now prove that each system of the above-mentioned family has an unusual phase, where the pair-correlation function obeys a power-law decay. For this, we consider the RG transformation defined by the renormalization of the basic cell (see, for $P=10$, $n=1$ of Fig. 1) (with reduced coupling constant $K^{(P)}$ between any pair of neighbor spins) into the single bond ($n=0$) linking the two roots with an effective reduced coupling constant $K'^{(P)}$. The RG recursive equation is obtained by imposing that the trace of $\exp[-\beta\mathcal{H}_1(K^{(P)})]$ over the internal spins (i.e., those different from the rooted ones) of $G^{(1)}$ is proportional to $\exp[-\beta\mathcal{H}_0(K'^{(P)})]$. This is equivalent to preserve the correlation function between σ_{RA} and σ_{RB} (Ref. 20) and it leads to

$$t'(t,P)=\frac{1-[(1-t^3)/(1+2t^3)]^P}{1+2[(1-t^3)/(1+2t^3)]^P}\ , \tag{3}$$

where we have used, for convenience, the thermal transmissivity variable[21] defined by

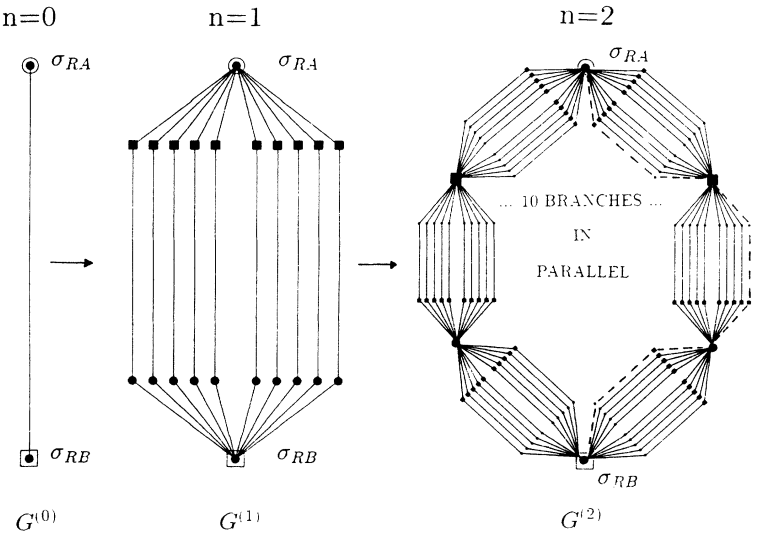

FIG. 1. First steps of construction of the studied diamond-type HL with $P=10$ branches. $G^{(0)}$, $G^{(1)}$, and $G^{(2)}$ are the corresponding graphs obtained at the respective levels $n=0,1,2$. The sites of the sublattice $\mathcal{A}$ ($\mathcal{B}$) are represented by circles (squares) and, in particular, the open circle (open square) is the root R_A (R_B). The broken line at $n=2$ indicates an arbitrary shortest path joining the roots.

$$t(K)\equiv\frac{1-e^{-3K}}{1+2e^{-3K}} . \tag{4}$$

Thus, if we fix P and t_n (the thermal transmissivity at the n level), we can obtain the transmissivities at previous levels by iterating Eq. (3), namely, $t_{n-1}=t'(t_n)$, $t_{n-2}=t'(t_{n-1})=t^{(2)}(t_n)$ and so on until $t_0=t^{(n)}(t_n)$, which corresponds to the equivalent transmissivity between the roots of the HL with P branches at the n level.[20] It is worth mentioning that our RG transformation [Eq. (3)] is an exact one due to the recursive construction of each HL and to the fact that the symmetries of the ground state are preserved for this choice of cells.[12]

The analysis of the flow diagrams of $t'(t,P)$ leads to the appearance of an unusual phase for finite $d_f(P)>d_c=3.025\ldots$, characterized by an attractor at a nonzero temperature $t_{\rm AF}^{(P)}$ (for $P=10$, $t_{\rm AF}^{(10)}=-0.477\ldots$), in the range $-\frac{1}{2}\le t<t_c^{(P)}$. We shall use the index notation AF to designate the unusual phase, despite the fact that the staggered magnetization is null along this phase (as we will see in Sec. V). For transmissivities above $t_c^{(P)}$ all points flow under successive RG iterations to the paramagnetic attractor $t_p=0$, defining thus the paramagnetic phase. In Fig. 2 it is shown the plots of the transmissivity $t_{\rm AF}^{(P)}$ and the critical transmissivity $t_c^{(P)}$ versus the fractal dimension $d_f(P)$. Notice that, as d_f diminishes tending to d_c, the fixed points $t_{\rm AF}^{(P)}$ and $t_c^{(P)}$ approach each other until they merge, for $d_f=d_c$, into a single marginal one. Observe also that, for $d_f\gg d_c$, the AF attractor converges to $T=0$. These behaviors confirm those suggested by Berker and Kadanoff[6] and are similar to those obtained,[12] in an exact way, for the $q=2$- and 4-state AF Potts model on the m-sheet Sierpinski gasket with side $b=4$.

