Large-N renormalization-group study of the commensurate dirty-boson problem

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We use a large-N renormalization-group (RG) method to study a model of interacting boson system with a quenched random potential. In the absence of impurities, the pure boson system has a critical point that describes the superfluid–Mott-insulator (SF–MI) transition. The SF–MI transition of d-dimensional bosons belongs to the $(d+1)$-dimensional XY model universality class. In this paper, we study the dirty-boson critical points in the neighborhood of this pure SF–MI critical point. In general, the on-site random potential in the original lattice model gives two types of randomness in the effective-field-theoretic action. One is the randomness in the effective on-site repulsion $w(\mathbf{x})$ and the other is the randomness of the chemical potential $u(\mathbf{x})$. It turns out that $d=2$ is the critical dimension for both types of disorder but the roles of these two types of disorder are reversed as $d=2$ is crossed. Applying $\epsilon=d-2$ expansion, we found coupled RG equations for both kinds of randomness which reveal several nontrivial critical points. All the weak random fixed points we found have three or more relevant directions. We conclude that the direct SF–MI transition is unlikely to occur near two dimensions.

I. INTRODUCTION

The problem of repulsively interacting bosons in a random potential has been the subject of intense research recently.[1-10] This so-called dirty-boson problem contains the essential difficulty of understanding interplay between interaction effect and randomness. One of the reasons why this problem is very challenging is that there is no sensible noninteracting limit for disordered bosons. That is, the zero interaction limit is pathological in the sense that the bosons will condense into the lowest localized state around a small region. For the metal-insulator transition of electrons, which has been understood better, disorder alone or interaction alone can induce localization of electrons and drive the electrons to the Anderson-insulator or the Mott-insulator (MI) state.[11] The interplay between these two effects has been studied during the last decade but there are still open problems.[11]

The dirty-boson problem has direct experimental realization of ^{4}He in Vycor glass and in other porous media.[12,13] This disordered interacting boson problem may be used to understand the superconductor-insulator transition in the disordered thin films[14] and short coherence-length superconductors.[15] Recently Wen and Wu[16] showed that the superfluid-Mott insulator transition of bosons with the Chern-Simons gauge field can describe the transition between quantum Hall (QH) states in the absence of disorder. Therefore, the dirty-boson problem with the Chern-Simons gauge field is intimately related to the quantum Hall transitions in the presence of random impurities. The QH–MI transition for the pure system is also studied by Chen, Fisher, and Wu[17] in which they studied fermions with Chern-Simons gauge field.

Following Fisher and co-workers,[1,2] we can write the Hamiltonian of the interacting lattice bosons in a random on-site potential as

$$H=H_0+H_1\ ,$$
$$H_0=-\sum_i(-J_0+\mu+\delta\mu_i)\hat{n}_i+\tfrac{1}{2}V\sum_i\hat{n}_i(\hat{n}_i-1)\ , \qquad (1)$$
$$H_1=-\sum_{i,j}J_{ij}(b_i^\dagger b_j+\text{H.c.})\ ,$$

where $\hat{n}_i=b_i^\dagger b_i$ and $b_i^\dagger$ is the boson creation operator at the site i. μ is the average chemical potential that fixes the boson density and $\delta\mu_i$ is the random on-site potential with zero average. J_{ij} is the hopping matrix element and $J_0=\sum_j J_{ij}$. In order to study the critical phenomena of the system, it is convenient to find the effective-theory. We will summarize the approach of Fisher and co-workers.[1,2] First the off-site hoping term in H_1 is decoupled by introducing the Hubbard-Stratanovich field Ψ_i, then the resulting action can be expanded in terms of Ψ_i. Since Ψ_i is linearly related to $\langle b_i\rangle$ for small $\langle b_i\rangle$, the field Ψ_i can be identified as a superfluid (SF) order parameter. It was shown[2] that the effective-field-theoretic action is given by

$$S=\int\frac{d^dk}{(2\pi)^d}\frac{d\omega}{2\pi}(k^2+\omega^2)|\Psi(k,\omega)|^2$$
$$+\sum_i(\overline{w}+w_i)|\Psi_i(\tau)|^2$$
$$+\sum_i\int d\tau(\overline{u}+u_i)\Psi_i^*(\tau)\partial_\tau\Psi_i(\tau)$$
$$+g\sum_i\int d\tau|\Psi_i(\tau)|^4\ , \qquad (2)$$

where w_i,u_i are random functions of i with zero average.

It is established[2] that SF–MI transitions of the pure system has two universality classes, the commensurate case and the incommensurate case. In the commensurate case, the superfluid density commensurates with a periodic potential. In this case, SF–MI transition is described

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by a tricritical point which belongs to the universality class of $(d+1)$-dimensional XY model with the dynamical exponent $z=1$. In the incommensurate case, the SF–MI transition happens on a line in the $\bar{w}-\bar{u}$ plane. It is argued[2] that the generic SF–MI transition should be the latter case (with $z=2$ at the transition) rather than the former case.

One natural question is that how disorder affects these two different SF–MI transitions. The destruction of superfluid in the presence of disorder brought the concept of the Bose-glass (BG) phase[4,18] in which bosons are localized by disorder. In seminal papers, Fisher and co-workers[1,2] suggest a scaling theory of SF–BG transition. They argued that superfluid-insulator transition should occur through the Bose-glass phase and the generic transition should be described by the action (2) with $\bar{u}+u_i \neq 0$ which does not have space-time isotropy. In the scaling theory, they postulated that the compressibility is totally due to the phonon mode and one of the main results of this assumption is that the dynamical exponent $z=d$. The simulation of the quantum rotor model[3] and some renormalization-group calculation[7] partially supported this picture although a recent quantum Monte-Carlo calculation[9] contradicts to these results. The earlier work of Ma, Halperin, and Lee[4] (MHL) was reexamined and the importance of the term that is linear in ω is emphasized. However, it was also argued that MHL theory may apply to the possible direct SF–MI transitions in the commensurate case (particle-hole symmetric case).

We can see that the dirty-boson effective action has the strict particle-hole symmetry if $\bar{u}+u_i=0$. It was also argued that the general commensurate case corresponds to the weaker particle-hole symmetric case,[2] i.e., $\bar{u}=0$ but $u_i \neq 0$. Are the transitions of particle-hole symmetric and asymmetric cases in the same universality class? Originally Fisher and co-workers[1,2] preferred that even arbitrarily weak disorder will induce the Bose-glass phase for both of the incommensurate and commensurate cases [Fig. 1(a)]. However, numerical calculations in Ref. 8 have not revealed the intervening Bose-glass phase in the commensurate case, although the superfluid–insulator transition indeed occurs through the Bose-glass phase in the incommensurate case [Fig. 1(b)]. Singh and Rokhsar[5] performed a real-space renormalization-group (RG) analysis for the commensurate case and they found the direct transitions from SF to MI when the disorder is sufficiently weak; the Bose glass is found beyond a threshold [Figs. 1(a) or 1(b) depending on the impurity strength]. Zhang and Ma[6] considered hard-core bosons with disorder. In this real-space renormalization-group analysis of a quantum spin-$\frac{1}{2}$ XY model with transverse random field (the hard-core boson model is mapped to this model), they concluded that commensurate and incommensurate cases are in the same universality class and the SF–insulator transition occurs always from the Bose-glass phase [Fig. 1(a)]. The conventional renormalization-group calculation by Weichman and Kim,[7] in which they used double dimensional expansion around $d=4$, partially supported the original picture[1,2] for the general dirty-boson problem although some technical problems exist.

