

Current density in a quantum Hall bar

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When current is forced through a quantum Hall bar it is understood that it flows primarily through edge states. These represent extra charges that in turn produce a more widely distributed current density that falls off rather slowly with distance and then drops rapidly as a result of screening by particles beyond the depletion region. We have solved numerically the self-consistent Schrödinger equation in the Hartree approximation for ideal samples with small widths and sharp boundaries, and combined these results with the Wiener-Hopf technique to obtain an accurate picture of the situation for macroscopically wide samples. Our results indicate that the redistribution of states in the bulk of a quantum Hall bar is a very important effect and that the bulk states contribute to a significant fraction of the total current.

I. INTRODUCTION

There has been a renewed interest in calculating the distribution of currents and charge in a quantum Hall bar. Chklovskii, Shklovskii, and Glazman¹ in their paper regarding partially occupied states solve the Poisson equation in the classical limit for a one-gate-induced confinement potential. Chklovskii, Matveev, and Shklovskii² analyze the same problem in a narrow channel. Dempsey, Gelfand, and Halperin study spin-split states near the edges using a Hartree-Fock variational scheme.³ Brey, Palacios, and Tejedor analyze the edge states varying the width of the boundaries.⁴ Pfannkuche and Hajdu⁵ thoroughly study, in mean-field approximation, the current and charge distribution for a small ideal Hall bar for various filling factors.

In many works the charge variation is solely due to changes in the population of Landau levels when they go through the Fermi energy. This seems to be a sensible approach (though not complete) for the big changes in the environment that occur at the edges due to the confining potential. It is assumed, though, that no modulation is possible in regions in which the Fermi energy lies in a mobility gap, and these are commonly referred to as incompressible. There is, however, the possibility of compression by varying the density of (Landau) states.

In this paper we analyze the situation of a Hall bar in equilibrium (no net current) and the changes resulting when there is a difference in the electrochemical potential between the two edges. In the following section we discuss the origin and effect of the edge currents and write down the Schrödinger equations in the Hartree approximation. In Sec. III we solve numerically the equations for a narrow Hall bar and show that the two edges are effectively decoupled, which permits us to extrapolate the results near one edge to a semi-infinite sample. In Sec. IV we use the results of Sec. III for the region near the edges of the bar, along with the Wiener-Hopf technique, to solve analytically the problem of a wide sample in the regions away from the edges.

II. EDGE FIELDS

There are two basic mechanisms for conduction under integer quantum Hall effect conditions:

(1) For each Landau level there is a continuum of edge states that carry the current responsible for the Landau diamagnetism. In equilibrium the currents in the two edges have the same magnitude but opposite directions so no net current flows. Now, if we apply a potential drop between the two edges then the electrochemical potentials will be different, resulting in a different population at the two edges and, therefore, a net current.

(2) If there is an electric field inside the sample then the Hall current density is proportional to the local electric field (averaged over some localization distance) and will have an exponentially small dependence on the conditions far away. A nice "experimental demonstration" is the high precision of the measured Hall resistances R_H despite the disturbances present near the ends of the bar.

(a) *Effect of the edge currents.* As we force current the population of current capable edge states is altered. This *extra* charge density associated with the *extra* edge current produces an *extra* electric field throughout the interior of the sample. These in turn induce an *extra* current density according to the second mechanism above. This fact is true whether the edge dividing filled and empty Landau levels is sharp or consists of regions of partially occupied states. Edge currents will also produce an *extra* magnetic field but its effect is less important.

