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Exact results for the level density and two-point correlation function of the transmission-matrix eigenvalues in quasi-one-dimensional conductors

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We study transport properties of phase-coherent quasi-one-dimensional disordered conductors in the diffusive regime, in terms of the eigenvalue distribution of the two-terminal transmission matrix. Using an expansion in inverse powers of the classical conductance, we calculate the average transmission eigenvalue density and the two-point correlation function for fluctuations in the density. A formula for the average value and the variance of a general linear statistic on the transmission eigenvalues is obtained. Our results confirm an earlier hypothesis, based on random-matrix theory, that fluctuations are universal in the diffusive regime and ultimately determined by level repulsion.

I. INTRODUCTION

Recent advances in the technology of microfabrication have enabled the experimental observation of a number of interesting quantum interference effects at low temperature. Two important examples are the weak-localization correction to the average Boltzmann conductance of thin metallic films[1-3] and the sample-specific reproducible fluctuations in the conductance of small metallic wires.[4-6]

The microscopic theory of these phenomena has been formulated on the basis of impurity-average Green-function techniques[7] and has explained many features of experiments on mesoscopic systems. In this theory, weak localization arises as a result of coherent backscattering of electrons diffusing through the sample in the presence of time-reversal symmetry, while infrared divergences in ladder diagrams lead to conductance fluctuations that are always of order e^2/h, independent of sample size (as long as it is smaller than the inelastic length) and degree of disorder (provided conduction is metallic), i.e., universal conductance fluctuations.

An alternative theoretical approach pioneered by Landauer,[8] called the scattering approach, relates transport coefficients of phase-coherent mesoscopic systems to asymptotic scattering states at the Fermi surface, described by the reflection and transmission matrices for a two-terminal sample. There are three different theories based on this approach.

The first theory,[9,10] called the local maximum-entropy approach, derives a Fokker-Planck equation for the evolution with sample length of the probability distribution of the transfer matrix of the system. The second theory,[11,12] called the global maximum-entropy approach, is built on ideas from classical random-matrix theory[13] and proposes an ansatz for the probability distribution of the transfer matrix of the whole sample. The third theory[14] proposes a Hamiltonian for the coupling between small slices of the sample and evaluates averages involving different elements of the scattering matrix using supersymmetry. Two basic simplifying features, common to all these approaches, are that the sample is assumed to have a quasi-one-dimensional geometry, and that two-terminal measurements are considered.

While the local maximum-entropy and the supersymmetric approaches constitute an exact macroscopic description of an underlying weak-scattering microscopic theory, the global maximum-entropy approach has been shown recently[15] to be an excellent, but approximate description, since the numerical value of the amplitude of the fluctuations in the conductance is not in precise agreement with that of diagrammatic calculations.

The scattering approach to two-probe quantum transport theory has been developed for a variety of observables in different physical systems, e.g., the conductance and shot-noise power of a phase-coherent conductor, the conductance of a disordered microbridge between normal metallic and superconducting leads, and the supercurrent-phase relationship of a point contact Josephson junction. All these quantities can be expressed as linear statistics[15] on the eigenvalues $\hat{T}_i=(1+\lambda_i)^{-1}$, $0\le\lambda_i<\infty$, of $tt^\dagger$, where t is the transmission matrix. Therefore, in general one is concerned with

$$A=\sum_i a(\lambda_i)\ , \tag{1.1}$$

where $a(\lambda_i)$ is an arbitrary smooth function.

In the diffusive regime, when the length L of the sample is much larger than the mean free path l, but much smaller than the localization length, one expects quantum interference to affect both the average and the variance of A, which in general can be obtained from the formulas

$$\langle A\rangle=\int_0^\infty a(\lambda)\rho(\lambda)d\lambda\ , \tag{1.2}$$

$$\mathrm{var}(A)=\int_0^\infty a(\lambda)a(\lambda')K(\lambda,\lambda')d\lambda\,d\lambda'\ , \tag{1.3}$$

in which $\rho(\lambda)$ and $K(\lambda,\lambda')$ are the average level density and two-point correlation function, respectively. It is clear from (1.2) and (1.3) that these functions contain all statistical information concerning weak-localization and universal mesoscopic fluctuations. The advantage of the random transfer-matrix approach is that, once one knows $\rho(\lambda)$ and $K(\lambda,\lambda')$, the behavior of many different linear statistics can be investigated with very little extra effort.

The present work is concerned with the exact calculation of both $\rho(\lambda)$ and $K(\lambda,\lambda')$ in the limit of a large number of transverse scattering eigenchannels, N, on scales for λ much larger than the mean level spacing. Using a moment expansion technique developed in the context of the local maximum-entropy approach,[16] we show that the average and the variance of a general linear statistic in a quasi-one-dimensional disordered conductor are given, respectively, by

$$\langle A \rangle = \tfrac{1}{2}\tilde{a}(0)G_B + \frac{1}{2\pi}\frac{\beta-2}{\beta}\int_0^\infty dk\, \tilde{a}(k)(1+e^{-k\pi}) - \frac{12-14\beta+3\beta^2}{2\beta^2\pi^2 G_B}\int_0^\infty dk\, \tilde{a}(k) k \sinh(k\pi) \times \ln(1-e^{-k\pi}) , \tag{1.4}$$