The correlation length critical exponent $\nu^{(P)}$ can be determined by

$$\nu^{(P)}=\frac{\ln L}{\ln r_c^{(P)}} , \tag{5}$$

where L is the chemical distance between the roots of the basic cell ($L=3$) and

$$r_c^{(P)}\equiv\left.\frac{dt'(t,P)}{dt}\right|_{t_c^{(P)}} . \tag{6}$$

The plot of $\nu^{(P)}$ as function of $d_f(P)$ can be seen in Fig. 3. Similar to Ref. 12, $\nu^{(P)}$ diverges for $d_f=d_c$ as a power law, namely, $\nu^{(P)}\sim[0.841\ldots][d_f(P)-d_c]^{-0.442\ldots}$ as d_f approaches d_c from above.

Let us now prove, through a procedure similar to [12], that the correlation function Γ_{AB} between the two roots of the HL with a finite number of P branches decays algebraically with distance along the unusual phase whose attractor occurs at $t_{\rm AF}^{(P)}$. We define the correlation function between the roots of the HL at the n level and at the temperature corresponding to the transmissivity t_n, $\Gamma_{AB}^{(n)}(t_n)$, by[22]

$$\Gamma_{AB}^{(n)}(t_n)\equiv\frac{3\langle\delta(\sigma_{RA},\sigma_{RB})\rangle_{n,P}-1}{2} , \tag{7}$$

where $\langle\cdots\rangle_{n,P}$ means the thermal average taken at the n level of a HL with P branches. On the other hand $\Gamma_{AB}^{(n)}(t_n)$ is related to the equivalent transmissivity between σ_{RA} and σ_{RB} through:[23]

$$\Gamma_{AB}^{(n)}(t_n)=t_n^{(n)}\equiv t^{(n)}(t_n) , \tag{8}$$

where, as already pointed out, $t_n^{(n)}$ is the nth iteration of Eq. (3) applied to t_n.

Expanding $t'(t_n)$ around the AF attractor we have

$$t'(t_n)\simeq t_{\rm AF}^{(P)}+r_{\rm AF}^{(P)}(t_n-t_{\rm AF}^{(P)}) \quad (t_n\simeq t_{\rm AF}^{(P)}) , \tag{9}$$

where

$$r_{\rm AF}^{(P)}\equiv\left.\frac{dt'(t,P)}{dt}\right|_{t_{\rm AF}^{(P)}} , \quad 0<r_{\rm AF}^{(P)}<1 .$$

The iteration of the above equation combined with Eq. (8)

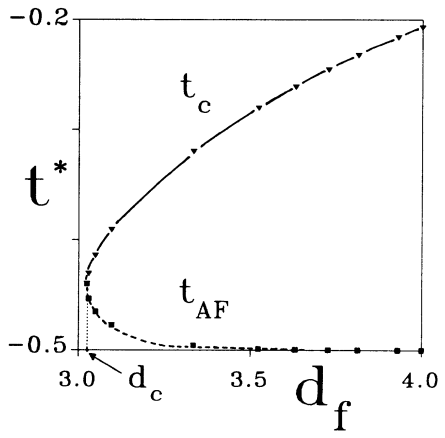

FIG. 2. The fixed points t^* at the critical transmissivity $t_c^{(P)}$ (represented by triangles) and at the attractor one $t_{\rm AF}^{(P)}$ of the unusual phase (represented by squares) as functions of the fractal dimension $d_f(P)$ of the diamond-type HL with P branches. The lines are guides for the eye.

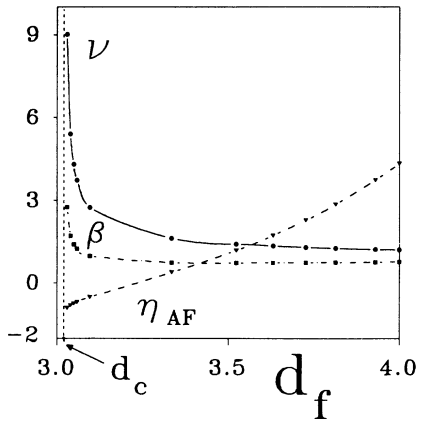

FIG. 3. The critical exponents of the correlation length $\nu^{(P)}$ (represented by small circles) and of the order parameter $\beta^{(P)}$ (represented by squares), as well as the correlation function exponent $\eta_{\rm AF}^{(P)}$ for the whole unusual phase (represented by triangles) vs the fractal dimension $d_f(P)$. The lines are guides for the eye.

leads in the fractal limit ($n \to \infty$) to

$$\Gamma_{AB} \equiv [\Gamma_{AB}^{(n)}(t_n) - \Gamma_{AB}^{(\infty)}(t_\infty)] \sim (r_{\rm AF}^{(P)})^n (t_n - t_{\rm AF}^{(P)}) \sim L_n^{-\epsilon^{(P)}} \tag{10}$$

with

$$\epsilon^{(P)} \equiv -\frac{\ln r_{\rm AF}^{(P)}}{\ln 3} , \tag{11}$$

where L_n is the chemical distance between the roots of the HL ($L_n = 3^n$). This result is in agreement with the suggestion of power-law decay made by Berker and Kadanoff.[6]

Assuming that, similar to the asymptotic behavior of $\Gamma_{AB}(L_{AB} \to \infty)$ in the d-dimensional Bravais lattices,

$$\Gamma_{AB}(L_{AB}) \sim L_{AB}^{-(d_f - 2 + \eta_{\rm AF})} \quad (L_{AB} \to \infty) \tag{12}$$

we obtain $\eta_{\rm AF}$ versus $d_f(P)$ as shown in Fig. 3 by triangles, which is similar to that found in Ref. 12. It is worth emphasizing that this $\eta_{\rm AF}$ exponent is valid for the whole unusual phase ($-\frac{1}{2} \le t < t_c^{(P)}$), but not necessarily for $t = t_c^{(P)}$ (see Sec. V B).