(b) *Obtaining the equations.* For a nearly uniform bar, long in the y direction and with a strong magnetic field in the $-z$ direction, the single-electron eigenstates can be written in the form $\psi_{nk}(x) \exp(iky)$, where ψ satisfies the equation

$$\left[-\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} + \frac{1}{2m^*} (eBx - \hbar k)^2 - e\phi(x) \right] \psi_{nk}(x) = E_{nk} \psi_{nk}(x), \quad (1)$$

where m^* is the effective mass of the electrons and we

have used the Landau gauge. This equation is correct under mean-field theory. Here the electrostatic potential is given by

$$\phi(x) = -\frac{e}{4\pi\epsilon\epsilon_0} \int [\rho(x') - \rho_{bk}(x')] u(x-x') dx', \quad (2)$$

where ρ_{bk} is a neutralizing background density and $u(x)$ will have a form similar to $-2 \ln |x-x'|$ but there will be some corrections due to screening by carriers beyond the depletion region; and the density of electrons is

$$\rho(x) = \sum |\psi(x)|^2, \quad (3)$$

where the sum is performed over occupied states.

III. NUMERICAL ANALYSIS OF A NARROW BAR

We solve (1) and (2) numerically. It is convenient to work in units of magnetic length $l = \sqrt{\hbar/eB}$ and magnetic energy $\hbar\omega_c = \hbar eB/m^*$. In these units, the equations to solve are

$$\left[-\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} (x-k)^2 + V(x) \right] \psi_{nk}(x) = E_{nk} \psi_{nk}(x), \quad (4)$$

$$V(x) = -2 \frac{l}{a} \int [\rho(x') - \rho_{bk}(x')] \ln |x-x'| dx', \quad (5)$$

$$\rho(x) = \frac{1}{2\pi} \sum_n \int |\psi_{nk}(x)|^2 dk, \quad (6)$$

where the sum is over occupied states and $a^* = (\epsilon m^*/m_e) a_0$ is the effective Bohr radius. For typical conditions in GaAs $a^* \simeq 100 \text{ \AA}$ and $l \sim a^*$ so we take $l = a^*$ in what follows. We also choose to work with the lowest Landau level approximation and neglect spin splitting so we perform the summation above for $n = 1$, therefore the sum turns into a factor $n = 2$. We consider samples of 20 and 40 magnetic lengths width limited by an infinite potential and ρ_{bk} is designed to cancel the electron density in the bulk and provide overall charge neutrality. In equilibrium we populate states with energies up to a certain Fermi energy and then take the system out of equilibrium by moving electrons from the states between $k_F - \delta k$ and k_F to those states between $-k_F - \delta k$ and $-k_F$ similarly as in Ref. 6. In this case we have a steady state and there is an electrochemical potential difference between the two ends, i.e., $\Delta\mu = E_{-k_F - \delta k} - E_{k_F - \delta k}$ that can be identified with the Hall voltage. We know of a similar calculation by Heinonen and Taylor⁷ where they use the interesting approach of minimizing a free energy with a current imposed as a constraint. We would like to point out that both methods should be equivalent though we do not arrive at the same results regarding the distributions of current and charge even when considering the effect of the different boundary conditions used. Follow-

ing a similar approach, Pfannkuche and Hajdu⁵ minimize a free energy for fixed total momentum p_y for various filling factors ν but only one sample size. Their results for integer filling factors are very similar to ours.

The current density can be calculated from

$$j_y(x) = -\frac{ne\omega_c}{2\pi} \int (k-x) |\psi_k(x)|^2 dk, \quad (7)$$

summing over occupied states, and the total current is just $I_y = \int j_y(x) dx$.

In Figs. 1–3 we show the results for the eigenenergies, charge density, and current density for the system in equilibrium and in nonequilibrium situations. The sample width is $20l$ and we used $E_F = \hbar\omega_c$ or equivalently $lk_F = 9.5$ and $l\delta k = 0.005$. Note that the total current is zero for the equilibrium case as there are the same number of current carrying states for the two directions. Also note that the current density is nonzero in the bulk due to the electric field. We verified that this δk is within the linear regime by comparing these results with similar ones calculated with smaller and bigger δk .

