Strong-coupling corrections to the Bardeen-Cooper-Schrieffer ratios for a d-wave superconductor

Hongguang Chi and J. P. Carbotte
Physics Department, McMaster University, Hamilton, Ontario, Canada L8S 4M1
(Received 27 September 1993)

We have calculated, for a d-wave superconductor, the strong-coupling corrections to the Bardeen-Cooper-Schrieffer (BCS) ratios: $\Delta C(T_c)/\gamma T_c, 2\Delta_0/T_c, \gamma T_c^2/H_c^2(0), h_c(0)$ as well as the ratios for the normalized slope in the specific-heat jump at T_c and London penetration depth. We have used a step-function approximation for the gap function and considered an Einstein spectral density for the bosons. We have obtained terms up to $O(T_c^4/\omega_E^4)$ (here T_c/ω_E is the strong-coupling parameter and ω_E is the Einstein energy). While our results are approximate for the strong-coupling cases, the weak-coupling results are exact. These weak-coupling ratios are no longer universal and depend on the d-wave considered. Our formulas are also applicable for some extended s-wave superconductors which have a basis function with a zero average over the Fermi surface.

I. INTRODUCTION

It is now commonly believed[1–5] that some heavy-fermion superconductors (e.g., UPt_3) are d-wave superconductors. There has been some evidence that the high-temperature oxide superconductors are also d-wave superconductors.[6–9] In the present study, we calculate the strong-coupling corrections to the Bardeen-Cooper-Schrieffer (BCS) ratios[10] such as $2\Delta_0/T_c$, $\Delta C(T_c)/\gamma T_c$, $\gamma T_c^2/H_c^2(0)$, etc., for a d-wave superconductor. These ratios are universal in the BCS theory of an isotropic strong-coupling superconductor and have been studied extensively.[11–18] Among these studies, the imaginary-axis approach of Marsiglio and Carbotte[16] is especially convenient. These authors showed quite amazingly that even for a strong-coupling isotropic superconductor, universal formulas for all these ratios with only a single strong-coupling parameter T_c/ω_{ln} (where ω_{ln} is the Allen-Dynes expression for the average phonon energy) can be obtained by fitting the derived analytic expressions [up to $O(T_c^2/\omega_{ln}^2)$ terms] to the experimental and numerical data. For a d-wave superconductor, these ratios are no longer universal even in the weak-coupling limit. One expects that in addition to the variable T_c/ω_{ln} the various ratios will also depend on the d wave considered. More recently, Millis, Sachdev, and Varma[19] and Williams and Carbotte[20] have calculated some of the properties of a d-wave superconductor stabilized by antiferromagnetic spin fluctuations. In this model the anisotropy is kept only in the numerator of the Eliashberg equations. Schachinger and Carbotte[21] have studied the dependence of the jump in the specific heat and the slope in the specific heat at T_c on the strong-coupling ratio T_c/ω_E (ω_E is the Einstein frequency for the boson mode), by solving the corresponding Eliashberg equations for a d-wave superconductor. They found large strong-coupling corrections to these two quantities. One of their interesting results is the nonmonotonic dependence of the slope in the specific heat at T_c on the strong-coupling ratioT_c/ω_E, which is qualitatively different from that of an isotropic superconductor.

In the present study, we follow the imaginary-axis approach[16] with the step-function approximation to the gap. For brevity, we have used a simple Einstein spectrum for the boson responsible for the superconductivity. Our objective is to understand the general dependence of these ratios on the strong-coupling variable T_c/ω_E and on the character of the d wave. We have obtained corrections up to $O(T_c/\omega_E)^4$ terms. The reader should note that implied in our use of the Eliashberg equations is the assumption that the system in its normal state has settled into a Fermi-liquid phase. This may not necessarily be the case. If not, the correct procedure to follow would depend on the nature of this new state and the usual small parameter T_c/T_F (where T_F is the Fermi energy) would need to be replaced by something else. In Sec. II we consider first the specific-heat jump and the slope at T_c. In Sec. III the correction to the gap-T_c ratio is studied. More ratios are calculated in Sec. IV. Conclusions are given in Sec. V.

II. STRONG-COUPLING CORRECTION TO $\Delta C(T_c)/\gamma T_c$

We begin with the Eliashberg equations written on the imaginary-frequency axis ($k_B=\hbar=1$),

$$\Delta_{\mathbf{k}}(\omega_n)Z_{\mathbf{k}}(\omega_n) = -\pi T \sum_m \left\langle \lambda_{\mathbf{k}\mathbf{k}'}(m-n)\frac{\Delta_{\mathbf{k}'}(\omega_m)}{[\omega_m^2+\Delta_{\mathbf{k}'}^2(\omega_m)]^{1/2}}\right\rangle , \quad (1)$$

$$Z_{\mathbf{k}}(\omega_n)=1+\frac{\pi T}{\omega_n}\sum_m \left\langle \lambda_{\mathbf{k}\mathbf{k}'}(m-n)\frac{\omega_m}{[\omega_m^2+\Delta_{\mathbf{k}'}^2(\omega_m)]^{1/2}}\right\rangle , \quad (2)$$

where $\Delta_{\mathbf{k}}(\omega_n)$ are the gaps and $Z_{\mathbf{k}}(\omega_n)$ are the renormalization factors defined at the Matsubara frequencies

$$\omega_n=\pi T(2n+1), \quad n=0,\pm 1,\pm 2,\ldots . \quad (3)$$

$\langle \cdots \rangle$ stands for the average taken over the Fermi surface S_F, i.e.,

$$\langle f(\mathbf{k})\rangle = \int_{S_F} \frac{dS_{\mathbf{k}}}{|v_{\mathbf{k}}|} f(\mathbf{k}) \Big/ \int_{S_F} \frac{dS_{\mathbf{k}}}{|v_{\mathbf{k}}|} . \tag{4}$$

Further, $\lambda_{\mathbf{kk}'}(m-n)$ has the form

$$\lambda_{\mathbf{kk}'}(m-n) = -\eta_{\mathbf{k}}\eta_{\mathbf{k}'}\lambda(m-n) , \tag{5}$$

with $\eta_{\mathbf{k}}$ as the basis function of the d wave and $\lambda(m-n)$ depending on the spectral density $A(\nu)$ through the relation

$$\lambda(m-n) = \int_0^\infty \frac{2\nu A(\nu)d\nu}{\nu^2+(\omega_m-\omega_n)^2} . \tag{6}$$

The basis function $\eta_{\mathbf{k}}$ satisfies the conditions

$$\langle \eta_{\mathbf{k}} \rangle = 0 , \tag{7}$$

$$\langle \eta_{\mathbf{k}}^2 \rangle = 1 . \tag{8}$$

It may be mentioned that some extended s-wave bases also satisfy Eqs. (7) and (8).

To make the analytic calculation possible, we follow the step-function approximation[14]

$$\Delta_{\mathbf{k}}(\omega_n) = \begin{cases} \Delta_0\eta_{\mathbf{k}} & \text{if } |\omega_n| < \omega_0, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Here ω_0 represents the maximum boson frequency in the system. We restrict our spectra to those in which the important boson frequencies are much higher than T_c and much less than ω_0. This allows an expansion in the strong-coupling parameter T/ν, which becomes T/ω_E when an Einstein spectral density is used.