IV. LOCAL AVERAGE QUANTITIES

In order to characterize better this unusual phase, we shall calculate in an exact way the order parameter and other thermodynamical quantities as well. For this, we shall use a recursive method developed initially for Ising models[15–17] and, afterwards, extended to the Potts model[18] on HL's. This technique allows one to obtain exact recursive relations for local average quantities, from which one can derive global thermodynamical functions such as the order parameter, the internal energy and the specific heat. Let us, in this section, focus on the calculation of local average quantities.

A. Recursive relations

Using the recursive method[18] with the boundary condition of fixing the rooted spin σ_{R_A} at the state 0, we obtain the following system of coupled equations:

$$\begin{aligned}
m_{\sigma_1}^{(n)} &= a_n m_{\mu_2}^{(k)} + b_n m_{\mu_1}^{(l)} + c_n \Delta_{\mu_1\mu_2}^{(n-1)} , \\
m_{\sigma_2}^{(n)} &= a_n m_{\mu_1}^{(l)} + b_n m_{\mu_2}^{(k)} + c_n \Delta_{\mu_1\mu_2}^{(n-1)} , \\
\Delta_{\mu_1\sigma_2}^{(n)} &= (1 + 2a_n - b_n) m_{\mu_1}^{(l)} + (b_n + 2c_n) \Delta_{\mu_1\mu_2}^{(n-1)} , \\
\Delta_{\sigma_1\sigma_2}^{(n)} &= (a_n + b_n)(m_{\mu_1}^{(l)} + m_{\mu_2}^{(k)}) + d_n \Delta_{\mu_1\mu_2}^{(n-1)} , \\
\Delta_{\mu_2\sigma_1}^{(n)} &= (1 + 2a_n - b_n) m_{\mu_2}^{(k)} + (b_n + 2c_n) \Delta_{\mu_1\mu_2}^{(n-1)} \quad (k, l < n) ,
\end{aligned} \tag{13}$$

where the spins σ_2 and σ_1 are aggregated at the n level, while the spins μ_1 and μ_2 were created at previous levels l ($l < n$). In Eq. (13) we chose to define the local magnetization $m_i^{(k)}$ at site i of the k-level HL with P branches by

$$m_i^{(k)} \equiv \langle \delta(\sigma_i, 0) \rangle_{k,P} - 1/3 \tag{14}$$

without the normalization factor 3/2, since it leads (as we will see in Sec. V A) to a normalized order parameter. The function $\Delta_{ij}^{(k)}$, a kind of correlation function, which appears in this technique as a natural variable in the case of the q-state Potts model (for $q \neq 2$),[18] is defined by

$$\Delta_{ij}^{(k)} \equiv 3 \langle \delta(\sigma_i, 0) \delta(\sigma_j, 0) \rangle_{k,P} - \langle \delta(\sigma_i, \sigma_j) \rangle_{k,P} . \tag{15}$$

In Eq. (13), the coefficients a_n, b_n, c_n, and d_n are functions of the coupling constant $K_n^{(P)}$ at the n level of the HL with P branches, which are given, expressed in terms of the transmissivity t_n, by

$$\begin{aligned}
a_n &\equiv \frac{t_n(1+t_n)}{1+t_n+t_n^2} , \quad b_n \equiv \frac{t_n^2}{1+t_n+t_n^2} , \\
c_n &\equiv \frac{t_n^3(1-t_n^2)}{(1+2t_n^3)(1+t_n+t_n^2)} , \\
d_n &\equiv \frac{t_n^2(1+3t_n+t_n^2-2t_n^3)}{(1+2t_n^3)(1+t_n+t_n^2)} ,
\end{aligned} \tag{16}$$

where the set $\{t_n, t_{n-1}, \ldots, t_0\}$ can be obtained from $t_{n-1} = t'(t_n)$ with $t'(t,P)$ given by Eq. (3).

Besides the coupled Eqs. (13), we also have the following recurrent relation (which will be used in the calculation of the internal energy in the next section):

$$\begin{aligned}
\langle \delta(\mu_1, \sigma_2) \rangle_{n,P} &= \langle \delta(\mu_2, \sigma_1) \rangle_{n,P} \\
&= \langle \delta(\sigma_1, \sigma_2) \rangle_{n,P} \\
&= e_n \langle \delta(\mu_1, \mu_2) \rangle_{n-1,P} + f_n
\end{aligned} \tag{17}$$

where

$$e_n \equiv b_n + 2c_n \quad \text{and} \quad f_n \equiv \frac{2a_n - b_n + 1}{3} . \tag{18}$$

We shall refer hereafter to the values of the coefficients k_n (where $k_n = a_n, b_n, c_n, d_n, e_n, f_n$) evaluated at the critical temperature t_c and at the attractor temperature $t_{\rm AF}$ as k_c and $k_{\rm AF}$, respectively.