Figures 4 and 5 show the difference between the nonequilibrium and equilibrium distributions for sample widths of 20 and 40 magnetic lengths. Note that there is a very important redistribution of charge and current and that a significant fraction of the latter is carried through bulk states. Note that the regions within a few magnetic lengths from the edges show almost no dependence on the sample width, showing that the interaction between the two edges is not important except for the change in electrochemical potential. For $20l$ width we have $I_{eq} = 10^{-7} ne\omega_c/2\pi$, $I_{neq} = -0.256 ne\omega_c/2\pi$, and $\Delta\mu = (0.254 \pm 0.007) \hbar\omega_c$, giving a Hall resistance $R_H = (0.992 \pm 0.027) h/ne^2$. In the case of $40l$ we have $I_{eq} = 10^{-7} ne\omega_c/2\pi$, $I_{neq} = -0.302 ne\omega_c/2\pi$, and $\Delta\mu = (0.305 \pm 0.005) \hbar\omega_c$, giving a Hall resistance $R_H = (1.010 \pm 0.016) h/ne^2$.

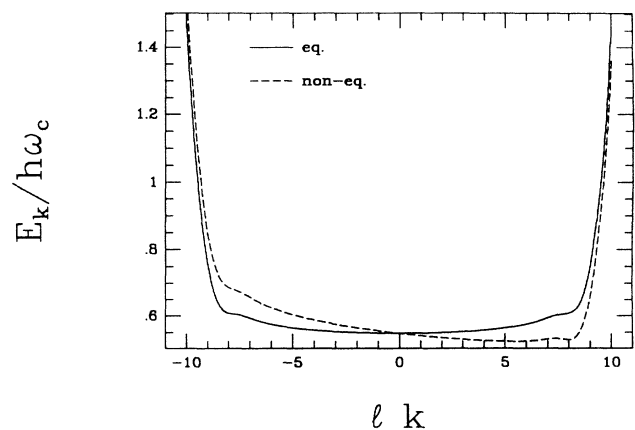


FIG. 1. Energies of states as a function of k . In equilibrium, states were populated up to E_F . The nonequilibrium state was achieved by transferring electrons from states near the right edge to states near the left one, therefore establishing an electrochemical potential difference between the two edges $\Delta\mu = E_{-k_F - \delta k} - E_{k_F - \delta k} = (0.254 \pm 0.007) \hbar\omega_c$.

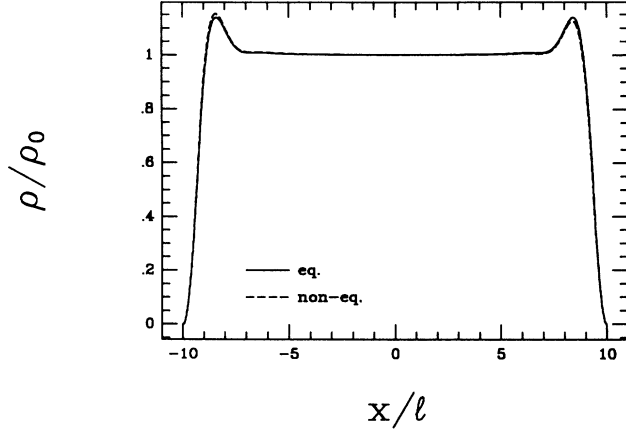


FIG. 2. Density of electrons. (Note: $\rho_0 = n/2\pi l^2$.)

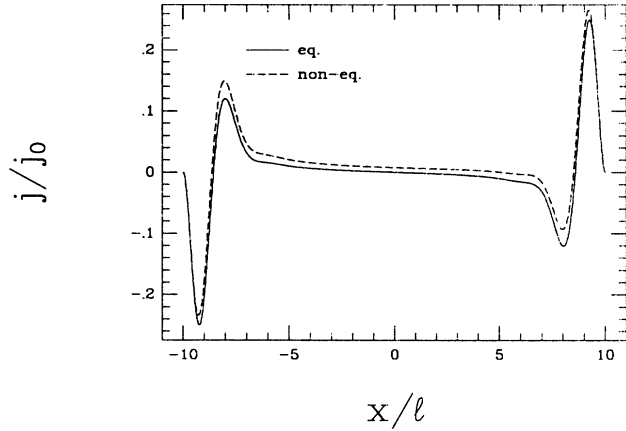


FIG. 3. Current density. In equilibrium the total current is $I_{\text{eq}} = 10^{-7} n e \omega_c / 2\pi$ while in the nonequilibrium state we have $I_{\text{neq}} = -0.256 n e \omega_c / 2\pi$, which along with the result for $\Delta\mu$ gives a Hall resistance $R_H = (0.992 \pm 0.027) h / n e^2$. (Note: $j_0 = n e \omega_c / 2\pi l$.)