For T near T_c, using Eqs. (5), (7), and (9), Eq. (2) becomes

$$Z_{\mathbf{k}}(\omega_n) = 1 + \frac{\pi T}{\omega_n}\eta_{\mathbf{k}} \sum_m \lambda(m-n)\omega_m \times \left[\frac{\alpha_3}{2}\frac{\Delta_0^2}{|\omega_m|^3} - \frac{3\alpha_5}{8}\frac{\Delta_0^4}{|\omega_m|^5} \right] , \tag{10}$$

with α_3 and α_5 defined as

$$\alpha_i = \langle \eta_{\mathbf{k}}^i \rangle , \tag{11}$$

$$\alpha_{il} = \langle \eta_{\mathbf{k}}^i \ln|\eta_{\mathbf{k}}| \rangle . \tag{12}$$

α_{il} will be needed later. After some lengthy calculations as shown in Appendix A, for an Einstein spectral density Eq. (1) becomes

$$1 = F(T) + G(T)\Delta_0^2 + J(T)\Delta_0^4 , \tag{13}$$

where

$$F(T) = \lambda[\varepsilon + (\tfrac{4}{3}-\varepsilon)u^2 + (\varepsilon - \tfrac{32}{15})u^4] , \tag{14}$$

$$G(T) = -\frac{\lambda\alpha_4}{2\gamma_1(\pi T)^2}\{1-(1+\gamma_1\varepsilon)u^2 + [\gamma_1(\tfrac{31}{12}-6\varepsilon)-1]u^4\} , \tag{15}$$

$$J(T) = \frac{\lambda\alpha_6}{\gamma_2(\pi T)^4}\left[1 - \left[\frac{3\gamma_2}{8\gamma_1}+1\right]u^2 + \left[\frac{3\gamma_2}{8}(\varepsilon+2\gamma_1^{-1}) + \frac{3\gamma_2}{2\gamma_1}+1\right]u^4\right] , \tag{16}$$

where ε and u are defined as

$$\varepsilon = \ln\frac{1.13\pi}{u} = \ln\frac{1.13\omega_E}{T} , \tag{17}$$

$$u = \frac{\pi T}{\omega_E} , \tag{18}$$

and

$$\gamma_1 = \frac{4}{7\xi(3)} = 0.4754 , \tag{19}$$

$$\gamma_2 = \frac{128}{93\xi(5)} = 1.327 . \tag{20}$$

Here $\xi(n)$ is the Riemann zeta function. In the derivation of Eqs. (13)–(16), $O((T/\omega_E)^6)$ and higher-order terms have been neglected.[22]

To calculate the specific-heat jump, we use the Bardeen-Stephen formula for the free energy,[23]

$$\frac{\Delta F}{N(0)} = -\pi T \sum_m \left\langle \{[\omega_m^2 + \Delta_{\mathbf{k}}^2(\omega_n)]^{1/2} - |\omega_m|\} \left[Z_{\mathbf{k}}(\omega_m) - Z_{\mathbf{k}}^N(\omega_m)\frac{|\omega_m|}{[\omega_m^2+\Delta_{\mathbf{k}}^2(\omega_m)]^{1/2}} \right] \right\rangle . \tag{21}$$

Here $N(0)$ is the density of single-particle states at the Fermi level in the normal phase; $Z_{\mathbf{k}}^N(\omega_m)$ is the renormalization factor for the normal state. From Appendix A, we have $Z_{\mathbf{k}}(\omega_n) = Z_{\mathbf{k}}^N(\omega_n) = 1$. With the help of Eqs. (9) and (11), Eq. (21) becomes

$$\frac{\Delta F}{N(0)} = -\tfrac{1}{2}[\alpha_4 C_2(T)\Delta_0^4 - \tfrac{4}{3}\alpha_6 C_3(T)\Delta_0^6] , \tag{22}$$

with

$$C_2(T) = \frac{7\xi(3)}{8(\pi T)^2} , \tag{23}$$

$$C_3(T) = \frac{93\xi(5)}{128(\pi T)^4} , \tag{24}$$

where higher-order terms have been neglected. Using Eq. (13), we obtain that

$$\Delta_0^2(T) = A_1 S + A_2 S^2 , \tag{25}$$

with

$$A_1 \equiv T_c |d\Delta_0^2/dT|_{T_c} = T_c F'/G , \tag{26}$$

$$A_2 \equiv \tfrac{1}{2}T_c^2 |d^2\Delta_0^2/dT^2|_{T_c} = -\frac{T_c^2 F'}{G}(F''/2F' - G'/G + F'J/G^2) , \tag{27}$$

$$S = 1 - T/T_c . \tag{28}$$

With the help of Eq. (25), Eq. (22) can be rewritten as

$$F_S-F_N=-\frac{N(0)}{2}(\alpha_4C_2^0A_1^2S^2+[2\alpha_4C_2^0A_1A_2+\alpha_4C_2'A_1^2-4\alpha_6C_3^0A_1^3/3]S^3)\ . \tag{29}$$

Here the superscript 0 means that it is calculated at T_c. From Eq. (29), the normalized specific-heat jump near T_c becomes

$$\Delta C(T_c)/\gamma T_c=-\frac{T}{\gamma T_c}\frac{d^2}{dT^2}\Delta F = f-q(1-T/T_c)\ , \tag{30}$$

with

$$f=(\gamma T_c^2)^{-1}N(0)\alpha_4C_2^0A_1^2\ , \tag{31}$$

$$q=\frac{3N(0)}{\gamma T_c^2}\left[\frac{4}{3}\alpha_6C_3^0A_1^3-\frac{5}{3}\alpha_4C_2^0A_1^2-2\alpha_4C_2^0A_1A_2\right]\ . \tag{32}$$

The quantity A_1 defined in Eq. (26) can be calculated from Eqs. (14) and (15) as

$$A_1=\frac{1}{C_2^0\alpha_4}\{1+[(2+\gamma_1)\varepsilon-8/3]u^2+S_8u^4\}\ , \tag{33}$$

with

$$S_8=\gamma_1(\gamma_1+2)\varepsilon^2-(\tfrac{23}{3}\gamma_1+2)\varepsilon+\tfrac{31}{12}\gamma_1+\tfrac{88}{15}\ . \tag{34}$$

Then the normalized specific-heat jump at T_c, $f\equiv\Delta C(T_c)/\gamma T_c$, is calculated from Eq. (31) as

$$f=\frac{1.43}{\alpha_4}[1+(48.9\varepsilon-52.6)(T_c/\omega_E)^2+(8.7\varepsilon^2-23.5\varepsilon+22.3)\pi^4(T_c/\omega_E)^4]\ . \tag{35}$$

The normalized slope of the specific heat at T_c,

$$g\equiv T_c\frac{d}{dT}\Delta(T_c)/\Delta C(T_c)=q/f\ ,$$

can be calculated from Eqs. (26), (27), (31), and (32); we obtain

$$g=2(2-\chi)\left[1+\pi^2\frac{8.85\varepsilon-9.43+(5.81-3.90\varepsilon)\chi}{2-\chi}(T_c/\omega_E)^2 -\pi^4\frac{(5.39\varepsilon^2-24.36\varepsilon+34.87)\chi+10.64\varepsilon^2+3.2\varepsilon-27.73}{2-\chi}(T_c/\omega_E)^4\right]\ . \tag{36}$$

Here χ, which is d-wave dependent, is defined as

$$\chi=\frac{\gamma_1^2}{\gamma_2}\frac{\alpha_6}{\alpha_4^2}=0.681\frac{\alpha_6}{\alpha_4^2}\ . \tag{37}$$

From Eq. (35) we note that the d-wave dependence of the jump at T_c appears only in the prefactor. Then the specific-heat jumps at T_c for different d waves will be proportional to each other. On the other hand, the slope of the specific heat at T_c, g, has a complex dependence on the d wave. The corresponding weak-coupling values are obtained by putting $T_c/\omega_E=0$, i.e.,

$$f|_{\rm WC}=1.43/\alpha_4\ ,$$

$$g|_{\rm WC}=2(2-\chi)=2(2-0.681\alpha_6/\alpha_4^2).$$

$f|_{\rm WC}=1.43/\alpha_4$ was obtained by Pokrovskii,[24] and $|_{\rm WC}$ denotes the weak-coupling limit.