In this paper, we are going to study a large-N generalization of the original action. By doing $1/N$ expansion, we can treat the interaction nonperturbatively in the coupling constant. Both types of disorder are assumed to be weak and we do the perturbation in the strength of two types of disorder. The large-N generalized action of the original dirty-boson model in the Eucledian space is given by

$$S=\int \frac{d^dk}{(2\pi)^d}\frac{d\omega}{2\pi}(\omega^2+v_0^2k^2+\bar{w})|\phi_i(k,\omega)|^2 + \int d^dx d\tau \left[(\bar{u}+u(\mathbf{x}))\phi_i^\dagger\partial_\tau\phi_i + w(\mathbf{x})\phi_i^\dagger\phi_i+\frac{g_0}{N}(\phi_i^\dagger\phi_i)^2\right], \tag{3}$$

where $i=1,\ldots,N$ and $N=1$ corresponds to the original model. $u(\mathbf{x})$ and $w(\mathbf{x})$ are Gaussian random functions of $\mathbf{x}$ with zero mean and their variances are given by $\langle u(\mathbf{x})u(\mathbf{y})\rangle = U_0\delta^d(\mathbf{x}-\mathbf{y})$ and $\langle w(\mathbf{x})w(\mathbf{y})\rangle = W_0\delta^d(\mathbf{x}$

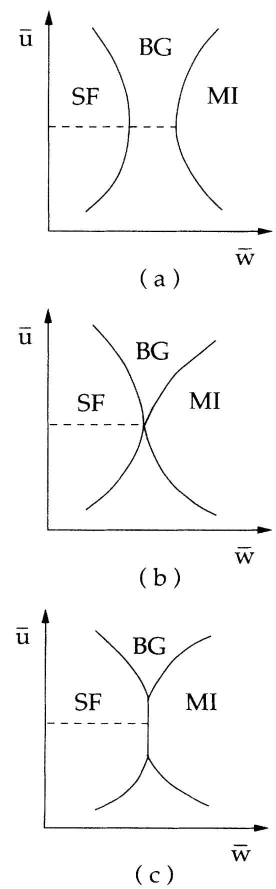

FIG. 1. Three possible phase diagrams for the dirty-boson system described by Eqs. (2) and (3). Our RG results favor the case (a). The boson density commensurates with the lattice in the MI phase and on the dotted line. In (a) there is no direct SF–MI transition. In (b) direct SF–MI transition is described by a tricritical point. Both of the BG and the MI phases are insulators, but the former is gapless and the latter has a finite gap. The commensurate dirty-boson model is defined on the dotted line and its extension in the MI phase.

$-\mathbf{y})$, respectively, where $\langle \cdots \rangle$ means the random average.

When $\bar{u}+u(\mathbf{x})=\bar{w}+w(\mathbf{x})=0$, (3) describes the multicritical point of SF–MI transition of the pure boson system. Near $d=2$, both of $\bar{u}$ and $\bar{w}$ terms are strongly relevant. In this paper, we will take $\bar{u}=\bar{w}=0$ and study the RG flow of U_0, W_0, and v_o. The physical meaning of setting $\bar{u}=0$ is the following. We tune the chemical potential μ to make the average boson density always commensurate with the lattice. Thus, we will call (3) with $\bar{u}=0$ the commensurate dirty-boson theory.

Our RG calculations are done at the critical point and with the renormalized mass term $\bar{w}=0$ in the course of the renormalization. Since the effect of the ϕ^4 term is calculated exactly at each order of $1/N$, the coupling constant g_0 is not renormalized. Following Ref. 19 and using dimensional regularization, we move an infrared scale μ in the renormalized theory with fixed bare parameters to obtain the RG flow of the renormalized parameters. Introducing dimensionless measures of disorder, $\tilde{U}=(U/v^2)\mu^{d-2}$ and $\tilde{W}=(W_v^2/g_0^2)\mu^{2-d}$, we performed $\epsilon=d-2$ expansion. The resulting renormalization-group equation up to $(1/N)^0$ order is found to be

$$\begin{aligned}
\frac{d\tilde{W}}{dl}&=\epsilon\tilde{W}+a\tilde{U}\tilde{W}+b\tilde{W}^2 ,\\
\frac{d\tilde{U}}{dl}&=-\epsilon\tilde{U}+b\tilde{W}\tilde{U}-a\tilde{U}^2 ,\\
\frac{d(\ln v)}{dl}&=-b\tilde{U}+c\tilde{W} ,
\end{aligned} \tag{4}$$

where $a=1/2\pi$, $b=128/\pi$, $c=64/\pi$ and l is the logarithmic measure of the RG flow. This is the central result of this paper.

Looking at $\epsilon\geq 0$ case, we can immediately see that there is only the pure fixed point which is given by $\tilde{U}^*=0, \tilde{W}^*=0$. At this trivial fixed point, $\tilde{U}$ is irrelevant and $\tilde{W}$ is relevant so that the RG flow goes to the strong randomness regime where our RG breaks down. However, for $\epsilon<0$, there are three fixed points. All of them have at least one relevant direction in the $\tilde{U}-\tilde{W}$ plane. Thus, including $\bar{u}$ and $\bar{w}$, these fixed points in the original theory (3) have at least three relevant directions. Therefore, in both cases, the direct SF–MI transition in the $\tilde{W}-\tilde{U}$ plane is unlikely to occur due to the absence of the weak random fixed point with two or less relevant directions. The superfluid–insulator transition is always governed by a strong random fixed point which cannot be reached by weak randomness expansion. More details of the RG flow will be discussed later.

The organization of this paper is as follows. In Sec. II, we consider a restricted model in which $u(\mathbf{x})=0$ and show the basic formalism we used. Here, we also examine some possible effects of long-range interactions. The RG calculation for the generic commensurate dirty-boson problem up to $(1/N)^0$ order is presented and the RG equation is calculated in Sec. III. We also discuss the results. In Sec. IV, the $1/N$ correction due to ϕ^4 interaction is considered. In Sec. V, we summarize and conclude this paper.

II. RENORMALIZATION-GROUP ANALYSIS FOR THE STRONG PARTICLE-HOLE SYMMETRIC MODEL

In this section, we consider a rather restricted model in which we set $u(\mathbf{x})=0$. This corresponds to the *strong* particle-hole symmetric case in the sense that more general model requires the particle-hole symmetry only in the average sense, i.e., $\langle u(\mathbf{x})\rangle=0$ and allows the local breaking of the symmetry at each site. We would like to consider more general long-range interactions $\int d^dx d^dy \phi^2(\mathbf{x})V(\mathbf{x}-\mathbf{y})\phi^2(\mathbf{y})$ with $V(\mathbf{x})\propto g_0/|\mathbf{x}|^{d-\lambda}$ and $V(q)=g_0/q^\lambda (0\leq\lambda<1/2)$. The $\lambda=0$ case corresponds to the usual short-range interaction. However, it should be mentioned that these are not the true long-range interactions of bosons because the true one should be $V(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x})\rho(\mathbf{x}')$ and $\rho(\mathbf{x})=i(\phi^\dagger\partial_0\phi-\partial_0\phi^\dagger\phi)$. But the long-range interaction between densities will induce the long-range interaction in the ϕ^4 term.

In order to perform the RG calculation in the critical theory, we need to introduce an arbitrary infrared mass parameter μ.[19] We will use the dimensional regularization method to calculate relevant divergent diagrams. The renormalization of the theory is given by the renormalization of two-point $\Gamma^{(2)}$ and four-point $\Gamma^{(4)}$ vertices. We will take the inverse of the full propagator as a two-point vertex and the two-boson scattering amplitude as a four-point vertex function. The relation between the bare theory and the renormalized theory is given by

$$\Gamma^{(N)}_{\text{bare}}(q,\omega;\Lambda)=Z^{-N/2}(\Lambda/\mu)\Gamma^{(N)}(q,\omega;\mu) , \tag{5}$$

where $\Gamma^{(N)}_{\text{bare}}$ and $\Gamma^{(N)}$ represent the bare and the renormalized vertices, respectively. We found that appropriate renormalization condition for $\Gamma^{(2)}$ can be chosen as

$$\begin{aligned}
\left.\frac{\partial}{\partial\omega^2}\Gamma^{(2)}\right|_{q=\mu,\lim_{\alpha\to 0}\omega=\alpha v\mu}&=1 ,\\
\left.\frac{\partial}{\partial q^2}\Gamma^{(2)}\right|_{q=\mu,\lim_{\alpha\to 0}\omega=\alpha v\mu}&=v^2 .
\end{aligned} \tag{6}$$

The renormalization condition for the scattering amplitude will be discussed later. Also, following standard procedure, we require the independence of the bare theory with respect to μ,

$$\mu\frac{d}{d\mu}|_\Lambda\Gamma^{(N)}_{\text{bare}}=0 , \tag{7}$$

where Λ is the mass parameter of the bare theory.