and

$$\mathrm{var}(A) = \frac{2}{\beta\pi^2}\int_0^\infty dk\, k(1-e^{-2\pi k})\tilde{a}^2(k) , \tag{1.5}$$

where

$$\tilde{a}(k) = \int_0^\infty dx \cos(kx) a[\sinh^2(x/2)] , \tag{1.6}$$

$G_B = Nl/L$ is the classical Boltzmann conductance, and β is a symmetry parameter ($\beta=1$ for systems with time-reversal symmetry and without spin-orbit scattering, $\beta=2$ for systems without time-reversal symmetry and $\beta=4$ for systems with time-reversal symmetry in the presence of spin-orbit coupling). Equations (1.4) and (1.5) show that weak localization [second term in (1.4)] and universal fluctuations are, in fact, a very general feature of quantum diffusion, and affect a large number of transport quantities, in addition to conductance. A short account of Eq. (1.5) has previously been given elsewhere.[17]

This paper is organized as follows. In Sec. II we define the scattering problem which is specified by attaching two semi-infinite ordered leads to the left and right of a disordered region. The leads serve to connect the sample to electron reservoirs wherein perfect phase randomization is supposed to occur. The multiple-scattering process is described by a diffusion equation, for the evolution with sample length, of the probability distribution of the transfer matrix of the system. Using an expansion in inverse powers of the classical conductance we obtain differential equations for generating functions, which turn out to be related to $\rho(\lambda)$ and $K(\lambda,\lambda')$ by means of Fredholm integral equations of the first kind. These equations are solved exactly in Sec. III and some examples of physical interest are studied. In Sec. IV we give an alternative derivation of $K(\lambda,\lambda')$ by considering Gaussian fluctuations in the level density. The global maximum-entropy result for $K(\lambda,\lambda')$ is recovered by solving the associated stochastic equation using an adiabatic approximation. The distinction between the fractional fluctuations of open and closed transmitting channels are shown to be a crucial difference between local and global maximum-entropy approaches. Conclusions and a summary are presented in Sec. V.

II. FORMULATION OF THE SCATTERING PROBLEM

Our model system consists of a disordered sample of finite length L and finite transverse cross-section W to which two perfectly ordered semi-infinite leads are connected. The leads support N scattering channels. The electron wave propagates coherently through the disordered sample and multiple elastic scattering can conveniently be described by a transfer-matrix M, which relates the wave amplitudes on the left to those on the right of the sample. It has been shown[9,10,18] that M performs a Brownian motion on its group manifold described by the Fokker-Planck equation

$$\frac{\partial P(\{\lambda\},\tau)}{\partial\tau} = \sum_{i=1}^N \left[-\frac{\partial}{\partial\lambda_i}\mathcal{D}_i^{(1)}(\{\lambda\}) + \frac{\partial^2}{\partial\lambda_i^2}\mathcal{D}_i^{(2)}(\{\lambda\})\right] P(\{\lambda\},\tau) , \tag{2.1a}$$

where $\tau = 2S/(\beta N + 2 - \beta)$ represents the diffusion time, S is the sample length in units of the mean free path, and $\mathcal{D}_i^{(1)}(\{\lambda\})$ and $\mathcal{D}_i^{(2)}(\{\lambda\})$ are, respectively, drift and diffusion coefficients given by

$$\mathcal{D}_i^{(1)}(\{\lambda\}) = 1 + 2\lambda_i + \beta \sum_{j(\neq i)} \frac{\lambda_i(1+\lambda_i)}{\lambda_i - \lambda_j} , \tag{2.1b}$$

$$\mathcal{D}_i^{(2)}(\{\lambda\}) = \lambda_i(1+\lambda_i) . \tag{2.1c}$$

The variables $\lambda_i \geq 0$ $(i=1,\ldots,N)$ appearing in (2.1) are the eigenvalues of the matrix $X = (M^\dagger M + (M^\dagger M)^{-1} - 2)/4$ and are related to the transmission eigenvalues $\hat{T}_i$ by $\lambda_i = 1/\hat{T}_i - 1$.

From (2.1) it has been shown[9] that the average of an arbitrary function $f(\{\lambda\})$ satisfies the following evolution equation:

$$\frac{\partial\langle f\rangle}{\partial\tau} = \left\langle \frac{1}{J}\sum_{i=1}^N \frac{\partial}{\partial\lambda_i}\left[\lambda_i(1+\lambda_i)J\frac{\partial f}{\partial\lambda_i}\right]\right\rangle , \tag{2.2}$$

in which $J = \prod_{i<j}|\lambda_i - \lambda_j|^\beta$.

We define the generating function

$$\Gamma_\eta = \sum_{i=1}^N \frac{1}{1+\lambda_i+\eta} , \tag{2.3}$$

in which $\eta > -1$ is an auxiliary parameter. Inserting (2.3) into (2.2) we find the evolution equations

$$2\partial_\tau\langle\Gamma_\eta\rangle = \frac{\partial}{\partial\eta}\left[\eta(1+\eta)\left[(2-\beta)\frac{\partial}{\partial\eta}\langle\Gamma_\eta\rangle - \beta\langle\Gamma_\eta^2\rangle\right]\right] \tag{2.4}$$

and

$$2\partial_\tau\langle\Gamma_\eta\Gamma_\xi\rangle = \frac{\partial}{\partial\eta}\left[\eta(1+\eta)\left[(2-\beta)\left\langle\Gamma_\xi\frac{\partial\Gamma_\eta}{\partial\eta}\right\rangle - \beta\langle\Gamma_\xi\Gamma_\eta^2\rangle - 4\frac{\langle\Gamma_\eta\rangle}{(\eta-\xi)^2}\right]\right] + (\eta\leftrightarrow\xi) . \tag{2.5}$$