To illustrate the dependence of these ratios on the strong-coupling parameter T_c/ω_E and the d wave, we consider the following two examples.

A. d-wave superconductor

The basis function is given by $n_{\mathbf{k}}=\sqrt{15/4}\sin^2\theta\cos2\phi$ (i.e., $\hat{k}_x^2-\hat{k}_y^2$). We consider a spherical Fermi surface. Then the various moments are easily calculated as $\alpha_4=\frac{15}{7}$, $\alpha_6=5.62$, and $\alpha_3=\alpha_5=0$. The other averages which will be needed later on are $\alpha_{2l}=0.287$, $\alpha_{4l}=0.956$, and $\langle|\eta_{\mathbf{k}}|\rangle=5.253$.

B. Extended s-wave superconductor

For concreteness we consider, as a possibility, an organic superconductor consisting of one-dimensional (1D) chains. A possible choice of $\eta_{\mathbf{k}}$ is $\sqrt{2}\cos k_y$, and k_y is the momentum in the direction perpendicular to the chains as in the work of the Suzumura and Schulz.[25] The average becomes $\langle\eta_{\mathbf{k}}^i\rangle=\pi^{-1}\int_0^\pi dk_y\eta_{\mathbf{k}}^i$. The various values of the average are calculated as $\alpha_4=1.5$, $\alpha_6=2.5$, $\alpha_3=\alpha_5=0$, $\alpha_{2l}=0.150$, $\alpha_{4l}=0.355$, and $\langle|\eta_{\mathbf{k}}|\rangle=0.9$.

In Figs. 1 and 2, we have plotted f and g as a function of the strong-coupling parameter T_c/ω_E for the d-wave (solid curve) and the extended s-wave (dashed curve) superconductors described above. One notes that f increases monotonically from the weak-coupling values 0.67 for the d wave and 0.96 for the extended s wave, which are similar to that of an isotropic superconductor[16] and agree qualitatively with the numerical calculation of Schachinger and Carbotte.[21] The results for the slope g are very interesting in that they start from the weak-coupling values 2.33 for the d wave and 2.5 for the extended s wave, and then increase initially and show a

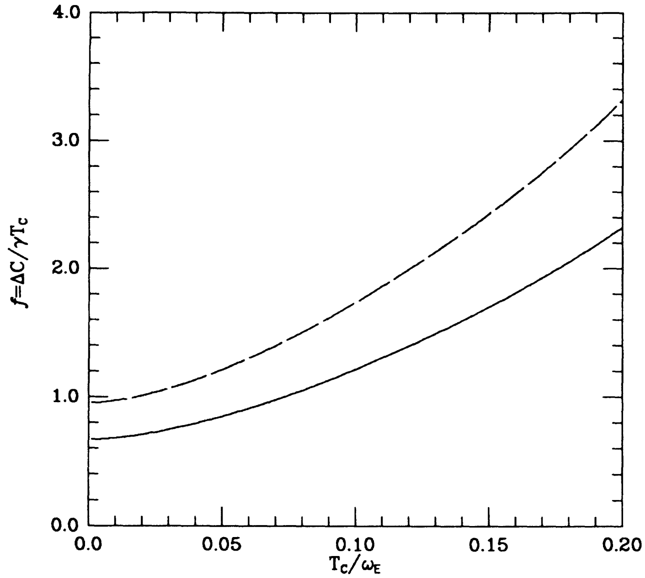

FIG. 1. Normalized specific-heat jump at T_c, $f \equiv \Delta C(T_c)/\gamma T_c$, as a function of the strong-coupling parameter T_c/ω_E. The solid curve is for a d-wave superconductor with a basis function $\eta_{\mathbf{k}}=\sqrt{15/4}\sin^2\theta\cos 2\phi$ ($\sim \hat{k}_x^2-\hat{k}_y^2$), and the dashed curve is for the extended s-wave superconductor with a basis $\sqrt{2}\cos k_y$ in the case of an organic superconductor. For details about the two bases, see Sec. II.

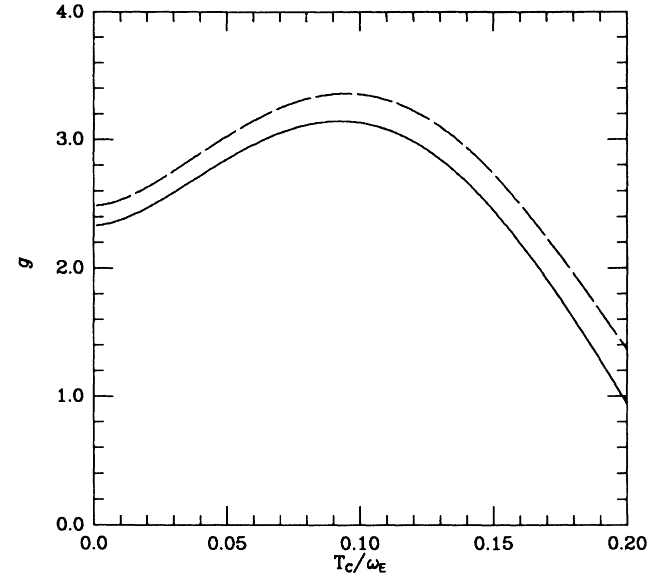

FIG. 2. Normalized slope in the specific-heat jump at T_c as a function of the strong-coupling parameter T_c/ω_E for the d-wave (solid curve) and extended s-wave (dashed curve) superconductors.

maximum. This feature is qualitatively different from that of an isotropic superconductor.[16] Our results agree with the numerical calculation of Ref. 21.