B. Local magnetization profiles

The successive iteration of Eqs. (13) and (3) allows us to obtain, for a fixed P, the local magnetization distribution for both sublattices $\mathcal{A}$ and $\mathcal{B}$ at any level n for all temperatures. In order to analyze this distribution, it is sufficient to focus on any one of the shortest paths between the two roots R_A and R_B (represented by a broken line in Fig. 1), since all of them are equivalent by symmetry. Similar to Ref. 15, 16, and 17, we identify each site of such a path by a pair (s,l) where l is the level at which the site appeared for the first time and s is the chemical distance from the site to the root R_A within the level l. These sites can be arranged over the interval [0,1] such that the pair (s,l) corresponds to the point $x \equiv s' \, 3^{-l}$ for a n-level HL (with $l = 1, 2, \ldots, n$ and $s = 1, 2, 4, 5, \ldots, 3^l - 1$).

In Fig. 4 we show the profiles of the local magnetizations m_{σ_i} versus x for a seventh-level HL with $P = 10$ branches at the critical temperature $T_c^{(10)}$ [Fig. 4(a)] and at the attractor temperature $T_{\rm AF}^{(10)}$ [Fig. 4(b)] with the boundary condition $m(x=0) = 2/3$ (a consequence of $\sigma_{R_A} = 0$). Similar profiles are obtained for other values of

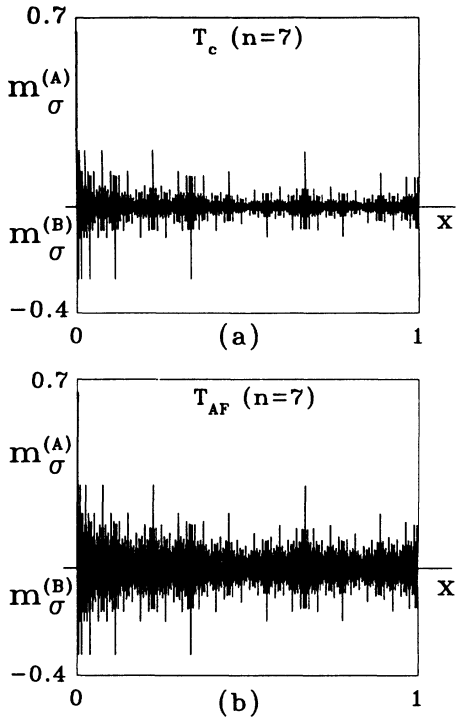

FIG. 4. The local magnetizations $m_\sigma^{(A)}$ and $m_\sigma^{(B)}$ of the spins along a shortest path between the roots R_A and R_B of the HL with $P=10$ branches at the $n=7$ level vs the position x. (a) and (b) were calculated at the critical temperature T_c and at the attractor temperature T_{AF}, respectively. The positive magnetizations $m_\sigma^{(A)}$ correspond to spins belonging to the sublattice $\mathcal{A}$ and the negative ones $m_\sigma^{(B)}$ refer to those of the sublattice $\mathcal{B}$.

P ($P>10$). Notice that positive (negative) magnetizations correspond to sites from the sublattice $\mathcal{A}$ ($\mathcal{B}$).

C. Multifractality

Covering the magnetization profile support at the n level with boxes of size $l_n=3^{-n}$ so that within each box there is one spin, one can define the local measure at the ith box, at the n-level HL with P branches, by

$$p_i^{(n)}\equiv\frac{|m_{\sigma_i}|}{\sum_j|m_{\sigma_j}|}\quad(i=1,2,\ldots,3^n+1)\ . \tag{19}$$

As the box width goes to zero, or equivalently the HL level goes to infinity, the measure at the box i scales as $p_i\sim l^{\alpha_i}$, where α_i is the Hölder exponent at this box. In the same limit, the number of boxes N_α with Hölder exponent between α and $\alpha+d\alpha$ scales as $N_\alpha\sim l^{-f(\alpha)}$.

Following the method of Chhabra and Jensen[24,25] we obtained the $f(\alpha)$ spectra for $P=10$ at the critical and at the AF attractor temperatures shown in Fig. 5. A detailed analysis near the points where $f(\alpha)$ vanishes shows that $df(\alpha)/d\alpha$ tends to infinity at these points for both temperatures, as usually occurs in deterministic fractals. We can see that at the AF attractor the $f(\alpha)$ spectrum is sharper reflecting the higher homogeneity in the magnetization distribution. Similar spectra are obtained for other values of P ($P>10$). Notice that in previous works[15–18] the multifractality associated with the magnetization profiles appeared only at the critical temperature, while here there are two distinct $f(\alpha)$ spectra: one at $T_c^{(P)}$ and the other at the whole unusual phase.