Let us start with the evaluation of the self-energy to the $(1/N)^0$ order. The $(1/N)^0$ order self-energy diagram is given by Fig. 2(a). The polarization bubble $\Pi_1(q,\omega=0)$ in Fig. 2(b) is calculated as

$$\begin{aligned}
\Pi_1(q,\omega=0)&=\int\frac{d^dk}{(2\pi)^d}\frac{d\nu}{2\pi}\frac{1}{\nu^2+v_0^2k^2}\frac{1}{\nu^2+v_0^2(k-q)^2}\\
&=\frac{c_1}{v_0^3}q^{d-3} ,
\end{aligned} \tag{8}$$

$$c_1=\frac{1}{(4\pi)^{(d+1)/2}}\frac{\Gamma[(3-d)/2]\Gamma^2[(d-1)/2]}{\Gamma(d-1)} .$$

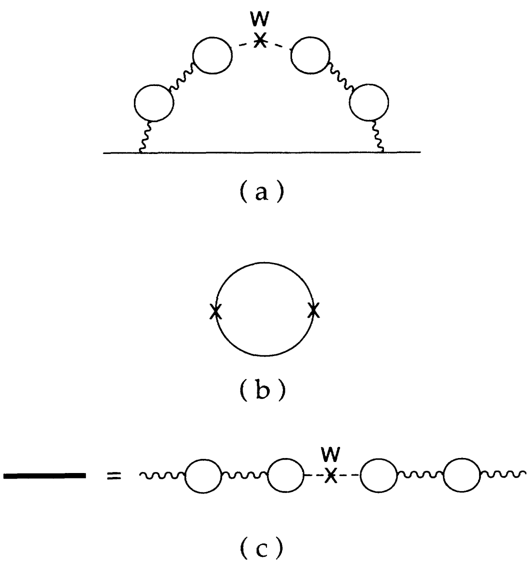

FIG. 2. (a) Self-energy correction due to the $w(\mathbf{x})$-type disorder. The wavy line represents the interaction and the dashed line with a cross $\times$ means the impurity average. W means $w(\mathbf{x})$-type disorder. (b) The polarization bubble Π_1. (c) The renormalized interaction (thick line) due to the $w(\mathbf{x})$-type disorder.

Assuming $3+\lambda > d$, the diagram in Fig. 2(c) can be approximated as

$$W_0\left[\frac{1}{1+\Pi_1(q,\omega=0)(g_0/q^\lambda)}\right]^2 \approx W_0\frac{v_0^6}{c_1^2g_0^2}q^{2(3+\lambda-d)} . \tag{9}$$

Using this result, we can evaluate the self-energy as

$$\Sigma_1(q,\omega)=\frac{d^dk}{(2\pi)^d}\frac{1}{\omega^2+v_0^2k^2}\left[\frac{W_0v_0^6}{c_1^2g_0^2}\right](k-q)^{2(3+\lambda-d)}$$
$$\approx\left[\frac{W_0v_0^6}{c_1^2g_0^2}\right]\left(c_2q^{4+2\lambda-d}+c_3\frac{\omega^2}{v_0^2}q^{2+2\lambda-d}\right) ,$$
$$c_2=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-2-\lambda)\Gamma(3+\lambda-d/2)\Gamma(d/2-1)}{\Gamma(d-3-\lambda)\Gamma(2+\lambda)} , \tag{10}$$
$$c_3=\frac{2+\lambda-d/2}{(4\pi)^{d/2}}\times\frac{\Gamma(d/2-2-\lambda)\Gamma(3+\lambda-d/2)\Gamma(d/2-2)}{\Gamma(d-3-\lambda)\Gamma(1+\lambda)} ,$$

where $v_0q \gg \omega$ is assumed. The bare two-point vertex $\Gamma^{(2)}_{\text{bare}}$ up to $(1/N)^0$ order is

$$\Gamma^{(2)}_{\text{bare}}=\omega^2+v_0^2q^2-\Sigma_1(q,\omega)$$
$$=\omega^2\left[1-\frac{W_0v_0^2}{g_0^2}\frac{c_3}{c_1^2}q^{2+2\lambda-d}\right]+v_0^2q^2\left[1-\frac{W_0v_0^2}{g_0^2}\frac{c_2}{c_1^2}q^{2+2\lambda-d}\right] . \tag{11}$$

Note that c_2 and c_3 diverge at $d=2+2\lambda$. Therefore, let us try $\epsilon=d-2-2\lambda$ expansion in order to handle these divergences. Also, for convenience, let us introduce dimensionless measures of the disorder $\tilde{W}_0=(W_0v_0^2/g_0^2)\Lambda^{2+2\lambda-d}$ in the bare theory and $\tilde{W}=(Wv^2/g_0^2)\mu^{2+2\lambda-d}$ in the renormalized theory. Adding appropriate counter terms to cancel the $1/\epsilon$ divergences from c_2 and c_3 and using the renormalization conditions (5) and (6), we get the following equations:

$$Z\approx 1+\tilde{W}\frac{\tilde{c}_3}{c_1^2}\ln\frac{\Lambda}{\mu} ,$$
$$v_0^2\approx Z^{-1}v^2\left[1-\tilde{W}\frac{\tilde{c}_2}{c_1^2}\ln\frac{\Lambda}{\mu}\right]^{-1} , \tag{12}$$

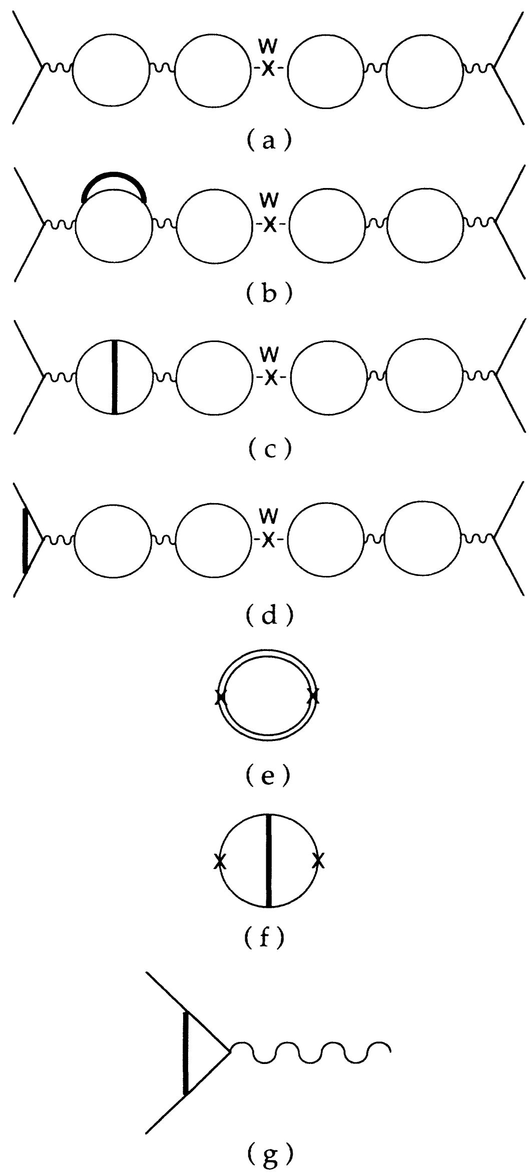

FIG. 3. (a), (b), (c), (d) are diagrams that contribute to the four-point function up to $\tilde{W}^2$ order. (e) The renormalized polarization bubble $\tilde{\Pi}_1$. The double line means the renormalized full propagator. (f) The polarization bubble Π_2. (g) The vertex V_1.