III. STRONG-COUPLING CORRECTION TO THE GAP-T_c RATIO

At zero temperature, we have the well-known replacement[26] $\omega_n \to \omega$, $\omega_m \to \omega'$, $2\pi T\sum_m \to \int_{-\infty}^{\infty} d\omega$; then, Eqs. (1) and (2) can be rewritten, by using Eq. (9), as

$$Z_{\mathbf{k}}(\omega)=1-\frac{1}{2\omega}\int_{-\infty}^{\infty} d\omega'\,\lambda(\omega'-\omega)\eta_{\mathbf{k}}\omega' \times\left\langle \frac{\eta_{\mathbf{k}'}}{(\omega'^2+\Delta_0^2\eta_{\mathbf{k}'}^2)^{1/2}}\right\rangle', \tag{38}$$

$$Z_{\mathbf{k}}(\omega)\Delta_0=\frac{1}{2}\int_{-\infty}^{\infty} d\omega'\,\lambda(\omega'-\omega)\left\langle \frac{\eta_{\mathbf{k}'}^2\Delta_0}{(\omega'^2+\Delta_0^2\eta_{\mathbf{k}'}^2)^{1/2}}\right\rangle' . \tag{39}$$

For an Einstein spectral density, $A(\nu)=A\delta(\nu-\omega_E)$; then, $\lambda(\omega'-\omega)$ defined in Eq. (16) becomes

$$\lambda(\omega'-\omega)=\lambda\frac{\omega_E^2}{\omega_E^2+(\omega'-\omega)^2}, \tag{40}$$

with

$$\lambda=2A/\omega_E . \tag{41}$$

Using Eqs. (38)–(40), an equation for Δ_0 is derived in Appendix B; it reads

$$1=\lambda[\ln 2\omega_E/\Delta_0-\alpha_{2l} +\tfrac{1}{2}(\alpha_4\ln 2\omega_E/\Delta_0-\alpha_{4l}-\alpha_4/2)(\Delta_0/\omega_E)^2 +\tfrac{3}{8}(\alpha_6\ln 2\omega_E/\Delta_0-\alpha_{6l}-7\alpha_6/12)(\Delta_0/\omega_E)^4] . \tag{42}$$

Here α_i and α_{il} are defined in Eqs. (11) and (12), respectively, and T_c is determined from Eq. (13) with $\Delta_0=0$, i.e.,

$$\lambda[\varepsilon+(\tfrac{4}{3}-\varepsilon)u^2+(\varepsilon-\tfrac{32}{15})u^4]=1 , \tag{43}$$

where ε and u are defined in Eqs. (17) and (18), respectively. The gap-T_c ratio can be solved from Eqs. (42) and (43) by repeated iterations and the result is

$$\frac{2\Delta_0}{T_c}=3.53\exp(-\alpha_{2l})[1+a_3\ln(\omega_E/b_3T_c)(T_c/\omega_E)^2+A_3(T_c/\omega_E)^4] , \tag{44}$$

where

$$a_3=\pi^2+\frac{3.53^2}{8}\alpha_4\exp(-2\alpha_{2l}) , \tag{45}$$

$$b_3=\frac{1}{1.13}\exp\left[\frac{(\alpha_4/2-\alpha_{2l}\alpha_4+\alpha_{4l})(3.53^2/8)e^{-2\alpha_{2l}}+4\pi^2/3}{a_3}\right] , \tag{46}$$

$$A_3=\tfrac{1}{2}[\pi^2(\varepsilon-\tfrac{4}{3})+\beta_1^2\beta_2]^2+\tfrac{1}{8}\beta_1^4\beta_2^2+\tfrac{1}{2}\beta_1^2\beta_2\pi^2(\varepsilon-\tfrac{4}{3})+\pi^4(\tfrac{32}{15}-\varepsilon)+\tfrac{1}{2}\beta_1^2\pi^2\alpha_4(\tfrac{4}{3}-\varepsilon)$$
$$+\tfrac{3}{8}\beta_1^4[\alpha_6(\varepsilon+\alpha_{2l})-\alpha_{2l}-7\alpha_6/12]-\tfrac{1}{4}\alpha_4\beta_1^4\beta_2\ , \tag{47}$$

with

$$\beta_1=1.76e^{-\alpha_{2l}}\ , \tag{48}$$

$$\beta_2=\alpha_4\varepsilon+\alpha_4\alpha_{2l}-\alpha_{4l}-\alpha_4/2\ . \tag{49}$$

From Eq. (44) one notes that the weak-coupling value of the ratio $2\Delta_0/T_c$ for a d-wave superconductor is given by

$$2\Delta_0/T_c|_{\rm WC}=3.53e^{-\alpha_{2l}}\ ,$$

which is always less than 3.53 for an isotropic superconductor. The dependence of $2\Delta_0/T_c$ on the d wave (through α_4, α_6, α_{2l}, and α_{4l}) is quite complicated. In Fig. 3, we have plotted $2\Delta_0/T_c$ as a function of the strong-coupling parameter T_c/ω_E for the d wave and extended s wave described in Sec. II. We note that in both cases $2\Delta_0/T_c$ increases monotonically as the coupling strength is increased. The degree of enhancement of $2\Delta_0/T_c$ is comparable with that for an isotropic superconductor.[16]

IV. MORE STRONG-COUPLING CORRECTIONS

In this section, we calculate three other ratios for a d-wave superconductor.

A. $\gamma T_c^2/H_c^2(0)$

To determine the correction to $\gamma T_c^2/H_c^2(0)$, $H_c(0)$ as the critical magnetic field at zero temperature, we first calculate the free-energy difference between the normal and superconducting phases. At $T=0$, Eq. (21) becomes

$$\frac{\Delta F}{N(0)}=-\left\langle\int_0^{\omega_E}d\omega[(\omega^2+\Delta_0^2\eta_{\mathbf{k}}^2)^{1/2}+\omega^2(\omega^2+\Delta_0^2\eta_{\mathbf{k}}^2)^{-1/2}-2\omega]\right\rangle$$
$$=-\frac{1}{2}\Delta_0^2\left[1-\frac{\alpha_4}{4}(\Delta_0/\omega_E)^2+\frac{\alpha_6}{8}(\Delta_0/\omega_E)^4\right]\ , \tag{50}$$

where we have used the fact that $Z_{\mathbf{k}}^N(\omega_n)=Z_{\mathbf{k}}(\omega_n)=1$ (see Appendix B) and Eq. (11). From Eq. (50), $H_c(0)$ can be obtained as

$$H_c(0)=[4\pi N(0)]^{1/2}\Delta_0[1-\tfrac{1}{8}\alpha_4(\Delta_0/\omega_E)^2+\tfrac{1}{16}(\alpha_6-\alpha_4^2/8)(\Delta_0/\omega_E)^4]\ . \tag{51}$$

Therefore, with Eqs. (51) and (44), $\gamma T_c^2/H_c^2(0)$ becomes

$$\frac{\gamma T_c^2}{H_c^2(0)}=0.168e^{2\alpha_{2l}}\left[1-a_4\ln\left[\frac{\omega_E}{b_4T_c}\right](T_c/\omega_E)^2+A_4(T_c/\omega_E)^4\right]\ , \tag{52}$$

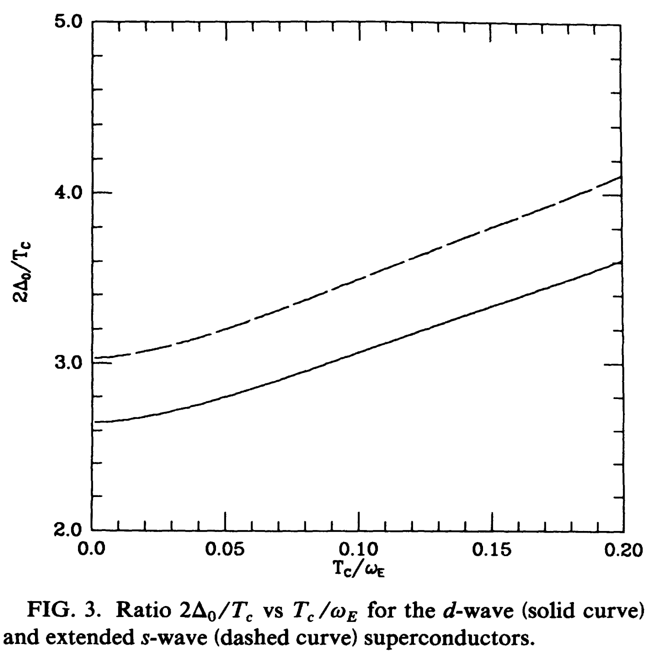

FIG. 3. Ratio $2\Delta_0/T_c$ vs T_c/ω_E for the d-wave (solid curve) and extended s-wave (dashed curve) superconductors.