We are interested in the metallic regime, which is defined by the inequalities $1 \ll S \ll N$. In this regime it makes sense to expand $\langle \Gamma_\eta \rangle$ and $\langle \Gamma_\eta \Gamma_\xi \rangle$ as

$$\langle \Gamma_\eta \rangle = \frac{N}{S} g(\eta) + \Psi(\beta) f(\eta) + h_\beta(\eta) \frac{S}{N} + O(S^2/N^2) \tag{2.6}$$

and

$$\langle \Gamma_\eta \Gamma_\xi \rangle = \frac{N^2}{S^2} g(\eta) g(\xi) + \frac{N}{S} \Psi(\beta)[g(\eta) f(\xi) + g(\xi) f(\eta)] + H_\beta(\eta,\xi) + O(S/N) , \tag{2.7}$$

where $\Psi(\beta) \equiv (\beta-2)/\beta$.

Inserting (2.6) and (2.7) into (2.4) and (2.5), taking N and S to infinity keeping their ratio G_B fixed, using the fact[16] that the third cumulant $\langle\langle \Gamma_\eta \Gamma_\xi \Gamma_\chi \rangle\rangle < O(1/G_B)$ and equating coefficients of each power in G_B, we find that $g(\eta)$, $f(\eta)$, $h_\beta(\eta)$ and $H_\beta(\eta,\xi)$ satisfy the following set of coupled differential equations:

$$\frac{dg(\eta)}{d\eta} + \frac{2\eta+1}{2\eta(\eta+1)} g(\eta) = \frac{1}{2\eta(\eta+1)} , \tag{2.8}$$

$$\frac{df(\eta)}{d\eta} + \frac{1+(2\eta+1)g(\eta)}{2\eta(\eta+1)g(\eta)} f(\eta) = \frac{(2\eta+1)g'(\eta)+2g(\eta)}{4\eta(\eta+1)g(\eta)} , \tag{2.9}$$

$$h_\beta(\eta) = -\frac{\partial}{\partial\eta}[\eta(1+\eta)\{\Psi^2(\beta) f'(\eta) + H_\beta(\eta,\eta)\}] , \tag{2.10}$$

$$\frac{\partial}{\partial\eta}\left[\eta(1+\eta)\left\{\Psi^2(\beta) f(\xi) g'(\eta) + 2g(\eta) H_\beta(\eta,\xi) - 2g(\eta) g(\xi) h_\beta(\eta) - g^2(\eta) h_\beta(\xi) + \frac{4}{\beta}\frac{g(\eta)}{(\eta-\xi)^2}\right\}\right] - h_\beta(\eta) g(\xi) + (\eta \leftrightarrow \xi) = 0 . \tag{2.11}$$

It is convenient to make the following change of variables:

$$\eta = \tfrac{1}{2}[\cosh(x)-1] , \quad \xi = \tfrac{1}{2}[\cosh(y)-1] , \tag{2.12}$$

and to define new functions

$$\hat{g}(x) = \sinh(x) g[\sinh^2(x/2)] , \tag{2.13}$$

$$\hat{f}(x) = \sinh(x) f[\sinh^2(x/2)] , \tag{2.14}$$

$$\hat{h}_\beta(x) = \sinh(x) h_\beta[\sinh^2(x/2)] , \tag{2.15}$$

$$\hat{H}_\beta(x,y) = \sinh(x)\sinh(y) H_\beta[\sinh^2(x/2), \sinh^2(y/2)] . \tag{2.16}$$

In term of the new variables we can rewrite (2.8)–(2.11) as

$$\frac{d\hat{g}(x)}{dx} = 1 , \tag{2.17}$$

$$\frac{d\hat{f}(x)}{dx} + \frac{1}{x}\hat{f}(x) = \frac{\sinh(2x)-2x}{2x\sinh^2(x)} , \tag{2.18}$$

$$\frac{d\hat{h}_\beta(x)}{dx} + \frac{2}{x}\hat{h}_\beta(x) = \frac{1}{x}\frac{d}{dx}\left[\frac{1}{\beta}\hat{Q}(x,x) + \Psi(\beta)\frac{3x^2+(x^2-3)\sinh^2(x)}{x^2\sinh^2(x)}\right] , \tag{2.19}$$

$$\frac{\partial}{\partial x}\left[2\Psi^2(\beta)\left[y\frac{2(1-x^2)\sinh^2(x)-4x^2+x\sinh(2x)}{2x^2\sinh^2(x)} - x\mathcal{L}(x)\mathcal{L}(y)\right] + y\hat{H}_\beta(x,x) - 2xy\hat{h}_\beta(x) - x^2\hat{h}_\beta(y) + \frac{16}{\beta}\frac{x\sinh(x)\sinh(y)}{[\cosh(x)-\cosh(y)]^2} + 2x\hat{H}_\beta(x,y)\right] + (x \leftrightarrow y) = 0 , \tag{2.20}$$

where

$$\frac{1}{\beta}\hat{Q}(x,y)\equiv\hat{H}_\beta(x,y)-x\hat{h}_\beta(y)-y\hat{h}_\beta(x)-\Psi^2(\beta)\mathcal{L}(x)\mathcal{L}(y)\ , \quad (2.21)$$

and

$$\mathcal{L}(x)\equiv\frac{d}{dx}\ln\left[\frac{\sinh(x)}{x}\right]. \quad (2.22)$$