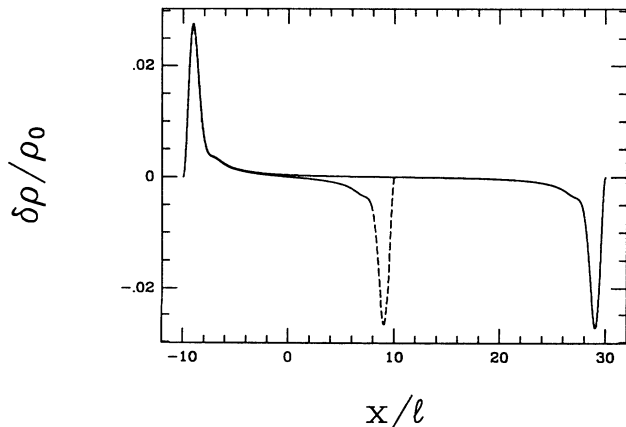


FIG. 4. Comparison between $\delta\rho$ for sample widths of $20l$ and $40l$. (Note: $\rho_0 = n/2\pi l^2$.)

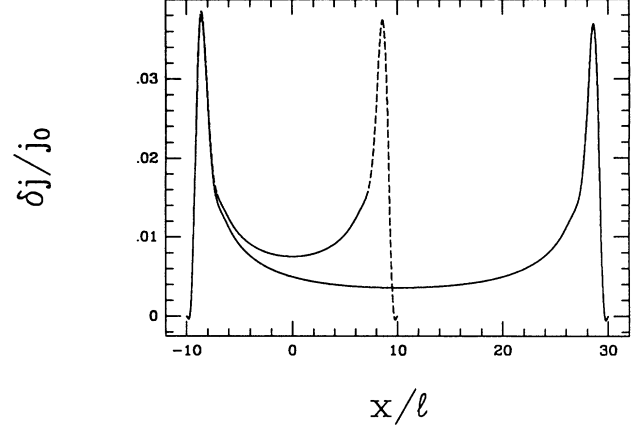


FIG. 5. Comparison between δj for sample widths of $20l$ and $40l$. (Note: $j_0 = n e \omega_c / 2\pi l$.)

IV. WIENER-HOPF ANALYSIS

We now consider the situation away from the edges where the electrostatic potential varies slowly on the scale of the *magnetic length* $l = \sqrt{\hbar/eB}$:

$$l^2 e \phi''(x) \ll \hbar \omega_c, \quad (8)$$

then the solutions of (1) are harmonic-oscillator-type wave functions with typical width $\sim l$ and center at

$$x_0 = l^2 k + \frac{m e \phi'(x_0)}{(eB)^2}, \quad (9)$$

and the mean velocity of electrons on those states is

$$\bar{u} = \frac{\phi'}{B}. \quad (10)$$

As the wave number k serves to label the different states, we can get the density of states (DOS) by differentiating it with respect to x_0 :

$$\text{DOS} \propto \left| \frac{dk}{dx_0} \right|.$$

As long as the energy of the states is far from the Fermi energy the occupation of each state remains constant, therefore the density of electrons is just the DOS times the number of occupied Landau levels n ,

$$\rho(x) = \frac{n e B}{h} - \frac{n m^* e \phi''(x)}{h e B}, \quad (11)$$

where the second term on the right-hand side represents a compression of the states by the electric field.