where $\tilde{c}_2=\epsilon c_2$ and $\tilde{c}_3=\epsilon c_3$. From μ independence of the bare parameters, we get

$$\mu\frac{\partial}{\partial\mu}(\ln Z)\approx-\tilde{W}\frac{\tilde{c}_3}{c_1^2}\,,$$
$$\mu\frac{\partial}{\partial\mu}v\equiv\beta(v)\approx\frac{1}{2}\left[\frac{\tilde{c}_2}{c_1^2}-\frac{\tilde{c}_3}{c_1^2}\right]\tilde{W}v\,. \tag{13}$$

Now we are going to renormalize the four-point function $\Gamma^{(4)}$. We will take the scattering amplitude of two bosons at $q_1=q_4=q$, $q_2=q_3=0$, $\omega_1=\omega_2=\omega_3=\omega_4=\omega$ as our $\Gamma^{(4)}$. First of all, let us calculate $\Gamma^{(4)}_{\rm bare}$ in the bare theory. $\Gamma^{(4)}_{\rm bare}$ in the $(1/N)^0$ order is the sum of $\Gamma^{(4)}_{\rm bare,1}$, $\Gamma^{(4)}_{\rm bare,2}$, $\Gamma^{(4)}_{\rm bare,3}$, and $\Gamma^{(4)}_{\rm bare,4}$, which are shown in Figs. 3(a), 3(b), 3(c), and 3(d), respectively. Let us introduce a renormalized polarization bubble $\tilde{\Pi}_1$, a new polarization bubble Π_2 and a vertex V_1 which are given by the diagrams in Figs. 3(e), 3(f), and 3(g). $\Gamma^{(4)}_{\rm bare}$ can be calculated symbolically as

$$\Gamma^{(4)}_{\rm bare}=\Gamma^{(4)}_{\rm bare,1}+\Gamma^{(4)}_{\rm bare,2}+\Gamma^{(4)}_{\rm bare,3}+\Gamma^{(4)}_{\rm bare,4}$$
$$\approx\left[\frac{1}{1+g_0(\tilde{\Pi}_1+\Pi_2)}\right]^2$$
$$\times W_0+2V_1\left[\frac{1}{1+g_0\tilde{\Pi}_1}\right]^2W_0\,. \tag{14}$$

The bubble $\tilde{\Pi}_1$ which is renormalized by the self-energy correction is given by

$$\tilde{\Pi}_1(q,\omega=0)=\int\frac{d^dk}{(2\pi)^d}\frac{d\nu}{2\pi}\frac{Z}{\nu^2+v^2k^2}\frac{Z}{\nu^2+v^2(k-q)^2}$$
$$=Z^2\frac{c_1}{v^3}q^{d-3}\,. \tag{15}$$

Evaluation of $\Pi_2(q,\omega=0)$ is a long task and the result is

$$\Pi_2(q,\omega=0)=\frac{W_0v_0^6}{g_0^2c_1^2}\int\frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{d\nu}{2\pi}\frac{1}{v_0^2(k-q)^2+\nu^2}\frac{1}{v_0^2k^2+\nu^2}\frac{1}{v_0^2(k-p)^2+\nu^2}\frac{1}{v_0^2(k-p-q)^2+\nu^2}\frac{1}{p^{2(d-3-\lambda)}}$$
$$=\frac{W_0}{v_0g_0^2}\frac{e}{c_1^2}q^{d-3}q^{2+2\lambda-d}\,, \tag{16}$$

where

$$e=\frac{1}{(4\pi)^{d+1/2}}\frac{\Gamma(1/2-\lambda)}{\Gamma(d-3-\lambda)}$$
$$\times\int_0^1dx_1\int_0^{1-x_1}dx_2x_2^{d/2-2-\lambda}(1-x_2)^{2-\lambda-d/2}$$
$$\times\int_0^1dy_1\int_0^{1-y_1}dy_2y_2^{d/2-2-\lambda}(x_2+y_2(1-x_2))^{-1/2}$$
$$\times[x_1(1-x_1)(1-x_2)y_2-x_1^2y_2^2x_2-2y_1y_2x_1x_2(1-x_2)$$
$$+y_1(1-y_1)x_2(1-x_2)^2]^{\lambda-1/2}\,. \tag{17}$$

We found that the $\lambda\neq0$ and $\lambda=0$ cases should be treated separately. First, for $\lambda=0$ case, it is found that the constant e does not diverge as $\epsilon\to0$ so that it can be dropped in the final RG equations. Now, let us look at the case of $\lambda\neq0$. The strategy is that we investigate the most divergent contributions in various limits and add up all of the contributions. Since the most divergent contribution comes from $x_2\to0$ or $y_2\to0$ limit in Eq. (17) and we want just this contribution, we can set $x_2=0$ or $y_2=0$ inside the square brackets in Eq. (17). We found that both ways give the same answer. Here we will take $x_2=0$ limit inside the square brackets and multiply by 2. Equation (17) becomes

$$e=\frac{2}{(4\pi)^{d+1/2}}\frac{\Gamma(1/2-\lambda)}{\Gamma(d-3-\lambda)}\int_0^1dx_1x_1^{\lambda-1/2}(1-x_1)^{\lambda-1/2}\int_0^{1-x_1}dx_2x_2^{d/2-2-\lambda}(1-x_2)^{2-\lambda-d/2}\int_0^1dy_1\int_0^{1-y_1}dy_2y_2^{d/2-3}$$
$$\approx\frac{2}{(4\pi)^{d+1/2}}\frac{\Gamma(1/2-\lambda)}{\Gamma(d-3-\lambda)(d/2-1-\lambda)(d/2-2)}\int_0^1dx_1x_1^{\lambda-1/2}(1-x_1)^{d/2-3/2}$$
$$\times\int_0^1dy_1(1-y_1)^{d/2-2}\,, \tag{18}$$

where only the leading divergence is taken in the second equation. The evaluation of the remaining integrals is straightforward and the result is

$$e=\frac{2\Gamma(1/2-\lambda)\Gamma(d/2-1/2)\Gamma(\lambda+1/2)\Gamma(d/2-1)}{(4\pi)^{d+1/2}(d/2-1-\lambda)(d/2-2)\Gamma(d-3-\lambda)\Gamma(d/2)\Gamma(\lambda+d/2)}$$
$$=\frac{2}{\epsilon}\frac{2}{(4\pi)^{d+1/2}}\frac{\Gamma(1/2-\lambda)\Gamma(1/2+\lambda+\epsilon/2)\Gamma(\lambda+1/2)\Gamma(-1+\lambda+\epsilon/2)}{\Gamma(-1+\lambda+\epsilon)\Gamma(1+\lambda+\epsilon/2)\Gamma(1+2\lambda+\epsilon/2)}\,. \tag{19}$$

$V_1(q_1=q_3=q,q_2=0,\omega_1=\omega_3=\omega,\omega_2=0)$ can be evaluated similarly as

$$V_1=\frac{W_0v_0^6}{g_0^2c_1^2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{v_0^2(q-p)^2+\omega^2}\frac{1}{v_0^2p^2+\omega^2}\frac{1}{p^{2(d-3-\lambda)}}\approx\frac{W_0v_0^2}{g_0^2}q^{2+2\lambda-d}\left[\frac{d_1}{c_1^2}+\left[\frac{\omega^2}{v_0^2q^2}\right]\frac{d_2}{c_1^2}\right], \tag{20}$$

$$\begin{aligned}d_1&=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-1-\lambda/2)\Gamma(2-d/2+\lambda)\Gamma(d/2-1)}{\Gamma(d-2-\lambda)\Gamma(1+\lambda)}=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\epsilon/2)\Gamma(1-\epsilon/2)\Gamma(\lambda+\epsilon/2)}{\Gamma(\lambda+\epsilon)\Gamma(1+\lambda)},\\ d_2&=d_1\lambda\left[\frac{2(\lambda-1)}{d-4}-\frac{d-3-\lambda}{d-2-\lambda}\right],\end{aligned} \tag{21}$$

where $v_0q\gg\omega$ is assumed. Note that $d_2=0$ for the short-range interaction ($\lambda=0$).