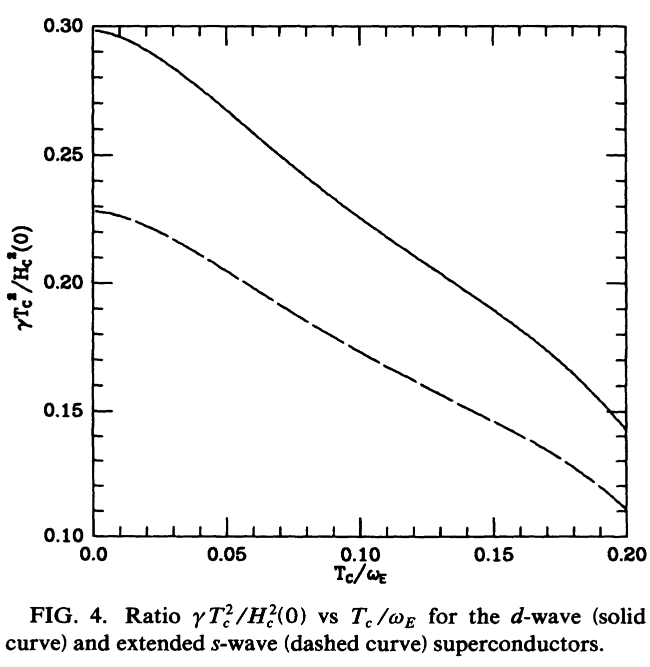

FIG. 4. Ratio $\gamma T_c^2/H_c^2(0)$ vs T_c/ω_E for the d-wave (solid curve) and extended s-wave (dashed curve) superconductors.

with

$$a_4=2a_3\ , \tag{53}$$

$$b_4=\exp\left[\frac{2a_3\ln b_3+(3.53^2/16)\alpha_4 e^{-2\alpha_{2l}}}{2a_3}\right]\ , \tag{54}$$

$$A_4=3a_3^2\left[\ln\left[\frac{\omega_E}{b_3T_c}\right]\right]^2-2A_3-\frac{1}{8}\left[\alpha_6-\frac{1}{2}\alpha_4^2\right](3.53/2)^4e^{-4\alpha_{2l}}\ . \tag{55}$$

Here one notes that in the weak-coupling limit

$$\gamma T_c^2/H_c^2(0)|_{\rm WC}=0.168\exp(2\alpha_{2l})$$

for a d-wave superconductor which is always larger than that for an isotropic superconductor, 0.168. In Fig. 4 we have plotted $\gamma T_c^2/H_c^2(0)$ versus the strong-coupling parameter T_c/ω_E for the d-wave and the extended s-wave superconductors described in Sec. II.

B. Critical magnetic field

The zero-temperature critical magnetic field $H_c(0)$ is given in Eq. (51) and its value for T near T_c can be obtained from Eq. (22) as

$$H_c(T)=\sqrt{4\pi N(0)}[\alpha_4C_2(T)\Delta_0^4-\tfrac{4}{3}\alpha_6C_3(T)\Delta_0^6]^{1/2}\ . \tag{56}$$

$T_c|H_c'(T_c)|$ can be calculated by using Eqs. (25) and (56); we have

$$h_c(0)\equiv\frac{H_c(0)}{T_c|H_c'(T_c)|}=0.576\sqrt{\alpha_4}e^{-\alpha_{2l}}\left[1-a_5\ln\left[\frac{\omega_E}{b_5T_c}\right](T_c/\omega_E)^2+A_5(T_c/\omega_E)^4\right]\ , \tag{57}$$

with

$$a_5=(\gamma_1+2)\pi^2-a_3\ , \tag{58}$$

$$b_5=\frac{1}{1.13}\exp\left[\frac{\frac{8}{3}\pi^2-(3.53^2/32)\alpha_4e^{-2\alpha_{2l}}-a_3\ln(1.13b_3)}{(\gamma_1+2)\pi^2-a_3}\right]\ , \tag{59}$$

$$A_5=l_1^2-S_8\pi^4-l_1\left[a_3\ln\left[\frac{\omega_E}{b_3T_c}\right]-\frac{1}{8}\alpha_4\beta_1^2\right]-\left[\frac{3}{8}\alpha_4\beta_1^2a_3\ln\left[\frac{\omega_E}{b_3T_c}\right]-\frac{1}{16}(\alpha_6-\alpha_4^2/8)\beta_1^4-A_3\right]\ , \tag{60}$$

and

$$l_1=(2\varepsilon+\gamma_1\varepsilon-8/3)\pi^2\ .$$

Here S_8, a_3, b_3, A_3, and β_1 are defined in Eqs. (34), (45), (46), (47), and (48), respectively.

From Eq. (56), one notes that in the weak-coupling limit,

$$h_c(0)|_{\rm WC}=0.576\sqrt{\alpha_4}e^{-\alpha_{2l}}\ ,$$

which could be either larger or smaller than the value of 0.576 for an isotropic superconductor. In Fig. 5 we have shown $h_c(0)$ against T_c/ω_E for the d wave and extended s wave used in previous sections. One notes the monotonically decreasing of $h_c(0)$ as the coupling strength is increased.

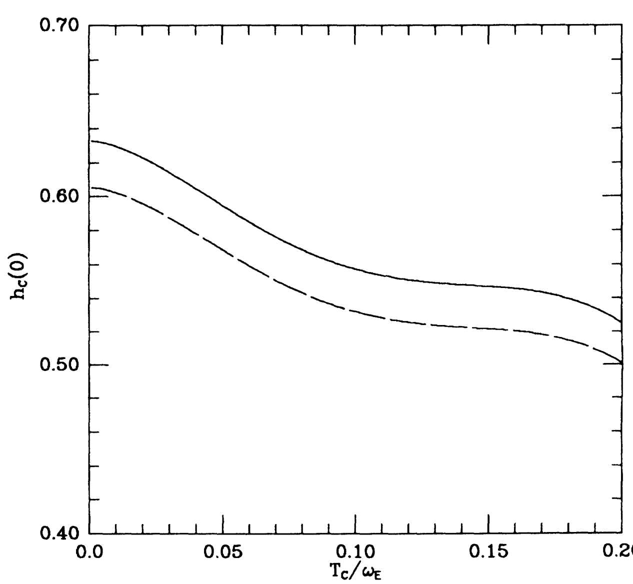

FIG. 5. Ratio $h_c(0)\equiv H_c(0)/T_c|H_c'(T_c)|$ vs T_c/ω_E for the d-wave (solid curve) and extended s-wave superconductors.