The functions $\hat{g}(x)$, $\hat{f}(x)$, $\hat{h}_\beta(x)$, and $\hat{Q}(x,y)$ must satisfy the conditions

$$\lim_{x\to0^+}\frac{\hat{g}(x)}{x}=1\ , \quad (2.23)$$

$$\lim_{x\to0^+}\frac{\hat{f}(x)}{x}=\frac{1}{3}\ , \quad (2.24)$$

$$\lim_{x\to0^+}\frac{\hat{h}_\beta(x)}{x}=\frac{12-14\beta+3\beta^2}{45\beta^2}\ , \quad (2.25)$$

$$\lim_{(x,y)\to(0^+,0^+)}\frac{\hat{Q}(x,y)}{xy}=\frac{2}{15}\ , \quad (2.26)$$

since $\Gamma_0=G$, i.e., the conductance in units of $2e^2/h$. Equations (2.17)–(2.20) can now easily be solved to give

$$\hat{g}(x)=x\ ,\quad \hat{f}(x)=\mathcal{L}(x)\ , \quad (2.27)$$

$$\hat{h}_\beta(x)=\Phi(\beta)\frac{x\sinh(2x)+2x^2-4\sinh^2(x)}{2x^3\sinh^2(x)}\ , \quad (2.28)$$

and

$$\hat{Q}(x,y)=8\left[\frac{4xy}{(x^2-y^2)^2}-\frac{\sinh(x)\sinh(y)}{[\cosh(x)-\cosh(y)]^2}\right], \quad (2.29)$$

where

$$\Phi(\beta)\equiv\frac{12-14\beta+3\beta^2}{2\beta^2}\ . \quad (2.30)$$

If we now use Eqs. (1.2) and (1.3) with $A=\Gamma_\eta$, we find, after transforming to the variables x and y, the following Fredholm integral equations for the transformed average level density, $\hat{\rho}(x)$, and two-point correlation function $\hat{K}(x,y)$:

$$2\int_0^\infty\frac{\sinh(x)\hat{\rho}(x')dx'}{\cosh(x)+\cosh(x')}=xG_B+\Psi(\beta)\mathcal{L}(x)+\hat{h}_\beta(x)/G_B\ , \quad (2.31)$$

$$4\int_0^\infty\int_0^\infty\frac{\sinh(x)\sinh(y)\hat{K}(x',y')dx'dy'}{[\cosh(x)+\cosh(x')][\cosh(y)+\cosh(y')]}=\frac{1}{\beta}\hat{Q}(x,y)\ . \quad (2.32)$$

These equations are solved exactly in the next section. One of the immediate consequences of (2.32) is that $\hat{K}(x,y)$ has a universal form, which is independent of G_B, and displays the same simple dependence on the symmetry parameter β predicted by classical random-matrix theory.[13]

III. EXACT SOLUTIONS OF THE INTEGRAL EQUATIONS AND APPLICATIONS

With the use of the identity

$$\frac{\sinh(x)}{\cosh(x)+\cosh(y)}=2\int_0^\infty dk\frac{\sin(kx)\cos(ky)}{\sinh(k\pi)} \quad (3.1)$$

one can show that the integral equation

$$\int_0^\infty\frac{\sinh(x)\psi(y)dy}{\cosh(x)+\cosh(y)}=\phi(x) \quad (3.2)$$

has a general solution of the form

$$\psi(x)=\frac{1}{2\pi i}[\phi(x+i\pi)-\phi(x-i\pi)]\ . \quad (3.3)$$

Using (3.3) we can solve (2.31) to find

$$\hat{\rho}(x)=\frac{G_B}{2}+\frac{\Psi(\beta)}{2}\left[\tfrac{1}{2}\delta(x)+\frac{1}{x^2+\pi^2}\right]+\frac{\Phi(\beta)}{2G_B}\frac{d}{dx}\left[\frac{\coth(x)(x^2+\pi^2)-2x}{(x^2+\pi^2)^2}\right]. \quad (3.4)$$

If we define $\hat{a}(x)\equiv a[\sinh^2(x/2)]$, we can use (3.4) to calculate the average of a general linear statistic as

$$\langle A\rangle\equiv\int_0^\infty\hat{\rho}(x)\hat{a}(x)dx=\frac{G_B}{2}\int_0^\infty\hat{a}(x)dx+\frac{\Psi(\beta)}{2}\left[\frac{\hat{a}(0)}{2}+\int_0^\infty\frac{\hat{a}(x)dx}{x^2+\pi^2}\right]-\frac{\Phi(\beta)}{2G_B}\int_0^\infty\frac{d\hat{a}(x)}{dx}\left[\frac{\coth(x)(x^2+\pi^2)-2x}{(x^2+\pi^2)^2}\right]dx\ , \quad (3.5)$$

which in Fourier space gives (1.4). It is important to note that (3.4) should apply only on scales large compared with the mean level spacing, which is of order $1/G_B$. For $x<1/G_B$ the last term becomes comparable with the first term and our asymptotic expansion breaks down. In practical applications it is sufficient to require that $\hat{a}(x)$ in Eq. (3.5) have smooth variations on scales of order of the mean level spacing.