Putting together all of these results, we can get the following $\Gamma^{(4)}_{\rm bare}(q_1=q_4=q,q_2=q_3=0,\omega_1=\omega_2=\omega_3=\omega_4=\omega)$,

$$\begin{aligned}\Gamma^{(4)}_{\rm bare}&\approx\left[\frac{1}{g_0(\tilde\Pi_1+\Pi_2)}\right]^2W_0+2V_1\left[\frac{1}{g_0\tilde\Pi_1}\right]^2W_0\\&\approx\frac{W_0v^6}{g_0^2c_1^2}q^{2(3-d)}Z^{-4}\left\{\left[1+\frac{W_0}{v_0g_0^2}\frac{e}{c_1^3}\frac{v^3}{Z^2}q^{2+2\lambda-d}\right]^{-2}+2\frac{W_0v_0^2}{g_0^2}q^{2+2\lambda-d}\left[\frac{d_1}{c_1^2}+\left[\frac{\omega^2}{v_0^2q^2}\right]\frac{d_2}{c_1^2}\right]\right\}.\end{aligned} \tag{22}$$

The renormalization condition for $\Gamma^{(4)}$ is taken as

$$\Gamma^{(4)}(q_1=q_4=\mu,q_2=q_3=0,\lim_{\alpha\to0}\omega_1=\omega_2=\omega_3=\omega_4=\alpha v\mu)=\frac{Wv^6}{g_0^2c_1^2}\mu^{2(3-d)}. \tag{23}$$

We introduce again $\tilde W=(Wv^2/g_0^2)\mu^{2+2\lambda-d}$ and $\tilde W_0=(W_0v_0^2/g_0^2)\Lambda^{2+2\lambda-d}$. If appropriate counter terms were added to cancel the $1/\epsilon$ divergences, the following equation can be obtained:

$$\begin{aligned}\tilde W_0&\approx\tilde WZ^2\left[\frac{v_0}{v}\right]^2\left[\frac{\mu}{\Lambda}\right]^{d-2-2\lambda}\\&\quad\times\left[1+2\tilde W\frac{\tilde e}{c_1^3}\ln\frac{\Lambda}{\mu}-2\tilde W\frac{\tilde d_1}{c_1^2}\ln\frac{\Lambda}{\mu}\right]\\&\approx\tilde W\left[\frac{\mu}{\Lambda}\right]^{d-2-2\lambda}\\&\quad\times\left[1+\frac{\tilde W}{c_1^2}\left[\tilde c_3+\tilde c_2+2\frac{\tilde e}{c_1}-2\tilde d_1\right]\ln\frac{\Lambda}{\mu}\right],\end{aligned} \tag{24}$$

where $\tilde e=\epsilon e$ and $\tilde d_1=\epsilon d_1$. Using the fact that the bare parameters are fixed, as we change μ, we get the following equation for $\tilde W$:

$$\mu\frac{\partial}{\partial\mu}\tilde W\equiv\beta(\tilde W)=-\epsilon\tilde W+\frac{1}{c_1^2}\left[\tilde c_2+\tilde c_3+2\frac{\tilde e}{c_1}-2\tilde d_1\right]\tilde W^2. \tag{25}$$

Let b be the reduction factor for the momentum scale from μ to μ/b, then the RG equations for $\tilde W$ and v up to $(1/N)^0$ order is

$$\frac{d\tilde W}{dl}=-\beta(\tilde W)=\epsilon\tilde W-\frac{1}{c_1^2}\left[\tilde c_2+\tilde c_3+2\frac{\tilde e}{c_1}-2\tilde d_1\right]\tilde W^2, \tag{26}$$

$$\frac{dv}{dl}=-\beta(v)=\frac{1}{2c_1^2}(\tilde c_3-\tilde c_2)v\tilde W,$$

where $l=\ln b$ is the logarithmic measure of the RG flow. $\tilde c_2$ and $\tilde c_3$ are given by

$$\begin{aligned}\tilde c_2&=-\frac{2}{(4\pi)^{d/2}}\frac{\Gamma(2-\epsilon/2)\Gamma(\lambda+\epsilon/2)}{\Gamma(-1+\lambda+\epsilon)\Gamma(2+\lambda)},\\ \tilde c_3&=-\frac{2-\epsilon}{(4\pi)^{d/2}}\frac{\Gamma(2-\epsilon/2)\Gamma(-1+\lambda+\epsilon/2)}{\Gamma(-1+\lambda+\epsilon)\Gamma(1+\lambda)}.\end{aligned} \tag{27}$$

Now, the RG equations for $0<\lambda<\frac{1}{2}$ up to $(1/N)^0$ order is

$$\begin{aligned}\frac{d\tilde W}{dl}&=-\beta(\tilde W)=\epsilon\tilde W-\frac{4}{c_1^2(4\pi)^{d/2}\Gamma(2+\lambda)}\tilde W^2,\\ \frac{dv}{dl}&=-\beta(v)=-\frac{1}{c_1^2}\frac{2}{(4\pi)^{d/2}\Gamma(2+\lambda)}v\tilde W,\end{aligned} \tag{28}$$

where the following coefficients are used:

$$\begin{aligned}\tilde c_2&=\frac{2}{(4\pi)^{d/2}}\frac{1-\lambda}{\Gamma(2+\lambda)},\\ \tilde c_3&=-\frac{2}{(4\pi)^{d/2}}\frac{1}{\Gamma(1+\lambda)},\\ \tilde d_1&=-\tilde c_3,\\ \frac{\tilde e}{c_1}&=\frac{4}{(4\pi)^{d/2}}\frac{1}{\Gamma(1+\lambda)}.\end{aligned} \tag{29}$$

For the usual short-range interactions $\lambda=0$,

$$\begin{aligned}\frac{d\tilde W}{dl}&=-\beta(\tilde W)=\epsilon\tilde W+\frac{128}{\pi}\tilde W^2,\\ \frac{dv}{dl}&=-\beta(v)=-\frac{64}{\pi}v\tilde W,\end{aligned} \tag{30}$$

where

$$c_1=\tfrac{1}{8},\quad\tilde c_2=1/\pi,\quad\tilde c_3=-1/\pi,\quad\tilde d_1=1/\pi \tag{31}$$

were used. $\tilde{\epsilon}=\epsilon e$ is the order of ϵ for the short-range interaction and is dropped in the RG equation.

The RG equations (28) for $\lambda\neq 0$ tell us that the disorder is relevant for $d>d_c=2+2\lambda$, irrelevant for $d<d_c$, and marginally irrelevant for $d=d_c$. For $d_c<d<3+\lambda$, the pure fixed point $\tilde{W}=0$ is unstable. However, there is a stable fixed point which is given by $\tilde{W}^*=\epsilon/Q$, where $Q=4/c_1^2(4\pi)^{d/2}\Gamma(2+\lambda)$. From $\beta(v)$, we can read off the dynamical exponent $z=1+(1/c_1^2)[2/(4\pi)^{d/2}][1/\Gamma(2+\lambda)]\tilde{W}^*$. More specifically, $z=1+\epsilon/2$ at this stable fixed point. For $d\le d_c$, the disorder is irrelevant and the pure fixed point is stable. Therefore, we can expect the direct superfluid–Mott-insulator transition for $d\le d_c$.