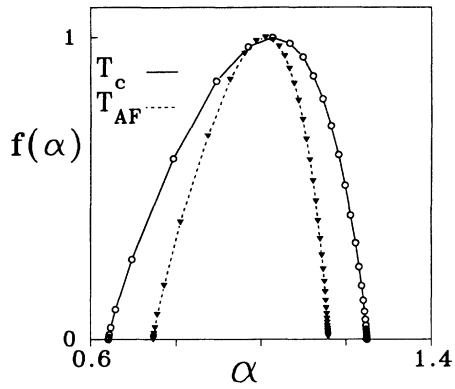

FIG. 5. The $f(\alpha)$ spectra at the critical temperature T_c (represented by circles) and at the attractor temperature T_{AF} (represented by triangles) for $P=10$. The lines are guides for the eye.

One can compute exactly the lowest Hölder exponent $\alpha_{\min}$, which is associated with the set of the largest measures, at both temperatures, namely,

$$\alpha_{\min}(t_c)=1-\frac{\beta}{\nu} \tag{20}$$

and

$$\alpha_{\min}(t_{AF})=\alpha_{\min}(t_c)+\left[\ln\left[\frac{1+2(b_{AF}-a_{AF})}{1+2(b_c-a_c)}\right]\right]\Big/\ln 3\ , \tag{21}$$

where β is the critical exponent associated with the global order parameter shown in Fig. 3. For $P=10$ we obtain $\alpha_{\min}(t_c)=0.642\ldots$ and $\alpha_{\min}(t_{AF})=0.747\ldots$.

The largest Hölder exponent $\alpha_{\max}$ associated with the set of smallest measures can also be calculated in an exact way at the critical temperature, namely,

$$\alpha_{\max}(t_c)=1+\frac{(\beta_M-\beta)}{\nu}\ , \tag{22}$$

where $\beta_M\equiv\ln y/\ln r_c$ satisfies the following equation:

$$[1+2a_cy+(a_c^2-b_c^2)y^2]\{b_c+[c_c(a_c+b_c)-d_cb_c]y\}$$
$$-c_c[(a_c-b_c)y^2-y][(b_c^2-a_c^2)y-(a_c+b_c)]=0\ , \tag{23}$$

which, for $P=10$, gives the value of $\alpha_{\max}(t_c)=1.250\ldots$. These exact results for the limit Hölder exponents are in excellent agreement with those obtained by the direct computation of the $f(\alpha)$ spectra: the relative errors, for $P=10$, are 0.09, 0.03, and 0.05 % for $\alpha_{\min}(t_c)$, $\alpha_{\min}(t_{AF})$,

and $\alpha_{\max}(t_c)$, respectively. These results exhibit the superiority of this method over the box counting method for computing the $f(\alpha)$ spectrum.

V. THERMODYNAMICAL QUANTITIES

Once we have calculated local average quantities, let us proceed in this section to the derivation of the global macroscopic functions.

A. Order parameter

The ordering scheme suggested by Monte Carlo (MC) simulations for the three-state AF Potts model on a bipartite Bravais lattice is the so called BSS, which consists of having predominance of one of the states on the sublattice $\mathcal{A}$ and of the other two states distributed with equal probabilities on the sublattice $\mathcal{B}$. A global order parameter per site describing this type of ordering may be defined[26,27,8] by

$$M=\frac{1}{N_s}\left\{\left|\sum_{i\subset A}\langle\delta(\sigma_i,0)\rangle-\sum_{i\subset B}\langle\delta(\sigma_i,0)\rangle\right|+\left|\sum_{i\subset A}\langle\delta(\sigma_i,1)\rangle-\sum_{i\subset B}\langle\delta(\sigma_i,1)\rangle\right|+\left|\sum_{i\subset A}\langle\delta(\sigma_i,2)\rangle-\sum_{i\subset B}\langle\delta(\sigma_i,2)\rangle\right|\right\}. \tag{24}$$

The sum over $i\subset\mathcal{A}$ ($i\subset\mathcal{B}$) refers to the sites at the sublattice $\mathcal{A}$ ($\mathcal{B}$) and N_s is the total number of sites on the lattice.

It is easy to show that Eq. (24) applied to the n level of the HL reduces, due to our symmetry-breaking condition, to

$$M_n^{(P)}=\frac{2D_n^{(P)}}{N_{sn}^{(P)}} \tag{25}$$

with

$$D_n^{(P)}\equiv\sum_{i\subset A}m_{\sigma_i}-\sum_{i\subset B}m_{\sigma_i} \tag{26}$$

and

$$N_{sn}^{(P)}=\left[\frac{2P}{3P-1}\right](3P)^n+\frac{4P-2}{3P-1}\ , \tag{27}$$

where $N_{sn}^{(P)}$ is the number of sites at the n-level HL with P branches. Note that the distinct order parameter proposed by Ono[13] reduces, in our case, to $\zeta_1=(3/4)M_n$ and $\zeta_2=0$ (see Definitions (4.1) of Ref. 13).