It is more convenient to solve (2.32) by using the identity

$$\hat{Q}(x,y)=32\int_0^\infty dk\frac{k\sin(kx)\sin(ky)}{e^{2k\pi}-1}\ , \quad (3.6)$$

which together with (3.1) yields

$$\hat{K}(x,y)=\frac{2}{\beta\pi^2}\int_0^\infty k(1-e^{-2k\pi})\cos(kx)\cos(ky)dk=\frac{1}{2\beta\pi^2}\frac{\partial^2}{\partial x\partial y}\ln\left[\frac{4\pi^2+(x-y)^2}{4\pi^2+(x+y)^2}\left[\frac{x+y}{x-y}\right]^2\right]. \quad (3.7)$$

Using (1.3) and (3.7) we can write the variance of a general linear statistic as

$$\mathrm{var}(A)\equiv\int_0^\infty dx\int_0^\infty dy\,\hat{a}(x)\hat{a}(y)\hat{K}(x,y)$$

$$=\frac{1}{2\beta\pi^2}\int_0^\infty dx\int_0^\infty dy\frac{d\hat{a}(x)}{dx}\frac{d\hat{a}(y)}{dy}\ln\left[\frac{4\pi^2+(x-y)^2}{4\pi^2+(x+y)^2}\left[\frac{x+y}{x-y}\right]^2\right]. \quad (3.8)$$

In Fourier space (3.8) reduces to (1.5).

It is also interesting to express $\hat{\rho}(x)$ and $\hat{K}(x,y)$ in terms of the original variables $\{\lambda\}$. The resulting expressions read as

$$\rho(\lambda)=\frac{G_B}{2\sqrt{\lambda(1+\lambda)}}+\frac{\Psi(\beta)}{2}\left[\tfrac{1}{2}\delta(\lambda)+\frac{1}{\sqrt{\lambda(1+\lambda)}(4r^2(\lambda)+\pi^2)}\right]$$

$$+\frac{\Phi(\beta)}{2G_B}\frac{d}{d\lambda}\left[\frac{1+2\lambda}{2\sqrt{\lambda(1+\lambda)}[4r^2(\lambda)+\pi^2]}-\frac{4r(\lambda)}{[4r^2(\lambda)+\pi^2]^2}\right], \quad (3.9)$$

and

$$K(\lambda,\lambda')=\frac{1}{2\pi^2\beta}\frac{\partial^2}{\partial\lambda\partial\lambda'}\ln\left[\frac{\pi^2+[r(\lambda)-r(\lambda')]^2}{\pi^2+[r(\lambda)+r(\lambda')]^2}\left[\frac{r(\lambda)+r(\lambda')}{r(\lambda)-r(\lambda')}\right]^2\right], \quad (3.10)$$

in which $r(\lambda)\equiv\ln(\sqrt{\lambda}+\sqrt{1+\lambda})$.

We now apply (1.4) and (1.5) to some cases of physical interest. The conductance G, in units of $2e^2/h$, of a disordered metal is given by

$$G=\sum_i a_G(\lambda_i), \quad (3.11)$$

where $a_G(\lambda)=(1+\lambda)^{-1}$. From (1.6) we get

$$\tilde{a}_G(k)=\frac{2\pi k}{\sinh(\pi k)}. \quad (3.12)$$

Inserting (3.12) into (1.4) and (1.5) yields

$$\langle G\rangle=G_B+\frac{\Psi(\beta)}{3}+\frac{2\Phi(\beta)}{45G_B}, \quad (3.13)$$

$$\mathrm{var}(G)=\frac{2}{15\beta}, \quad (3.14)$$

in agreement with Refs. 9, 16, and 18. The shot-noise power P, in units of $4e^3|v|/h$ (v is the applied potential), of a phase-coherent conductor is given by[19]

$$P=\sum_i a_P(\lambda_i), \quad (3.15)$$

where $a_P(\lambda)=\lambda(1+\lambda)^{-2}$. Equation (1.6) then gives

$$\tilde{a}_P(k)=\frac{2\pi k(1-2k^2)}{3\sinh(\pi k)}, \quad (3.16)$$

and thus

$$\langle P\rangle=\frac{G_B}{3}+\frac{\Psi(\beta)}{45}-\frac{2\Phi(\beta)}{105G_B}, \quad (3.17)$$

$$\mathrm{var}(P)=\frac{46}{2835\beta}, \quad (3.18)$$

in agreement with Refs. 16 and 19. The conductance G_{NS}, in units of $2e^2/h$, of a disordered microbridge between a normal metal and a superconductor is a linear statistic for the orthogonal case ($\beta=1$) and satisfies[20]

$$G_{NS}=\sum_i a_{NS}(\lambda_i), \quad (3.19)$$

where $a_{NS}(\lambda)=2(1+2\lambda)^{-2}$. Using (1.6) we find

$$\tilde{a}_{NS}(k)=\frac{k\pi}{\sinh(k\pi/2)}, \quad (3.20)$$

thus substituting (3.20) into (1.4) and (1.5) we get

$$\langle G_{NS}\rangle=G_B+\frac{4}{\pi^2}-1+\frac{4}{\pi^2G_B}\left[\frac{12}{\pi^2}-1\right], \quad (3.21)$$