Now let us look at the short-range interaction case $\lambda=0$. From Eq. (30), we can see that the disorder is relevant for $d\ge 2$ and irrelevant for $d<2$. Therefore, there is only the unstable pure fixed point and the RG flow goes to the strong disorder regime for $d=2$ and slightly larger than two. For $d<2$, the pure fixed point becomes stable. There is also an unstable fixed point which is given by $\tilde{W}^*=|\epsilon|\pi/128$. The dynamical exponent at the unstable fixed point is $z=1+|\epsilon|/2$.

III. RENORMALIZATION-GROUP ANALYSIS FOR THE COMMENSURATE DIRTY-BOSON MODEL

Now we are going to study the generic model (3) with $\bar{u}=\bar{w}=0$. Here, we consider the usual short-range interaction $\lambda=0$. The disorder characterized by $u(\mathbf{x})$ in Eq. (3) will be considered in addition to the $w(\mathbf{x})$-type disorder. This means that we have to consider more diagrams that are generated by this new disorder. The additional self-energy correction due to $u(\mathbf{x})$-type disorder is given by the diagram in Fig. 4,

$$\Sigma_2(q,\omega)=-U_0\omega^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{\omega^2+v_0^2(q-k)^2}=\frac{U_0}{v_0^d}c_4\omega^2\omega^{d-2}\ ,\qquad c_4=-\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\ . \tag{32}$$

The new bare two-point vertex $\Gamma^{(2)}_{\rm bare}$ up to $(1/N)^0$ order is

$$\Gamma^{(2)}_{\rm bare}=\omega^2+v_0^2q^2-\Sigma_1(q,\omega)-\Sigma_2(q,\omega)=\omega^2\left[1-\frac{W_0v_0^2}{g_0^2}\frac{c_3}{c_1^2}q^{2-d}-\frac{U_0}{v_0^2}c_4\left[\frac{\omega}{v_0}\right]^{d-2}\right]+v_0^2q^2\left[1-\frac{W_0v_0^2}{g_0^2}\frac{c_2}{c_1^2}q^{2-d}\right]\ . \tag{33}$$

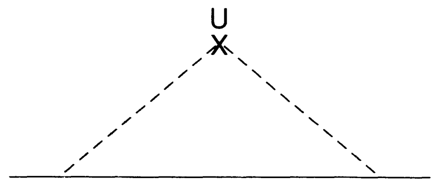

FIG. 4. The self-energy correction due to the $u(\mathbf{x})$-type disorder. The dashed line with a cross $\times$ means the impurity average. U means the $u(\mathbf{x})$-type disorder.

It is clear that we need to do an $\epsilon=d-2$ expansion in order to handle the divergences. Let us introduce additional dimensionless measures of the disorder $\tilde{U}_0=(U_0/v_0^2)\Lambda^{d-2}$ and $\tilde{U}=(U/v^2)\mu^{d-2}$ in the bare and the renormalized theory. Adding appropriate counter terms and using (5), (6), and (7), we can again obtain the following equations for Z and v:

$$\mu\frac{\partial}{\partial\mu}(\ln Z)\approx-\tilde{W}\frac{\tilde{c}_3}{c_1^2}+\tilde{c}_4\tilde{U}\ ,\qquad \mu\frac{\partial}{\partial\mu}v\equiv\beta(v)\approx\tfrac{1}{2}\left[\frac{\tilde{c}_2}{c_1^2}-\frac{\tilde{c}_3}{c_1^2}\right]\tilde{W}v+\tilde{c}_4\tilde{U}v\ . \tag{34}$$

There can be 5 additional diagrams that contribute to $\Gamma^{(4)}_{\rm bare}$. Let us identify $\Gamma^{(4)}_{\rm bare,5}$, $\Gamma^{(4)}_{\rm bare,6}$, $\Gamma^{(4)}_{\rm bare,7}$, $\Gamma^{(4)}_{\rm bare,8}$, and $\Gamma^{(4)}_{\rm bare,9}$ as the diagrams in Figs. 5(a), 5(b), 5(c), 5(d), and 5(e), respectively. First of all, let us evaluate the new vertex V_2 of Fig. 6(a)

$$V_2=-U_0\omega^2\int\frac{d^dp}{(2\pi)^d}\frac{1}{v_0^2(q-p)^2+\omega^2}\frac{1}{v_0^2p^2+\omega^2}\approx-\frac{U_0}{v_0^2}\left[d_3q^{d-4}\frac{\omega^2}{v_0^2}+d_4q^{d-6}\frac{\omega^4}{v_0^4}\right]\ ,$$

$$d_3=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)\Gamma^2(d/2-1)}{\Gamma(d-2)}\ ,\qquad d_4=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)\Gamma(d/2-2)\Gamma(d/2-1)}{\Gamma(d-4)}\ , \tag{35}$$

where $v_0q\gg\omega$ is assumed. The bubble Π_3 of Fig. 6(b) is given by

$$\Pi_3(q,\omega=0)=-U_0\int\frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{d\nu}{2\pi}\nu^2\frac{1}{v_0^2(k-q)^2+\nu^2}\frac{1}{v_0^2k^2+\nu^2}\times\frac{1}{v_0^2(k-p)^2+\nu^2}\frac{1}{v_0^2(k-p-q)^2+\nu^2}=-\frac{U_0}{v_0^5}fq^{d-3}q^{d-2}\ , \tag{36}$$

where

$$f=\frac{1}{2}\frac{\Gamma(5/2-d)}{(4\pi)^{d+1/2}}\int_0^1 dx\int_0^1 dy\int_0^1 dz\frac{z^{1-d/2}(1-z)^{1-d/2}}{[x(1-x)z+y(1-y)(1-z)]^{5/2-d}} \tag{37}$$

is a convergent integral. We can see that the bubble Π_3 is smaller by the factor ϵ (setting $d=2+\epsilon$) than Π_1 and Π_2 so that the contribution from $\Gamma^{(4)}_{\mathrm{bare},9}$ is higher order in ϵ and we can neglect it.

Using the calculated V_2, we can evaluate $\Gamma^{(4)}_{\mathrm{bare}}$ as

$$\begin{aligned}\Gamma^{(4)}_{\mathrm{bare}}&=\sum_{i=1}^{9}\Gamma^{(4)}_{\mathrm{bare},i}\\ &\approx\left[\frac{1}{1+g_0(\tilde\Pi_1+\Pi_2+\Pi_3)}\right]^2W_0+2(V_1+V_2)\left[\frac{1}{1+g_0\tilde\Pi_1}\right]^2W_0-U_0\omega^2(1+2V_1+2V_2)\\ &\approx\left[\frac{1}{g_0(\tilde\Pi_1+\Pi_2)}\right]^2W_0+2(V_1+V_2)\left[\frac{1}{g_0\tilde\Pi_1}\right]^2W_0-U_0\omega^2(1+2V_1+2V_2)\\ &\equiv\Gamma^{(4)}_{\mathrm{bare},w}+\Gamma^{(4)}_{\mathrm{bare},u}\ ,\end{aligned}\tag{38}$$

where

$$\begin{aligned}\Gamma^{(4)}_{\mathrm{bare},w}&=\frac{W_0v^6}{g_0^2c_1^2}q^{2(3-d)}Z^{-4}\\ &\quad\times\left[\left[1+\frac{W_0}{v_0g_0^2}\frac{e}{c_1^3}\frac{v^3}{Z^2}q^{2-d}\right]^{-2}+2\frac{W_0v_0^2}{g_0^2}\frac{d_1}{c_1^2}q^{2-d}\right.\\ &\quad\left.-2\frac{U_0}{v_0^2}\left[d_3q^{d-4}\frac{\omega^2}{v_0^2}+d_4q^{d-6}\frac{\omega^4}{v_0^4}\right]\right]\ ,\\ \Gamma^{(4)}_{\mathrm{bare},u}&=-U_0\omega^2\left[1-2\frac{U_0}{v_0^2}\left[d_3q^{d-4}\frac{\omega^2}{v_0^2}+d_4q^{d-6}\frac{\omega^4}{v_0^4}\right]\right.\\ &\quad\left.+2\frac{W_0v_0^2}{g_0^2}\frac{d_1}{c_1^2}q^{2-d}\right]\ ,\end{aligned}\tag{39}$$