C. London limit penetration depth

To calculate the London penetration depth $\lambda_L(T)$, let us look at the response of the system to a static magnetic field, represented by the vector potential $\mathbf{A}$,

$$j_i = -\frac{c}{4\pi}\sum_j K_{ij} A_j \ .$$

The kernel K_{ij} can be calculated in terms of the single-particle Green's functions using standard many-body theory.[26] It reads

$$\frac{c}{4\pi}K_{ij} = \frac{3n_e e^2 T}{4mc}\sum_{\omega_n}\int d\Omega_{\mathbf{k}}\hat{k}_i\hat{k}_j \frac{\Delta_{\mathbf{k}}^2(\omega_n)}{[\omega_n^2+\Delta_{\mathbf{k}}^2(\omega_n)]^{3/2}} \ , \quad (61)$$

where n_e is the density of electrons and $\hat{k}_i$ is the unit vector along the ith axis. Using Eqs. (9) and (61), we have

$$Y_i = \lambda_i^{-2} = \lambda_L^{-2}\left[\frac{3}{2}T\sum_{n=0}^{\infty}\int d\Omega_{\mathbf{k}}\hat{k}_i^{\,2}\frac{\Delta_0^2\eta_{\mathbf{k}}^2}{(\omega_n^2+\Delta_0^2\eta_{\mathbf{k}}^2)^{3/2}}\right] , \quad (62)$$

with $\lambda_L^{-1} = [4\pi n_e e^2/mc]^{1/2}$. From Eq. (62), we have

$$\frac{Y_i(0)}{T_c|Y_i'(T_c)|} = \frac{\alpha_4}{6|\alpha_{2\hat{k}_i}|}\{1-(2\varepsilon+\gamma_1\varepsilon-\tfrac{8}{3})\pi^2(T_c/\omega_E)^2 + [(2\varepsilon+\gamma_1\varepsilon-\tfrac{8}{3})^2-S_8]\pi^4(T_c/\omega_E)^4\} \ , \quad (63)$$

where Eq. (33) has been used and the $\alpha_{2\hat{k}_i}$ are defined as

$$\alpha_{2\hat{k}_i} = \langle \eta_{\mathbf{k}}^2 \hat{k}_i^{\,2}\rangle \ . \quad (64)$$

$\alpha_{2\hat{k}_1} = \frac{3}{7}$ and $\alpha_{2\hat{k}_3} = \frac{1}{7}$ for the d wave described are in Sec. II. Then $[Y_1(0)/T_c|Y_1^{s\prime}(T_c)|]|_{\mathrm{WC}} = \frac{5}{6}$ and $[Y_3(0)/T_c|Y_3'(T_c)|]|_{\mathrm{WC}} = \frac{5}{2}$. As the ratios for transverse and longitudinal directions are proportional to each other, we only show the result for $Y_1(0)/T_c|Y_1'(T_c)|$ in Fig. 6.

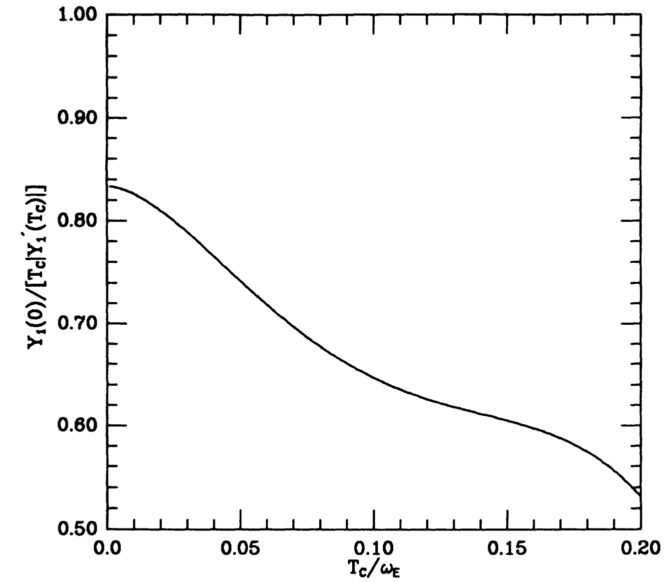

FIG. 6. Ratio $Y_1(0)/T_c|Y_c'(T_c)|$ (with $Y_1 \equiv \lambda_1^{-2}$ and λ_1 as the transverse London penetration depth for a field perpendicular to the symmetry axis of the order parameter) vs T_c/ω_E for the d-wave superconductor.

V. CONCLUSIONS

We have calculated, for a d-wave superconductor, the strong-coupling corrections to the Bardeen-Cooper-Schrieffer (BCS) ratios $f=\Delta C(T_c)/\gamma T_c$, $2\Delta_0/T_c$, $\gamma T_c^2/H_c^2(0)$, and $h_c(0)$ as well as the ratios for the normalized slope in the specific-heat jump at T_c, $g=T_c(d/dT)\Delta(T_c)/\Delta C(T_c)$, and London penetration depth. We have used a step-function approximation for the gap function and considered an Einstein spectral density for the bosons. We have obtained terms up to $O(T_c^4/\omega_E^4)$. In the weak-coupling limit, our results are exact and our formula for $\Delta C(T_c)/\gamma T_c$ agrees with the earlier works and the formulas for other ratios are new. These weak-coupling ratios are no longer universal and depend on the d wave considered. Furthermore, in the weak-coupling limit, $\Delta C(T_c)/\gamma T_c$ and $2\Delta_0/T_c$ for a d-wave superconductor are always less than that for an isotropic superconductor, while $\gamma T_c^2/H_c^2(0)$ for a d-wave superconductor is always larger than that for an isotropic one. Among the strong-coupling ratios, the d-wave dependence appears in the prefactor for the specific-heat jump at T_c and London penetration depth, and is complex for other ratios. The magnitudes of the strong-coupling corrections of these ratios are comparable with that for an isotropic superconductor. An nonmonotonic dependence of the slope in the specific heat is obtained, which agrees with the recent numerical results. Our formulas are also applicable to some extended s-wave superconductors having a basis function satisfying $\langle \eta_{\mathbf{k}}\rangle = 0$ and $\langle \eta_{\mathbf{k}}^2\rangle = 1$. These ratios for the strong-coupling correction have been illustrated as a function of the strong-coupling parameter T_c/ω_E for a d-wave superconductor and for an extended s-wave superconductor.

ACKNOWLEDGMENTS

This research was supported in part by the National Sciences and Engineering Research Council of Canada (NSERC), by the Canadian Institute for Advanced Research (CIAR), and by the Ontario Center for Materials Research (OCMR).