$$\mathrm{var}(G_{NS})=\frac{16}{15}-\frac{48}{\pi^4}. \quad (3.22)$$

Finally, we consider the supercurrent-phase relationship $I(\phi)$, in units of $e\Delta/\hbar$ (Δ is the energy gap), in a point contact Josephson junction, which is a linear statistic on the eigenvalues of the transmission matrix associated with nonsuperconducting states. According to Ref. 21 we have (with $\beta=1$)

$$I(\phi)=\tfrac{1}{2}\sum_i\frac{a_G(\lambda_i)\sin(\phi)}{\sqrt{1-a_G(\lambda_i)\sin^2(\phi/2)}}$$

$$=\tfrac{1}{2}\sin(\phi)\sum_{n=0}^{\infty}T_{n+1}u_n\sin^{2n}(\phi/2), \quad (3.23)$$

where $T_n\equiv\sum_i a_G^n(\lambda_i)$ and u_n is related to the Legendre polynomial $P_n(x)$ by $u_n=(-1)^nP_{2n}(0)$. Using the generating function Γ_η we can show that

$$\langle T_m\rangle=A_mG_B+B_m\Psi(\beta)+C_m\Phi(\beta)/G_B, \quad (3.24)$$

$$\mathrm{cov}(T_m,T_n)=\frac{K_{n,m}}{\beta}, \quad (3.25)$$

where A_m, B_m, C_m, and $K_{n,m}$ are defined by the expansions

$$g(\eta)=\sum_{m=0}^{\infty}(-1)^m A_{m+1}\eta^m , \quad (3.26)$$

$$f(\eta)=\sum_{m=0}^{\infty}(-1)^m B_{m+1}\eta^m , \quad (3.27)$$

$$h_\beta(\eta)=\Phi(\beta)\sum_{m=0}^{\infty}(-1)^m C_{m+1}\eta^m , \quad (3.28)$$

and

$$Q(\eta,\xi)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}K_{n+1,m+1}\eta^n\xi^m , \quad (3.29)$$

in which $Q(\eta,\xi)$ is related to $\hat{Q}(x,y)$ by

$$\hat{Q}(x,y)=\sinh(x)\sinh(y)Q[\sinh^2(x/2),\sinh^2(y/2)] . \quad (3.30)$$

Using (3.24) we can write the average of (3.23) as

$$\langle I(\phi)\rangle = I^{(0)}(\phi)G_B + I^{(1)}(\phi) + I^{(2)}(\phi)/G_B , \quad (3.31)$$

where

$$I^{(0)}(\phi)=\tfrac{1}{2}\sin(\phi)\sum_{n=0}^{\infty}A_{n+1}u_n\sin^{2n}(\phi/2) = \cos(\phi/2)\tanh^{-1}[\sin(\phi/2)] , \quad (3.32)$$

$$I^{(1)}(\phi)=-\tfrac{1}{2}\sin(\phi)\sum_{n=0}^{\infty}B_{n+1}u_n\sin^{2n}(\phi/2) = -\frac{\sin(\phi)}{6}[1+\tfrac{7}{15}\sin^2(\phi/2)+\tfrac{71}{210}\sin^4(\phi/2) + \tfrac{521}{1890}\sin^6(\phi/2)+\cdots] , \quad (3.33)$$

and

$$I^{(2)}(\phi)=\tfrac{1}{4}\sin(\phi)\sum_{n=0}^{\infty}C_{n+1}u_n\sin^{2n}(\phi/2) = \frac{\sin(\phi)}{90}[1+\tfrac{5}{7}\sin^2(\phi/2)+\tfrac{23}{35}\sin^4(\phi/2) + \tfrac{1315}{2079}\sin^6(\phi/2)+\cdots] , \quad (3.34)$$

while for the variance we find

$$\mathrm{var}[I(\phi)]=\tfrac{1}{4}\sin^2(\phi)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}K_{n+1,m+1}u_n u_m\sin^{2n}(\phi/2)\sin^{2m}(\phi/2) = \frac{\sin^2\phi}{30}[1+\tfrac{62}{63}\sin^2(\phi/2)+\tfrac{3631}{3780}\sin^4(\phi/2)+\tfrac{9773}{10395}\sin^6(\phi/2)+\cdots] . \quad (3.35)$$

A quantity of particular interest is the variance of the critical supercurrent $I_c \equiv \max I(\phi) = I(\phi_c)$. It has been shown in Ref. 22 that $\mathrm{rms}(I_c)=\mathrm{rms}[I(\phi_c)]$, where $\phi_c \approx 1.97$. Since all coefficients in (3.35) are very close to 1 we can evaluate the sum approximately to find

$$\mathrm{var}[I(\phi)] \approx \tfrac{2}{15}\sin^2(\phi/2) , \quad (3.36)$$

and therefore $\mathrm{rms}(I_c)\approx 0.30$.

IV. GAUSSIAN FLUCTUATIONS IN THE LEVEL DENSITY

The objective of this section is to show that $K(\lambda,\lambda')$ results entirely from Gaussian fluctuations in the level density, defined as

$$\sigma(\lambda)\equiv\sum_i \delta(\lambda-\lambda_i) , \quad (4.1)$$

on scales that are large compared with the mean level spacing. This alternative way of deriving $K(\lambda,\lambda')$ bears a closer relation with the theory of universal fluctuations of linear statistics developed by Beenakker[15] in the context of the global maximum-entropy approach. In fact, we recover Beenakker's results simply by solving the exact stochastic equation for $\sigma(\lambda)$ in an adiabatic approximation.