Using the set of recurrent Eq. (13), one is able to show that $D_n^{(P)}$ also follows a recurrence equation given by

$$D_n^{(P)}-D_{n-1}^{(P)}=P(a_n-b_n)\left[\frac{1}{(a_{n-1}-b_{n-1})}-2\right]\times(D_{n-1}^{(P)}-D_{n-2}^{(P)})\ . \tag{28}$$

Iterating this equation one arrives at the following exact expression for the order parameter per site as function of the temperature (implicit in the parameters a and b)

$$M_n^{(P)}=\left[\frac{N_{s1}^{(P)}}{N_{sn}^{(P)}}\right]M_1^{(P)}+\left[\left[\frac{N_{s1}^{(P)}}{N_{sn}^{(P)}}\right]M_1^{(P)}-\left[\frac{N_{s0}^{(P)}}{N_{sn}^{(P)}}\right]M_0^{(P)}\right]\sum_{i=2}^{n}P^{i-1}\left[\frac{b_i-a_i}{b_1-a_1}\right]\prod_{j=1}^{i-1}[1+2(b_j-a_j)]\ . \tag{29}$$

For a fixed point t^* of the renormalization, and for large n, this expression reduces to

$$M_n^{(P)}=\frac{2(3P-1)(1-t^*)(b^*-a^*)}{3\{P[1+2(b^*-a^*)]-1\}}\left[\frac{1+2(b^*-a^*)}{3}\right]^n, \tag{30}$$

where a^* and b^* are the coefficients a_n and b_n calculated at a fixed point t^*. Evaluating this expression at the fixed points we verify that $\lim_{n\to\infty}M_n^{(P)}(t_c^{(P)})=\lim_{n\to\infty}M_n^{(P)}(t_{\rm AF}^{(P)})=0$, which proves the vanishing of the order parameter per site $M^{(P)}$ for the whole range of temperatures, namely,

$$M^{(P)}(t)\equiv\lim_{n\to\infty}M_n^{(P)}(t)=0\ ,\quad\frac{-1}{2}\le t\le 0\ ,\quad\forall P\ge 10\ . \tag{31}$$

In Fig. 6 we show, for $P=10$, the behavior of the order parameter per site $M_n^{(10)}$ as function of temperature for different levels of the HL, exhibiting an abrupt decay as the level n increases.

Although $M_n^{(P)}$ vanishes in the thermodynamical limit for all temperatures, let us show that it exists a non-null critical exponent $\beta^{(P)}$. First of all, we verified that, for a fixed temperature T, $M_n^{(P)}$ behaves as

$$M_n^{(P)}(T)\sim A(T,P)L_n^{-\theta(T,P)}\ , \tag{32}$$

where $L_n=3^n$ is the linear size of the n-level HL with P branches, $A(T,P)$ is a finite constant, and $\theta(T,P)$ is a temperature dependent positive exponent. Besides that, we have also observed that the inflexion point $T_n^{*(P)}$ (where $\partial^2M_n^{(P)}/\partial T^2|_{T_n^{*(P)}}=0$) approaches $T_c^{(P)}$ as $n\to\infty$ according to

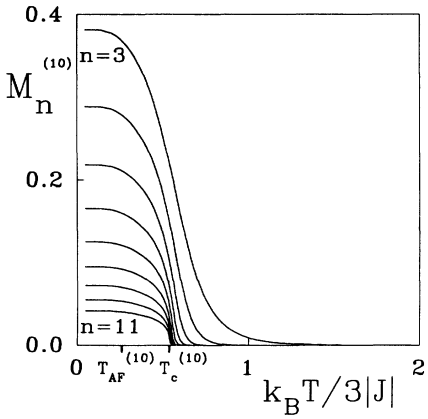

FIG. 6. The order parameter per site $M_n^{(10)}$ as function of temperature for consecutive levels n, $3 \le n \le 11$, of the HL with $P=10$ branches. $T_c^{(10)}$ and $T_{AF}^{(10)}$ correspond, respectively, to the critical and attractor temperatures in the $n \to \infty$ limit.

$$\delta T_n^{(P)} \equiv \frac{T_n^{*(P)} - T_c^{(P)}}{T_c^{(P)}} \sim A^*(P) L_n^{-1/\nu^{(P)}} , \tag{33}$$

where $\nu^{(P)}$ has the value obtained in Fig. 3 for the correlation length critical exponent. $T_n^{*(P)}$ plays, therefore, the role of the rounding temperature, which appears in the finite-size scaling theory (see, e.g., Ref. 28) as the temperature at which the plot of a given quantity for a finite size departs significantly from the corresponding thermodynamical limit.

Combining expressions (32) and (33) we obtain the following asymptotic behavior for $M_n^{(P)}$ in the neighborhood of T_c:

$$M_n^{(P)}(T_n^{*(P)}) \sim B(P)(\delta T_n^{(P)})^{\beta^{(P)}} , \tag{34}$$

where the numerical estimate for $\beta^{(P)}$ is

$$\beta^{(P)} = \nu^{(P)} \theta(T_c^{(P)}, P) , \tag{35}$$

which gives for $P=10$ the approximate value of 0.982. Combining Eq. (34) with the recursive equation for $D_n^{(P)}$ [Eq. (28)] we obtain the following exact value for $\beta^{(P)}$

$$\beta^{(P)} = \frac{\ln\{3/[1+2(b_c-a_c)]\}}{\ln r_c^{(P)}} \quad (\beta^{(10)} = 0.982\ldots) , \tag{36}$$

which agrees very well with the numerical estimate [Eq. (35)]. In Fig. 3 we show the plot of the exact value of $\beta^{(P)}$ as function of $d_f(P)$. The asymptotic behavior of β in the neighborhood of d_c is $\beta^{(P)} \sim [0.342\cdots](d_f(P) - d_c)^{-0.387\ldots}$.