APPENDIX A: GAP EQUATION FOR T NEAR T_c

We consider a d-wave system with $\alpha_3=\alpha_5=0$; then, Eq. (10) becomes

$$Z_{\mathbf{k}}(\omega_n) = 1 \ . \quad \text{(A1)}$$

Using Eqs. (5), (8), (9), (11), and (A1), Eq. (1) becomes

$$1 = \pi T\sum_m \lambda(m-n)\left[\frac{1}{|\omega_m|} - \frac{\alpha_4}{2}\frac{\Delta_0^2}{|\omega_m|^3} + \frac{3\alpha_6}{8}\frac{\Delta_0^4}{|\omega_m|^5}\right] . \quad \text{(A2)}$$

We will follow Marsiglio and Carbotte[16] closely. We first write $\lambda(m-n)$ of Eq. (16) as

$$\lambda(m-n)=\int_0^\infty d\nu\frac{2\nu A(\nu)}{\omega_m^2+a_n^2}\left[1-\frac{2\omega_m\omega_n}{\omega_m^2+a_n^2}+\frac{4\omega_m^2\omega_n^2}{(\omega_m^2+a_n^2)^2}+\cdots\right], \tag{A3}$$

with

$$a_n^2=\omega_n^2+\nu^2 . \tag{A4}$$

Using (A3), Eq. (A2) becomes

$$1=\int_0^\infty d\nu\, 2\nu A(\nu)[(P_1+Q_1)-\tfrac{1}{2}\alpha_4(P_2+Q_2)\Delta_0^2+\tfrac{3}{8}\alpha_6(P_3+Q_3)\Delta_0^4] , \tag{A5}$$

where

$$P_i=\sum_{m=0}^{\infty}\frac{2\pi T}{\omega_m^{2i-1}}\frac{1}{\omega_m^2+a_n^2} , \tag{A6}$$

$$Q_i=\sum_{m=0}^{\infty}\frac{2\pi T}{\omega_m^{2i-3}}\frac{4\omega_n^2}{(\omega_m^2+a_n^2)^3} . \tag{A7}$$

P_i and Q_i ($i=1,2,3$) are calculated as

$$P_1=\frac{1}{a_n^2}F_1 , \tag{A8}$$

$$P_2=\frac{2}{a_n^2}C_2(T)-\frac{1}{a_n^4}F_1 , \tag{A9}$$

$$P_3=\frac{8}{3a_n^2}C_3(T)-\frac{2}{a_n^4}C_2(T)+\frac{1}{a_n^6}F_1 , \tag{A10}$$

$$Q_1=\frac{\omega_n^2}{2a_n^2}\left[F_3+\frac{i}{a_n}F_2\right] , \tag{A11}$$

$$Q_2=4\omega_n^2\left[\frac{1}{a_n^6}F_1+\frac{1}{16a_n^4}F_3-\frac{5i}{8a_n^5}F_2\right] , \tag{A12}$$

$$Q_3=4\omega_n^2\left[\frac{2C_2(T)}{a_n^6}+\frac{3}{a_n^8}F_1-\frac{5}{8}\frac{F_3}{a_n^6}+2\left[\frac{7}{4}\right]^2\frac{i}{a_n^7}F_2\right] , \tag{A13}$$

with

$$F_1=\tfrac{1}{2}[\psi(y_+)+\psi(y_-)]-\psi(\tfrac{1}{2}) , \tag{A14}$$

$$F_2=\frac{1}{2(2\pi T)}[\psi^{(1)}(y_+)-\psi^{(1)}(y_-)] , \tag{A15}$$

$$F_3=\frac{1}{2(2\pi T)^2}[\psi^{(2)}(y_+)+\psi^{(2)}(y_-)] , \tag{A16}$$

and

$$y_\pm=\frac{1}{2}\pm\frac{ia_n}{2\pi T} . \tag{A17}$$

Here $\psi(x)$ is the digamma function and $\psi^{(m)}(x)$ are the polygamma functions. To remain consistent, we need the terms in P_i and Q_i up to $O(T/\nu)^4$. For brevity, let us consider an Einstein spectral density, i.e.,

$$A(\nu)=A\delta(\nu-\omega_E) . \tag{A18}$$

Then Eq. (A5) becomes

$$1=\lambda\omega_E^2\left[P_1+Q_1-\frac{\alpha_4}{2}(P_2+Q_2)\Delta_0^2+\frac{3}{8}\alpha_6(P_3+Q_3)\Delta_0^4\right] , \tag{A19}$$

where the ν appeared in P_i and Q_i has been replaced by ω_E. As small n values are dominant, we chose[16] $n=1$. Then we can rewrite Eq. (A17) as

$$y_\pm=\tfrac{1}{2}\pm iy , \tag{A20}$$

$$y=\frac{(\omega_E^2+\pi^2T^2)^{1/2}}{2\pi T} . \tag{A21}$$

To expand P_i and Q_i up to the $O(T_c/\omega_E)^4$ term, we first expand them up to term of order y^{-4} and then to the term $O(T_c/\omega_E)^4$. The expansions of F_1, F_2, and F_3 are obtained as

$$F_1=\ln\frac{1.13\pi}{u}+\frac{1}{3}u^2-\frac{7}{15}u^4 , \tag{A22}$$

$$F_2/a_n=-\frac{i}{(2\pi T)^2}\left[4u^2-\frac{8}{3}u^4\right] , \tag{A23}$$

$$F_3=\frac{u^2}{\pi^2T^2} , \tag{A24}$$

with

$$u=\pi T/\omega_E . \tag{A25}$$

Using Eqs. (A22)–(A24) in Eqs. (A8)–(A13) and (A19), we obtain Eqs. (13)–(16).

APPENDIX B: GAP EQUATION AT ZERO TEMPERATURE

Substituting Eq. (40) into Eq. (38), we have

$$Z_{\mathbf{k}}(\omega)=1-\frac{\lambda\eta_{\mathbf{k}}}{2\omega}\int_{-\infty}^{\infty}\frac{\omega_E^2}{\omega_E^2+(\omega'-\omega)^2}\omega'\left\langle\frac{\eta_{\mathbf{k}'}}{(\omega'^2+\Delta_0^2\eta_{\mathbf{k}'}^2)^{1/2}}\right\rangle' . \tag{B1}$$

Then the normal-state renormalization factor $Z_{\mathbf{k}}^N(\omega)$ is equal to 1. To carry out the integration, we make the expansion

$$\frac{\omega_E^2}{\omega_E^2+(\omega'-\omega)^2}=\frac{\omega_E^2}{\omega'^2+a^2}\left[1+\frac{2\omega\omega'}{\omega'^2+a^2}+\frac{4\omega^2\omega'^2}{(\omega'^2+a^2)^2}+\frac{8\omega^3\omega'^3}{(\omega'^2+a^2)^3}+\cdots\right], \tag{B2}$$

with

$$a^2=\omega_E^2+\omega^2 . \tag{B3}$$

Using Eq. (B2), Eq. (B1) becomes

$$Z_{\mathbf{k}}(\omega)=1-2\lambda\omega_E^2\eta_{\mathbf{k}}\langle\eta_{\mathbf{k}'}(A_1+A_2)\rangle' , \tag{B4}$$

where

$$A_1=\int_0^\infty d\omega'\frac{\omega'^2}{\sqrt{(\omega'^2+\Delta_0^2\eta_{\mathbf{k}'}^2}}\frac{1}{(\omega'^2+a^2)^2}=\frac{1}{2a^2}-\frac{\Delta_0^2\eta_{\mathbf{k}'}^2}{2a^4}\left[\ln\frac{2a}{\Delta_0|\eta_{\mathbf{k}'}|}-1\right]+O(a^{-6}) , \tag{B5}$$