We define new variables $\{x\}$ and t through the relations

$$\lambda_n=\tfrac{1}{2}[\cosh(2x_n t)-1] , \quad (4.2)$$

$$\tau=2t/\beta . \quad (4.3)$$

The resulting Langevin equation for each x_n can be obtained by using (4.2) and (4.3) in (2.1). We find

$$t\frac{dx_n}{dt}=-x_n+\frac{1}{\beta}\coth(2x_n t) + \sum_{m(\neq n)}\frac{\sinh(2x_n t)}{\cosh(2x_n t)-\cosh(2x_m t)}+\frac{\Gamma_n(t)}{\sqrt{2\beta}} , \quad (4.4)$$

where $\Gamma_n(t)$ is the usual Langevin force satisfying

$$\langle\Gamma_n(t)\Gamma_m(t')\rangle=2\delta_{nm}\delta(t-t') . \quad (4.5)$$

Assuming Gaussian fluctuations around the equilibrium position $\{x^{(0)}\}$, we find that $\delta x_n \equiv x_n - x_n^{(0)}$ satisfies

$$\frac{d(\delta x_n)}{dt}=-\sum_m \gamma_{nm}(t)\delta x_n+\frac{\Gamma_n(t)}{t\sqrt{2\beta}} , \quad (4.6)$$

where

$$\gamma_{nm}(t)=\frac{\pi}{t^2(N+1)}\sum_q q\coth\left[\frac{q\pi}{2t}\right]\sin(nq)\sin(mq) , \quad (4.7)$$

in which $q=(N+1)^{-1}\pi l$ and $l=1,2,\ldots,N$. Note that the linearization of the force term in Eq. (4.6) is exact only for $q \ll 1/t$, which is the regime that determines the behavior of linear statistics derived from smooth functions $a(\lambda)$.

The normal coordinates

$$\delta\tilde{x}_q = \left[\frac{2}{N+1}\right]^{1/2} \sum_n \delta x_n \sin(nq) \tag{4.8}$$

satisfy a set of decoupled Langevin equations

$$\frac{d(\delta\tilde{x}_q)}{dt} = -\gamma_g(t)\delta\tilde{x}_q + \frac{\tilde{\Gamma}_q(t)}{t\sqrt{2\beta}} , \tag{4.9}$$

where

$$\langle \tilde{\Gamma}_q(t)\tilde{\Gamma}_{q'}(t')\rangle = 2\delta_{qq'}\delta(t-t') \tag{4.10}$$

and

$$\gamma_q(t) = \frac{q\pi}{2t^2}\coth\left[\frac{q\pi}{2t}\right] . \tag{4.11}$$

It is instructive at this stage to introduce an adiabatic approximation in which we suppose that $\delta\tilde{x}_q$ is solely determined by the rapid fluctuations of the Langevin noise, so that

$$\frac{d(\delta\tilde{x}_q)}{d\tau} = -\gamma_q(t)\delta\tilde{x}_q + \frac{\tilde{\Gamma}_q(\tau)}{t\sqrt{2\beta}} . \tag{4.12}$$

Equation (4.12) can be integrated to yield

$$\delta\tilde{x}_{q,t}(\tau) = \delta\tilde{x}_{q,t}(0)e^{-\tau\gamma_q(t)} + \frac{1}{t\sqrt{2\beta}}\int_0^\tau e^{-(\tau-\tau')\gamma_q(t)}\tilde{\Gamma}_q(\tau')d\tau' . \tag{4.13}$$

For $\tau \gg 1/\gamma_q(t)$ one finds

$$\langle \delta\tilde{x}_{q,t}^2\rangle_0 = \frac{1}{\beta q\pi}\tanh\left[\frac{q\pi}{2t}\right] , \tag{4.14}$$

where $\langle\ \rangle_0$ stands for adiabatic average.

Going beyond the adiabatic approximation we get from (4.9) the exact expression

$$\langle \delta\tilde{x}_{q,t}^2\rangle = \frac{1}{\beta q\pi}(1-e^{-q\pi/t}) . \tag{4.15}$$

From the relation

$$\mathrm{var}(A) = \sum_{n,m} \frac{d}{dx}a(\lambda[x])\bigg|_{x=x_n} \frac{d}{dx'}a(\lambda'[x'])\bigg|_{x'=x_m} \times\langle \delta x_n \delta x_m\rangle , \tag{4.16}$$

Equation (1.3) and the definition

$$\hat{K}(x,y) \equiv \tfrac{1}{4}\sinh(x)\sinh(y)K[\sinh^2(x/2),\sinh^2(y/2)] , \tag{4.17}$$

we find

$$\hat{K}(x,y) = \frac{4}{\pi}\int_0^\infty dk\, k^2\cos(kx)\cos(ky)\lim_{t\to 0} t\langle \delta\tilde{x}_{q,t}^2\rangle|_{q=2kt} . \tag{4.18}$$

Inserting (4.14) into (4.18) we get

$$\hat{K}_0(x,y) = \frac{2}{\beta\pi^2}\int_0^\infty k\tanh(k\pi)\cos(kx)\cos(ky)dk = \frac{1}{\beta\pi^2}\frac{\partial^2}{\partial x\partial y}\ln\left[\frac{\sinh(x/2)+\sinh(y/2)}{\sinh(x/2)-\sinh(y/2)}\right] , \tag{4.19}$$

which in terms of the original $\{\lambda\}$ variables gives

$$K_0(\lambda,\lambda') = \frac{1}{\pi^2\beta}\frac{\partial^2}{\partial\lambda\partial\lambda'}\ln\left[\frac{\sqrt{\lambda}+\sqrt{\lambda'}}{\sqrt{\lambda}-\sqrt{\lambda'}}\right] , \tag{4.20}$$

in agreement with Ref. 15. Inserting (4.15) into (4.18) we recover (3.7) and (3.10).