B. Internal energy and specific heat

The calculation of the internal energy of the n level HL with P branches involves, according to Eq. (1), the thermal average of $\delta(\sigma_i, \sigma_j)$ over all the spins at the n level. Since this average is the same for all bonds [Eq. (17)], it follows that the dimensionless internal energy per bond at the n level $E_{bn}^{(P)}$ is given by

$$E_{bn}^{(P)} = \frac{\langle \mathcal{H}_n^{(P)} \rangle_{n,P}}{3|J_n^{(P)}| N_{bn}^{(P)}} = \langle \delta(\sigma_i, \sigma_j) \rangle_{n,P} , \tag{37}$$

where $N_{bn}^{(P)}$ is the number of bonds in the n level HL with P branches. $E_{bn}^{(P)}$ satisfies [see Eq. (17)] the following recurrence equation:

$$E_{bn}^{(P)} = e_n E_{bn-1}^{(P)} + f_n . \tag{38}$$

Successive iterations of the above equation leads to the following exact expression for the dimensionless internal energy per site in the fractal limit

$$E_s^{(P)} \equiv \lim_{n\to\infty} \frac{\langle \mathcal{H}_n^{(P)} \rangle_{n,P}}{3|J_n^{(P)}| N_{sn}^{(P)}} = \frac{(3P-1)}{2P} \left[\sum_{i=1}^{\infty} f_i \prod_{j=i+1}^{\infty} e_i + f_\infty \right] . \tag{39}$$

The dimensionless specific heat per site in the HL with P branches $C_s^{(P)}$ defined by

$$C_s^{(P)} \equiv \lim_{n\to\infty} \frac{1}{3k_B N_{sn}^{(P)}} \frac{\partial \langle \mathcal{H}_n^{(P)} \rangle_{n,P}}{\partial T} \tag{40}$$

can be obtained by differentiating the recurrence for $E_{bn}^{(P)}$ [Eq. (38)] and iterating it successively, leading to the plots shown in Fig. 7.

The absence of latent heat and the continuity of the order parameter at the critical temperature shows that the transition is continuous. Similarly to the Kosterlitz-Thouless phase (see, e.g., Ref. 29), the specific heat does not diverge at this temperature.

The maximum of the specific heat occurs at a temperature $T_{\max}^{(P)}$ lower than the critical temperature $T_c^{(P)}$. A similar behavior is found in the $q=3$ anisotropic Potts model with ferromagnetic and antiferromagnetic interactions in the two respective directions of the square lattice[30] (whose ground-state degeneracy is infinite) and in the F model of an antiferroelectric[31] (which exhibits an unconventional type infinite-order transition).

Assuming a critical behavior for the energy in the

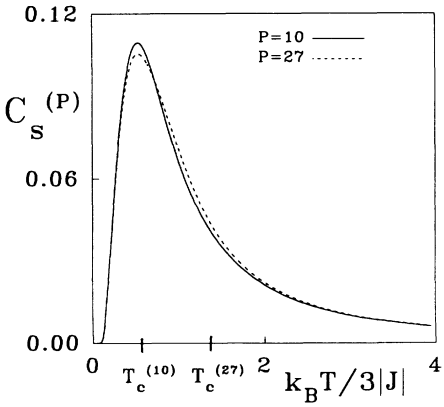

FIG. 7. The dimensionless specific heat per site $C_s^{(P)}$ of the HL's with $P=10$ (full line) and $P=27$ (broken line) branches as functions of temperature.

neighborhood of the critical temperature like $E_{bn}^{(P)} \sim (\delta t_n^{(P)})^{\sigma(P)}$ the recurrent Eq. (38) leads to the following exact expression for the specific-heat critical exponent $\alpha^{(P)}$

$$\alpha^{(P)}=1-\sigma^{(P)}=1+\frac{\ln e_c}{\ln r_c^{(P)}} . \tag{41}$$

On the other hand, it follows from the definition of $e_c^{(P)}$ as function of $t_c^{(P)}$ [see Eqs. (16) and (18)] and from Eq. (3) that

$$r_c^{(P)}=3Pe_c^{(P)} . \tag{42}$$

Combining Eqs. (1), (5), (41), and (42) we derive that

$$d_f(P)\nu^{(P)}=2-\alpha^{(P)} , \tag{43}$$

which *proves* that the hyperscaling law for Bravais lattices continues to be valid (replacing the spatial dimension by the fractal one) for the whole family of HL's considered herein. Such law has already been numerically verified for the ferromagnetic Ising[32,17] and Potts models[18] on a number of HL's.