If we are interested in the changes that occur when a Hall current is imposed, we need only to consider

$$\delta\rho(x) = -\frac{n m^* e \delta\phi''(x)}{h e B}, \quad (12)$$

where the δ 's denote the values minus the equilibrium ones.

This equation is valid only at distances from the edge

which are larger than a few times the magnetic length l . For a semi-infinite system with an edge on its left, we then consider the edge to be at $x = -\lambda \sim l$ and we suppose that the charge density there can be calculated by other means. We denote this edge charge as $S(x)$. The equations to solve are

$$\delta\rho(x) = -\frac{nm^*}{heB}e\delta\phi''(x), \quad x > 0, \quad (13)$$

$$\begin{aligned} \delta\phi(x) = & -\frac{e}{4\pi\epsilon\epsilon_0} \int_0^\infty \delta\rho(x')u(x-x')dx' \\ & -\frac{e}{4\pi\epsilon\epsilon_0} \int_{-\infty}^0 S(x')u(x-x')dx'. \end{aligned} \quad (14)$$

Analysis of the equations above by Thouless⁸ shows that a perturbative expansion is not completely satisfactory and that it is necessary to go beyond it. A similar problem was considered by MacDonald *et al.*⁹ and Thouless¹⁰ but the boundary conditions in those papers are not, however, the ones we want to consider here, since it was assumed that a uniform electrostatic field was imposed. We now obtain an analytical solution to this problem using the Wiener-Hopf method¹¹ for $S(x)$ known.

Let us change notation from the electrostatic potential $\phi(x)$ to the potential energy for electrons $V(x) = -e\phi(x)$ and in this section consider this convenient set of dimensionless quantities:

$$\begin{aligned} x &\longrightarrow \tilde{l}\tilde{x}, & \delta\rho &\longrightarrow (n/\tilde{l}^2)\delta\tilde{\rho}, \\ S &\longrightarrow (n/\tilde{l}^2)\delta\tilde{S}, & V &\longrightarrow (e^2n/2\epsilon\epsilon_0\tilde{l})\tilde{V}, \end{aligned} \quad (15)$$

where $\tilde{l} = nl^2/a_B^*$.

Note that for usual conditions both \tilde{l} and l have the same order of magnitude. Then

$$\delta\tilde{\rho}(\tilde{x}) = \frac{d^2\tilde{V}(\tilde{x})}{d\tilde{x}^2}, \quad \tilde{x} > 0, \quad (16)$$

$$\begin{aligned} \tilde{V}(\tilde{x}) = & \frac{1}{2\pi} \int_0^\infty \delta\tilde{\rho}(\tilde{x}')u(\tilde{x}-\tilde{x}')d\tilde{x}' \\ & + \frac{1}{2\pi} \int_{-\infty}^0 \delta\tilde{S}(\tilde{x}')u(\tilde{x}-\tilde{x}')d\tilde{x}'. \end{aligned} \quad (17)$$

We drop the tilde from now on.

A. Fourier transformation

If we Fourier transform the equations above by defining

$$\begin{aligned} f(k) &= \int_0^\infty \delta\rho(x)e^{-ikx}dx, & S(k) &= \int_{-\infty}^0 S(x)e^{-ikx}dx, \\ \phi_r(k) &= \int_0^\infty V(x)e^{-ikx}dx, & \phi_l(k) &= \int_{-\infty}^0 V(x)e^{-ikx}dx, \\ Q(k) &= \int_{-\infty}^\infty u(x)e^{-ikx}dx, \end{aligned} \quad (18)$$

then (16) transforms to

$$f(k) = -k^2\phi_r(k) - ikV(0) - V'(0), \quad (19)$$

$$\phi_r(k) + \phi_l(k) = \frac{Q(k)}{2\pi}[f(k) + S(k)]. \quad (20)$$

We can eliminate ϕ_r to get

$$\begin{aligned} [k^2\phi_l - ikV(0) - V'(0)] + S(k) \\ = \left(1 + \frac{k^2Q(k)}{2\pi}\right)[f(k) + S(k)]. \end{aligned} \quad (21)$$