$$A_2=\int_0^\infty d\omega'\frac{4\omega^2\omega'^4}{\sqrt{\omega'^2+\Delta_0^2\eta_{\mathbf{k}'}^2}}\frac{1}{(\omega'^2+a^2)^4}=\frac{1}{3}\frac{\omega^2}{a^4}+O(a^{-6}) , \tag{B6}$$

with Eqs. (B5), (B6), and (B3), Eq. (B4) becomes

$$Z_{\mathbf{k}}(\omega)=1-\frac{\lambda\omega_E^2\eta_{\mathbf{k}}\Delta_0^2}{(\omega_E^2+\omega^2)^2}\left[\alpha_3\left[\ln\frac{2(\omega_E^2+\omega^2)^{1/2}}{\Delta_0}-1\right]-\alpha_{3l}\right] , \tag{B7}$$

where α_3 and α_{3l} are defined in Sec. II. We consider the cases of $\alpha_3=\alpha_{3l}=0$; then,

$$Z_{\mathbf{k}}(\omega)=1 . \tag{B8}$$

In fact, for $\alpha_3\neq0$ and $\alpha_{3l}\neq0$, $Z_{\mathbf{k}}(0)=1+O(\Delta_0/\omega_E)^2\simeq1$. Therefore $Z_{\mathbf{k}}(\omega)=1$ can be taken as a general result. With Eq. (B2), Eq. (39) can be rewritten (using $\omega=0$) as

$$\Delta_0=\lambda\omega_E^2\Delta_0\langle\eta_{\mathbf{k}'}^2A_3\rangle' , \tag{B9}$$

where

$$\begin{aligned}A_3&=\int_0^\infty\frac{d\omega'}{\omega'^2+\omega_E^2}\frac{1}{\sqrt{\omega'^2+\Delta_0^2\eta_{\mathbf{k}'}^2}}\\&=\frac{1}{2\omega_E}\frac{1}{\sqrt{\omega_E^2-\Delta_0^2\eta_{\mathbf{k}'}^2}}\ln\left[\frac{\omega_E+\sqrt{\omega_E^2-\Delta_0^2\eta_{\mathbf{k}'}^2}}{\omega_E-\sqrt{\omega_E^2-\Delta_0^2\eta_{\mathbf{k}'}^2}}\right]\\&=\frac{1}{\omega_E^2}\left[\varepsilon_1+\frac{1}{2}\left[\frac{\Delta_0\eta_{\mathbf{k}'}}{\omega_E}\right]^2\left[\varepsilon_1-\frac{1}{2}\right]+\frac{3}{8}\left[\frac{\Delta_0\eta_{\mathbf{k}'}}{\omega_E}\right]^4\left[\varepsilon_1-\frac{7}{12}\right]+O\left[\frac{\Delta_0}{\omega_E}\right]^6\right] .\end{aligned} \tag{B10}$$

Substituting Eq. (B10) into Eq. (B9), we obtain Eq. (42).

[1]F. Steglich, Phys. Scr. T **29**, 15 (1990).
[2]K. Miyake, S. Schmit-Rink, and C. M. Varma, Phys. Rev. B **34**, 6554 (1986).
[3]D. J. Scalapino, E. Loh, and J. E. Hirsch, Phys. Rev. B **34**, 8190 (1986).
[4]K. Miyake, J. Magn. Magn. Mater. **63&64**, 411 (1987).
[5]C. J. Pethick and D. Pines, Phys. Rev. Lett. **57**, 118 (1986); S. Schmit-Rink, K. Miyake, and C. M. Varma, *ibid.* **57**, 2575 (1986); A. J. Millis, S. Sachdev, and C. M. Varma, Phys. Rev. B **37**, 4975 (1988).
[6]G. Kotliar, Phys. Rev. B **37**, 3664 (1988).
[7]A. J. Millis, H. Monien, and D. Pines, Phys. Rev. B **42**, 167 (1990); P. Monthoux, A. V. Balatsky, and D. Pines, Phys. Rev. Lett. **67**, 3448 (1991); A. J. Millis, Phys. Rev. B **45**, 13 047 (1992), H. Monien, P. Monthoux, and D. Pines, *ibid.* **43**, 275 (1991); P. Monthoux and D. Pines, Phys. Rev. Lett. **69**, 961 (1992); D. Pines, Physica C **185-189**, 120 (1991); P. Monthoux and D. Pines, Phys. Rev. B **47**, 6069 (1993).
[8]N. Bulut, D. Hone, D. J. Scalapino, and N. E. Bickers, Phys. Rev. Lett. **64**, 2723 (1990); N. Bulut and D. J. Scalapino, Phys. Rev. B **45**, 2371 (1992); N. E. Bickers, S. R. White, and D. J. Scalapino, Phys. Rev. Lett. **62**, 961 (1989).
[9]S. Wermbter and L. Tewordt, Phys. Rev. B **43**, 10 530 (1991); St. Lenck and J. P. Carbotte, *ibid.* **46**, 14 850 (1992); E. J.

Nicol, C. Jiang, and J. P. Carbotte, *ibid.* **47**, 8131 (1993).

[10]J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, New York, 1964).

[11]B. T. Geilikman and V. Z. Kresin, Fiz. Tverd. Tela (Leningrad) **1**, 3294 (1965) [Sov. Phys. Solid State **7**, 2659 (1966)]; B. T. Geilikman, V. Z. Kresin, and W. F. Masharov, J. Low Temp. Phys. **18**, 241 (1975).

[12]V. Z. Kresin and V. P. Parkhomenko, Fiz. Tverd. Tela (Leningrad) **16**, 3663 (1974) [Sov. Phys. Solid State **16**, 2180 (1975)].

[13]N. F. Masharov, Fiz. Tverd. Tela (Leningrad) **16**, 2342 (1974) [Sov. Phys. Solid State **16**, 1524 (1975)].

[14]C. R. Leavens and J. P. Carbotte, Can. J. Phys. **49**, 724 (1971).

[15]B. Mitrovic, H. G. Zarate, and J. P. Carbotte, Phys. Rev. B **29**, 184 (1984).

[16]F. Marsiglio and J. P. Carbotte, Phys. Rev. B **33**, 6141 (1986); **41**, 8765 (1990).

[17]F. Marsiglio, J. P. Carbotte, and J. Blezius, Phys. Rev. B **41**, 6457 (1990).

[18]J. P. Carbotte, Rev. Mod. Phys. **62**, 1027 (1990).

[19]A. J. Millis, S. Sachdev, and C. M. Varma, Phys. Rev. B **37**, 4975 (1988).

[20]P. J. Williams and J. P. Carbotte, Phys. Rev. B **39**, 2180 (1989).

[21]E. Schachinger and J. P. Carbotte, Phys. Rev. B **43**, 10 279 (1991).

[22]P. B. Allen and B. Mitrovic, in *Solid State Physics,* edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1982), Vol. 37, p. 1.

[23]J. Bardeen and M. Stephen, Phys. Rev. **136**, A1485 (1964).

[24]V. L. Pokrovskii, Zh. Eksp. Teor. Fiz. **40**, 641 (1961) [Sov. Phys. JETP **13**, 447 (1961)].

[25]Y. Suzumura and H. J. Schultz, Phys. Rev. B **39**, 11 398 (1989).

[26]A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1975).