In order to understand the fundamental differences between (4.19) and (3.10) it is useful to discuss differences between open and closed channels. From (3.11) we see that only the channels for which $a_G(\lambda_i) \approx O(1)$ can make a finite contribution to the average conductance. Thus we say that a channel i is open if $\lambda_i \ll 1$, while it is closed if $\lambda_i \gg 1$. There are three different regimes: The ballistic regime, where all channels are open, the diffusive regime, where a finite fraction of channels are closed, and the insulating regime, where all channels are closed. We shall consider only the diffusive regime.

The relative contribution of the open channels to the variance of a linear statistic can be obtained by using the form of $K(\lambda,\lambda')$ in (3.10) for $\lambda,\lambda' \ll 1$. We find

$$K(\lambda,\lambda')|_{\mathrm{open}} = K_0(\lambda,\lambda') \tag{4.21}$$

and therefore

$$\mathrm{var}(A)|_{\mathrm{open}} = \frac{2}{\beta\pi^2}\int_0^\infty dk\, k\tanh(k\pi)\tilde{a}^2(k) . \tag{4.22}$$

We get, for instance,

$$\mathrm{var}(G)|_{\mathrm{open}} = \frac{1}{8\beta} . \tag{4.23}$$

For the closed channels we take (3.10) with $\lambda,\lambda' \gg 1$. The resulting two-point correlation function is

$$K(\lambda,\lambda')\bigg|_{\mathrm{closed}} = \frac{1}{2\pi^2\beta}\frac{\partial^2}{\partial\lambda\partial\lambda'} \times\ln\left[\frac{4\pi^2+(\ln\lambda-\ln\lambda')^2}{(\ln\lambda-\ln\lambda')^2}\right] , \tag{4.24}$$

thus

$$\mathrm{var}(A)|_{\mathrm{closed}} = \frac{1}{\beta\pi^2}\int_0^\infty dk\, k(1-e^{-2k\pi})|\tilde{a}(k)|^2 , \tag{4.25}$$

where

$$\tilde{a}(k) \equiv \int_{-\infty}^\infty dx\, a(e^x)e^{ikx} . \tag{4.26}$$

Applying (4.25) to the conductance yields

$$\mathrm{var}(G)|_{\mathrm{closed}} = \frac{1}{6\beta} . \tag{4.27}$$

If we use the global maximum-entropy result (4.20) for the two-point correlation function we find that

$$K_0(\lambda,\lambda')|_{\rm open}=K_0(\lambda,\lambda')=K_0(\lambda,\lambda')|_{\rm closed}\ , \tag{4.28}$$

which, of course, implies

$$\mathrm{var}(A)_0|_{\rm open}=\mathrm{var}(A)_0=\mathrm{var}(A)_0|_{\rm closed}\ . \tag{4.29}$$

On the other hand, for the local maximum-entropy approach we find inequalities

$$\mathrm{var}(A)|_{\rm open}<\mathrm{var}(A)<\mathrm{var}(A)|_{\rm closed}\ . \tag{4.30}$$

This crucial difference between global and local maximum-entropy approaches has an important consequence: It is not possible to account for the geometrical dependence of universal fluctuations via suitably chosen average level densities. A similar conclusion has also been drawn in Ref. 15.

V. SUMMARY AND CONCLUSIONS

We have calculated exactly the average level density and two-point correlation function of the transmission-matrix eigenvalues of a phase-coherent quasi-one-dimensional conductor in the limit of a large number of transverse scattering channels. As a consequence, we have derived exact expressions for the average and variance of a general linear statistic on the transmission eigenvalues. We have applied the formulas to a number of examples of particular physical interest. The results show that weak-localization and universal mesoscopic fluctuations are, in fact, very general phenomena that apply to a large number of transport observables in the mesoscopic regime.

Using the stochastic evolution equation of the level density we have shown that the two-point correlation function is completely determined by Gaussian fluctuations, at scales that are large compared with the mean level spacing. A crucial difference between global and local maximum-entropy approaches has been discussed. The contributions of open and closed channels to the fluctuations of linear statistics is qualitatively different in these theories, which, in turn, implies that the basic assumption of the global maximum-entropy approach that information about the geometric structure could be included in the average level density cannot be justified.

We conclude by remarking that since an increasing number of transport observables for a variety of systems has been shown to be linear statistics on the transmission eigenvalues, we expect our results to be of considerable general interest. Extensions to more elaborate geometries, while appearing possible, do not seem to be straightforward and require further research.

We recently learned of related work by Beenakker,[23] in which the second term of Eq. (1.4) has been obtained using a different method.

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