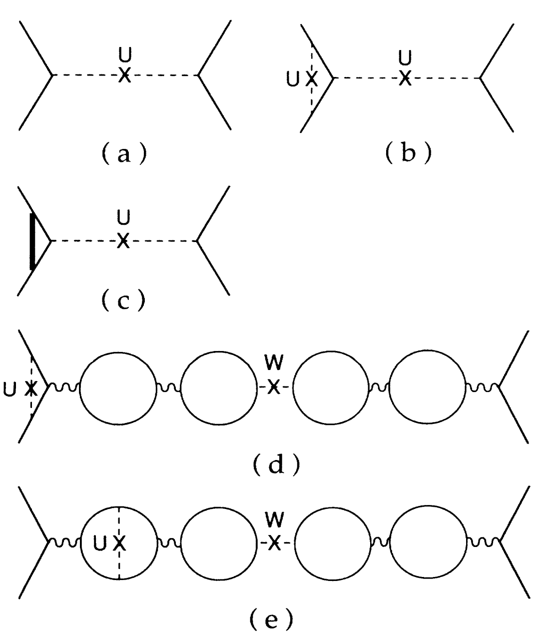

FIG. 5. Additional diagrams that contribute to the four-point function in the generic commensurate dirty-boson model up to second order in the impurity strength.

where $d_2=0$ for the short-range interaction is used.

We take the following renormalization condition for $\Gamma^{(4)}=\Gamma^{(4)}_w+\Gamma^{(4)}_u$:

$$\begin{aligned}\Gamma^{(4)}_w\Big|_{q_1=q_4=\mu,q_2=q_3=0,\lim_{\alpha\to0}\omega_1=\omega_2=\omega_3=\omega_4=\alpha v\mu}&=\frac{Wv^6}{g_0^2c_1^2}\mu^{2(3-d)}\ ,\\ \frac{\partial\Gamma^{(4)}_u}{\partial\omega^2}\Big|_{q_1=q_4=\mu,q_2=q_3=0,\lim_{\alpha\to0}\omega_1=\omega_2=\omega_3=\omega_4=\alpha v\mu}&=-U\ .\end{aligned}\tag{40}$$

Adding appropriate counter terms and using the renormalization condition, we obtain the following equations:

$$\begin{aligned}\tilde W_0&\approx\tilde WZ^2\left[\frac{v_0}{v}\right]^2\left[\frac{\mu}{\Lambda}\right]^{d-2}\left[1+2\frac{\tilde W}{c_1^2}\left[\frac{\tilde e}{c_1}-\tilde d_1\right]\ln\frac{\Lambda}{\mu}\right]\\ &\approx\tilde W\left[\frac{\mu}{\Lambda}\right]^{d-2}\left[1+\frac{\tilde W}{c_1^2}(\tilde c_3+\tilde c_2+2\frac{\tilde e}{c_1}-2\tilde d_1)\ln\frac{\Lambda}{\mu}\right.\\ &\quad\left.-\tilde U\tilde c_4\ln\frac{\Lambda}{\mu}\right]\ ,\\ \tilde U_0&\approx\tilde UZ^{-2}\left[\frac{v}{v_0}\right]^2\left[\frac{\mu}{\Lambda}\right]^{2-d}\left[1-2\tilde W\frac{\tilde d_1}{c_1^2}\ln\frac{\Lambda}{\mu}\right]\\ &\approx\tilde U\left[\frac{\mu}{\Lambda}\right]^{2-d}\left[1-\frac{\tilde W}{c_1^2}(\tilde c_3+\tilde c_2+2\tilde d_1)\ln\frac{\Lambda}{\mu}\right.\\ &\quad\left.+\tilde U\tilde c_4\ln\frac{\Lambda}{\mu}\right]\ .\end{aligned}\tag{41}$$

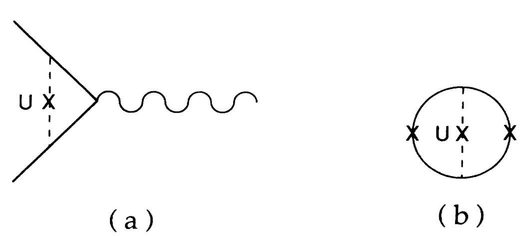

FIG. 6. (a) The vertex V_2. (b) The polarization bubble Π_3.

Using μ independence of the bare parameters, we obtain the following equations for $\tilde{W}$ and $\tilde{U}$:

$$\mu\frac{\partial}{\partial\mu}\tilde{W}\equiv\beta(\tilde{W})=-\epsilon\tilde{W}+\frac{1}{c_1^2}\left[\tilde{c}_2+\tilde{c}_3+2\frac{\tilde{e}}{c_1}-2\tilde{d}_1\right]\times\tilde{W}^2-\tilde{c}_4\tilde{U}\tilde{W}\ , \qquad (42)$$

$$\mu\frac{\partial}{\partial\mu}\tilde{U}\equiv\beta(\tilde{U})=\epsilon\tilde{U}-\frac{1}{c_1^2}(\tilde{c}_3+\tilde{c}_2+2\tilde{d}_1)\tilde{W}\tilde{U}+\tilde{c}_4\tilde{U}^2\ ,$$

where $\tilde{c}_4=\epsilon c_4$ and for small ϵ, the coefficients of these equations are given by

$$c_1=1/8\ ,\quad \tilde{c}_2=1/\pi\ ,\quad \tilde{c}_3=-1/\pi\ ,$$
$$\tilde{c}_4=1/(2\pi)\ ,\quad \tilde{d}_1=1/\pi\ . \qquad (43)$$

We can ignore $\tilde{e}$ which is the order of ϵ. Now the RG equations for $\tilde{W}$, $\tilde{U}$, and v are given by

$$\frac{d\tilde{W}}{dl}=-\beta(\tilde{W})=\epsilon\tilde{W}+\frac{1}{2\pi}\tilde{U}\tilde{W}+\frac{128}{\pi}\tilde{W}^2\ ,$$
$$\frac{d\tilde{U}}{dl}=-\beta(\tilde{U})=-\epsilon\tilde{U}+\frac{128}{\pi}\tilde{W}\tilde{U}-\frac{1}{2\pi}\tilde{U}^2\ , \qquad (44)$$
$$\frac{dv}{dl}=-\beta(v)=-\frac{64}{\pi}v\tilde{W}-\frac{1}{2\pi}v\tilde{U}\ ,$$

where l is again the logarithmic measure of the RG flow. The RG flow is drawn in Fig. 7(a) for $\epsilon\geq 0$, (b) for $\epsilon<0$.

For $d<2(\epsilon<0)$, we can see that there are three fixed points which are given by $(\tilde{W}^*,\tilde{U}^*)=(0,0),(0,2\pi|\epsilon|)$, $(\pi|\epsilon|/128,0)$. The dynamical exponents at these fixed points are given by $z=1,1+|\epsilon|,\ 1+\frac{1}{2}|\epsilon|$, respectively.

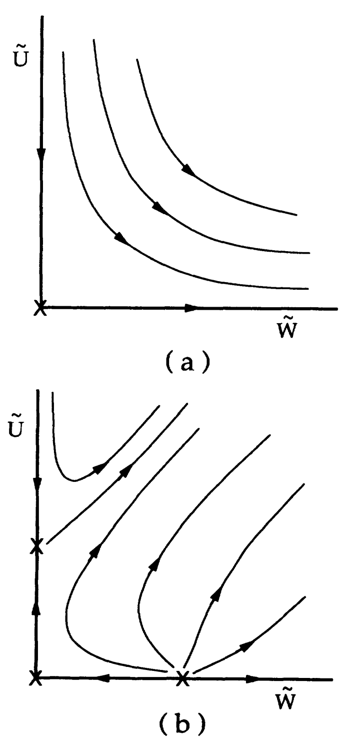

FIG. 7. RG flows (a) for $2\leq d<3$ and (b) for $d<2$.