If we consider k as a complex variable we can observe that f and ϕ_r are regular and bounded (RB) functions in the lower half (complex) plane (LHP) while S and ϕ_l have the same properties but in the upper half plane (UHP) (for consistency we must require the domain of analyticity to include a strip around the real axis for all functions). Note that Q will not, in general, be analytic; usually Q will have singularities at the imaginary axis. Now, the goal is to write the equation in such a way that one side is RB on the LHP and the other side is RB on the UHP. In order to do so we write

$$\frac{N(k)}{D(k)} = \left[1 + \frac{k^2Q(k)}{2\pi}\right], \quad (22)$$

asking that N (D) be RB on the LHP (UHP). We now multiply (21) by $D(k)$ and then decompose again,

$$L(k) + U(k) = N(k)S(k). \quad (23)$$

Now L (U) is RB on the LHP (UHP). Finally we arrange (21) as

$$\begin{aligned} [k^2\phi_l - ikV(0) - V'(0)]D(k) + S(k)D(k) \\ = N(k)f(k) + L(k). \end{aligned} \quad (24)$$

Note that the left-hand side of the equation is composed by functions which are regular on the UHP while the right-hand side is regular on the lower half plane. Then each function must be the analytic continuation of the other, therefore both must be equal to an entire function. Asymptotic analysis for $|k| \rightarrow \infty$ shows that this entire function must actually vanish everywhere so we obtain the following expression for the Fourier transform of the density of electrons:

$$f(k) = -\frac{L(k)}{N(k)}, \quad (25)$$

from where we can obtain the charge density, potential, and current density. The Appendix shows the explicit formulas for L and N .

B. Combination with the results of Sec. III

As we have seen, the conditions near the edges are independent of one another and we can, therefore, extrapolate the behavior near one of the edges to a semi-infinite system. We have found that (12) holds more than four magnetic lengths away from the edge, so that we can apply the method above to this problem if we include all the charge density within $4l$ of the edge in $S(x)$. We use

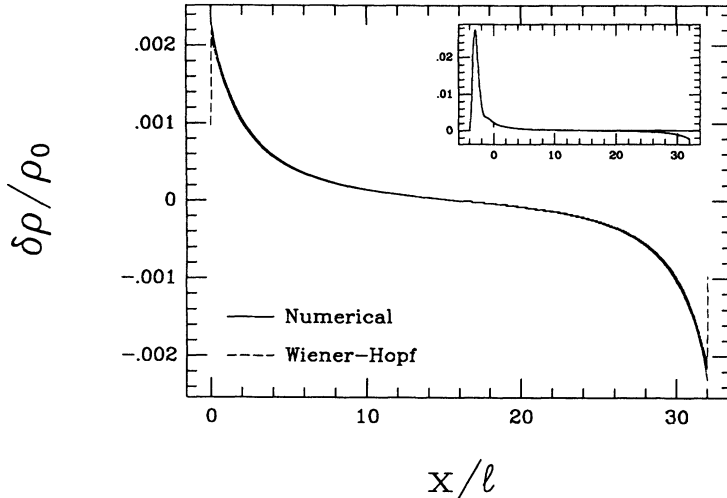


FIG. 6. Extra charge density obtained numerically in Sec. III and the linear superposition of the Wiener-Hopf solutions produced by each edge charge (Sec. IV). The inset shows the edge charge density for $x < 0$ which becomes $S(x)$ and the induced charge density for $x > 0$. (Note: $\rho_0 = n/2\pi l^2$).

the numerical data corresponding to a sample $40l$ wide.

Figure 6 shows the previous numerical results along with the semi-infinite sample solution calculated with the Wiener-Hopf method. Here we present a superposition of the induced charges produced by both edges calculated independently. The agreement is very good and demonstrates the applicability of the procedure. The inset shows the edge charge for $x < 0$ and the induced charge it produces for $x > 0$.