Although there is no proof that the scaling laws valid for bravais lattices continue to hold for fractals, if we assume that the Rushbrooke scaling relation [which, in fact, has been verified for the Ising ferromagnet on the Wheatstone-bridge HL (Ref. 33)] and the Fisher one are valid for the considered family of HL's, then one would obtain a value for the η critical exponent at t_c which is different from $\eta_{\rm AF}^{(P)}$ shown in Fig. 3. This would lead to a discontinuity of η at the critical temperature (for example, for $P=10$, $\eta_c \simeq -0.38$, and $\eta_{\rm AF} \simeq -0.51$).

C. Entropy

Following a procedure similar to Ref. 34, we obtained an exact expression for the entropy per site $S_s^{(P)}$ given by

$$S_s^{(P)}(T)=\frac{k_B(3P-1)}{2}\sum_{i=0}^{\infty}\frac{1}{(3P)^{i+1}}\left\{\ln[3(1+\lambda_i+\lambda_i^2)]-\lambda_i\ln\lambda_i\left[\frac{1+2\lambda_i}{1+\lambda_i+\lambda_i^2}\right]\right\} , \tag{44}$$

where λ_i is the ratio of restricted partition functions ($\lambda_i \equiv P_i/Q_i$ in the notation of Ref. 34), which satisfies the following recurrence equation:

$$\lambda_{i+1}=\left[\frac{2+6\lambda_i+\lambda_i^3}{3(1+\lambda_i+\lambda_i^2)}\right]^P \quad [\lambda_0 \equiv \exp(3J_\infty^{(P)}/k_BT)] . \tag{45}$$

If one uses the thermodynamical relationship $C_s^{(P)}(T)=T\partial S_s^{(P)}/\partial T$ one would recover the plots shown in Fig. 7, as expected. From Eq. (44) we obtain the following analytic expression for the residual entropy s_0:

$$s_0^{(P)} \equiv S_s^{(P)}(T=0)=\frac{k_B(3P-1)}{2}\sum_{i=0}^{\infty}\frac{\ln[3(1+\lambda_i+\lambda_i^2)]}{(3P)^{i+1}} \tag{46}$$

whose value for $P=10$ is 0.5496..., which corresponds to approximately 50% of the entropy at the $T\to\infty$ limit [$S_s(T=\infty)=\ln 3$]. Notice that this is a relatively big value if we compare it with the $q=3$ AF Potts model on the simple cubic lattice where it is found that there is long-range order and that $s_0 \simeq 0.34 S_s(T\to\infty)$.[27] Such large residual entropy is usually characteristic of models with no long-range order as, for example, the Ising AF on the triangular lattice,[2] where $s_0 \simeq 0.49 S_s(T\to\infty)$.

VI. CONCLUSIONS

We present an exactly soluble model defined in a family of diamond hierarchical lattices, which has an unusual phase like that one suggested by Berker and Kadanoff for complex systems with nonzero residual entropy per spin. In this phase, the pair-correlation function decays algebraically with a temperature-independent η exponent. We calculate, in an exact way, the multifractal local magnetizations and a number of thermodynamical functions together with their corresponding critical exponents.

We prove that the BSS order parameter vanishes for all range of temperatures. We believe that the vanishing of this order parameter along the unusual phase is due mainly to the multiplicity of spin configurations, which generates a tendency towards equiprobability for the three states. We also prove the hyperscaling law for the studied family of fractals.

As far as we know, there has been no report in the literature concerning: (i) *exact* calculations of any thermodynamical quantity along the mentioned unusual phase, in particular, of the order parameter, (ii) the *temperature dependence* of the $f(\alpha)$ spectrum characterizing the multifractality of the local average magnetizations—the $f(\alpha)$ calculated at the critical temperature differs from that at the unusual phase, (iii) the *discontinuity* at the critical temperature of the critical exponent η, which was derived assuming that certain scaling laws continue to hold for this family of fractal systems.

Finally, we would like to point out the found similarities and differences between the studied systems and the ferromagnetic XY model on the square lattice (see, e.g., Refs. 14 and 29). The similarities are (i) the algebraic decay of correlations along the low-temperature phase, (ii) the vanishing of the order parameter, (iii) the absence of divergence in the specific heat. The differences are (i) in the XY model η varies continuously with the temperature, while in our family of systems, if the scaling laws are valid, η jumps to a different value at T_c, (ii) the correlation length ξ diverges, as T_c is approached from above, as an exponential law in the XY model in contrast to the power law obtained in the considered fractal family, (iii) the peak of the specific heat in the XY model occurs above the critical temperature, contrarily to what hap-

pens in the studied family of systems. As the considered fractal family is somewhat similar to the bidimensional XY model, it would be interesting to look for a certain type of topological excitations, the vortices, which at low-temperatures occur for the XY model in tightly bound pairs that unbind at the critical temperature. Although these vortices were initially defined for a continuous spin model, Kolafa[35] extended them to the three-state Potts antiferromagnet. The search for these vortices in the considered family of HL's would require the use of another technique, as, for example, Monte Carlo simulations, since the symmetry-breaking condition used in our recursive method does not allow us to distinguish the state 1 from the state 2.

ACKNOWLEDGMENTS

We acknowledge Constantino Tsallis for useful remarks. We thank the Brazilian agency Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for financial support.

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