An asymptotic analysis shows that $\delta\rho(x) \rightarrow 1/x^2$ as $x \rightarrow \infty$. Therefore $j_y(x) \rightarrow 1/x$ and the total current carried by the bulk states diverges logarithmically with the sample width for a fixed edge charge. Actually there is a cutoff in the range of the current due to screening by mobile carriers outside the depletion region. For realistic conditions this distance is much larger than the magnetic length, therefore we get $I_{\text{int}} \sim \delta N_{\text{edge}}(ne^3/2\pi\epsilon_0\epsilon h)\ln(l_{\text{max}}/l)$ and δN_{edge} is the density of added electrons at the edge. In our case and considering $\ln(l_{\text{max}}/l) \sim 3$ we have

$$I_{\text{int}} \approx 0.115 \frac{ne\omega_c}{2\pi}, \quad (26)$$

while the calculated edge current is

$$I_{\text{edge}} \approx 0.072 \frac{ne\omega_c}{2\pi}, \quad (27)$$

which shows the importance of the interior states in carrying the current in agreement with Ref. 8.

V. CONCLUSIONS

We have calculated the distribution of currents and charges in a quantum Hall bar using the Hartree approximation without many artificial assumptions. We have found a method that combines a numerical self-consistent approach (to find the structure of the edge states) with an analytical solution (in the interior of the sample). This solution shows that a significant fraction of the current flows through states away from the edges.

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APPENDIX: CALCULATION OF $N(K)$ AND $L(K)$

1. General method

We first take the logarithm of Eq. (22),

$$\ln N - \ln D = \ln M \equiv \ln \left[1 + \frac{k^2 Q(k)}{2\pi} \right], \quad (A1)$$

and then we compute $\ln N$ by integrating along the discontinuities on the positive imaginary axis,

$$\begin{aligned} \ln N = & -\frac{1}{2\pi i} \int_{K_0}^R \frac{\ln M(iy + \epsilon) - \ln M(iy - \epsilon)}{iy - k} idy \\ & + \underbrace{\frac{1}{2\pi i} \int_{\circ R} \frac{\ln M(z)}{z - k} dz}_{c_R}. \end{aligned} \quad (A2)$$

Note that $\ln D(k) = -\ln N(-k)$.

For L and U we compute L by integrating NS around the positive imaginary axis

$$L(k) = \frac{1}{2\pi i} \int_{K_0}^{\infty} \frac{id\beta}{i\beta - k} S(i\beta) [N(i\beta + \epsilon) - N(i\beta - \epsilon)]. \quad (A3)$$

2. Explicit results for the Coulomb interaction

For the simple Coulomb interaction for line charges

$$u(x) = -2 \ln |x| \implies Q(k) = \frac{2\pi}{k} \text{sgn}(\text{Re}k), \quad (A4)$$

and using this in Eq. (A2) we get

$$N(k) = [1 + |k|]^{1/2} \exp\{\phi(k)\}, \quad (\text{A5})$$

where

$$\phi(k) = -\frac{k \ln k - k}{\pi} - \frac{1}{\pi} \int_0^k \frac{t^2 \ln t}{1 - t^2} dt, \quad (\text{A6})$$

which has the following property:

$$\phi(k) = \pi/4 - \phi(1/k). \quad (\text{A7})$$

The discontinuity of N along the imaginary axis to use

in Eq. (A3) is

$$\begin{aligned} \Delta N(i\beta) &\equiv [N(i\beta + \epsilon) - N(i\beta - \epsilon)] \\ &= \frac{2i\beta}{[\beta^2 + 1]^{1/4}} \exp\{\mathcal{Q}(\beta)\}, \end{aligned} \quad (\text{A8})$$

where

$$\mathcal{Q}(\beta) = \frac{1}{\pi} \ln \beta \tan^{-1} \beta - \frac{1}{\pi} \int_0^\beta \frac{\tan^{-1} t}{t} dt, \quad (\text{A9})$$

that satisfies

$$\mathcal{Q}(\beta) = \mathcal{Q}(1/\beta). \quad (\text{A10})$$

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