Note that all of these fixed points are essentially unstable because they have at least one relevant direction in the $\tilde{W}-\tilde{U}$ plane. Among these three fixed points, $(\tilde{W}^*,\tilde{U}^*)=(0,2\pi|\epsilon|)$ is a nontrivial fixed point that has the least number (one) of relevant directions. However, we need to fine tune the strength of the $w(\mathbf{x})$-type disorder in order to reach this fixed point. For $2\leq d<3(\epsilon\geq 0)$, there is only the trivial fixed point $(\tilde{W}^*,\tilde{U}^*)=(0,0)$. At this fixed point, $\tilde{U}$ is relevant and $\tilde{W}$ is irrelevant so that the RG flow goes to the strong randomness regime where our RG scheme breaks down.

Therefore, in both cases, there is no stable nontrivial random fixed point near the pure XY fixed point. This means that the SF–insulator transition should be described by possible strong random fixed points. The conclusion we can deduce from these results is that, near $d=2$, the direct SF–MI transition is unlikely to happen in the whole $\overline{w}-\overline{u}$ plane even for the weak disorder. This result essentially supports the original picture of Fisher and co-workers[1,2] [Fig. 1(a)] that the SF–insulator transition should always occur from the Bose-glass phase rather than from the Mott insulator even for the commensurate case.

IV. $1/N$ CORRECTION

In this section, we are going to investigate the effects of the $1/N$ correction to the RG equation. The $1/N$ correction due to the ϕ^4 interaction in the previous commensurate dirty-boson model will be considered. The most important $1/N$ correction that can affect the RG equation enters in the coefficient of the $\tilde{W}$ term in Eq. (44). This $1/N$ correction can be read off from the scaling dimensions of $\phi^\dagger\phi$ at the pure fixed point:[20]

$$[\phi^\dagger\phi]=2-\eta\ , \qquad (45)$$

where

$$\eta=\frac{32}{3\pi^2}\frac{1}{2N}\ . \qquad (46)$$

As a result, the critical dimension for $\tilde{W}$ is changed from 2 to $2-2\eta$. If the convention of $\epsilon=d-2$ was taken, the RG equation becomes

$$\frac{d\tilde{W}}{dl}=(\epsilon+2\eta)\tilde{W}+\frac{1}{2\pi}\tilde{U}\tilde{W}+\frac{128}{\pi}\tilde{W}^2\ ,$$
$$\frac{d\tilde{U}}{dl}=-\epsilon\tilde{U}+\frac{128}{\pi}\tilde{W}\tilde{U}-\frac{1}{2\pi}\tilde{U}^2\ , \qquad (47)$$
$$\frac{dv}{dl}=-\frac{64}{\pi}v\tilde{W}-\frac{1}{2\pi}v\tilde{U}\ .$$

Note that the density ρ has no anomalous dimension[16] and the RG equation for $\tilde{U}$ is not affected at this order.

Now there can be three possible RG flows that depend on the dimensionality of the system and N. For $\epsilon=d-2\geq 0$, there is only one fixed point which is the unstable trivial fixed point $(\tilde{W}^*,\tilde{U}^*)=(0,0)$. The RG flow is given by Fig. 7(a) and it flows to the strong disorder regime. For $\epsilon<0$ and $|\epsilon|<2\eta$, there can be two fixed points which are $(\tilde{W}^*,\tilde{U}^*)=(0,0)$, $(0,2\pi|\epsilon|)$ with the dynamical exponents $z=1,1+|\epsilon|$. But all of them are

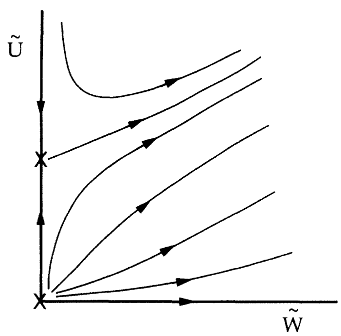

FIG. 8. RG flows which has the $1/N$ correction for $d<2$ and $|\epsilon|<2\eta$.

unstable and the RG flow is given by Fig. 8. On the other hand, for $\epsilon<0$ and $|\epsilon|>2\eta$, there is one more fixed point which is given by $(\tilde{W}^*,\tilde{U}^*)=[\pi/128(|\epsilon|-2\eta),0]$ with $z=1+\frac{1}{2}(|\epsilon|-2\eta)$. All of the three fixed points are still unstable and the RG flow goes to the strong disorder regime. The RG flow is again given by Fig. 7(b). Therefore, after the inclusion of the $1/N$ correction, there is no stable weak random fixed point. The direct SF–MI transition is unlikely in the whole $\bar{w}-\bar{u}$ plane.

V. SUMMARY AND CONCLUSION

We study a large-N generalization of the commensurate dirty-boson problem. The $1/N$ expansion allows us to treat interaction effects properly. On the other hand, the disorder is assumed to be weak and the perturbation in the strength of the randomness is performed. In order to understand the behaviors of bosons in this model, we need two types of disorder, i.e., the random coefficient of the density term $\phi^\dagger\partial_0\phi$, $u(\mathbf{x})$, and the random coefficient of the quadratic term $\phi^\dagger\phi, w(\mathbf{x})$.

For a restricted model with $u(\mathbf{x})=0$ which has an additional particle-hole symmetry, we introduce more general long-range interactions $V(q)=g_0/q^\lambda,(0\le\lambda<1/2)$, (which is not a genuine long-range interaction as mentioned in Sec. II). The critical dimension for $w(\mathbf{x})$-type disorder is found to be $d_c=2+2\lambda$ and we performed the $\epsilon=d-2-2\lambda$ expansion. It is found that the $\lambda=0$ and $\lambda\neq0$ cases show different behaviors and they are not continuously related in the $\epsilon=d-d_c$ expansion.

First, let us look at the case of $\lambda\neq0$. For $d>d_c$, the pure fixed point is unstable, but there is a stable fixed point which governs the transition. For $d\le d_c$, the pure fixed point is stable and the disorder is irrelevant. Therefore, we expect the direct SF–MI transition. For $d\le d_c$, the critical point is the same as that for the pure system. For $d>d_c$, SF–MI transition is described by a new nontrivial fixed point.

For the short-range interaction ($\lambda=0$), the disorder is relevant for $d\ge2$ and irrelevant for $d<2$. There is only the unstable pure fixed point and the RG flow goes to the strong disorder regime for $2\le d<3$. For $d<2$, the pure fixed point becomes stable and there is also an unstable nontrivial fixed point. Therefore, we expect a direct SF–MI transition for $d<2$ but none for $2\le d<3$.

We consider the usual short-range interaction for the general case of $u(\mathbf{x})\neq0$ but $\bar{u}=0$ (the commensurate dirty-boson problem). The critical dimension of both types of disorder is found to be $d=2$ [at $(1/N)^0$ order] and we performed the $\epsilon=d-2$ expansion ($d<3$). For $d<2$, we have three fixed points and they have at least one relevant direction in the $\tilde{W}-\tilde{U}$ plane. In the case of $2\le d<3$, the RG flow is governed by the $w(\mathbf{x})$-type disorder and it flows to the strong disorder regime which cannot be reached by the perturbation in the strength of the randomness. Therefore, we expect that the direct SF–MI transition is unlikely to happen near two dimensions.

The effects of the $1/N$ correction are considered. For the commensurate dirty-boson model, the $1/N$ correction due to the ϕ^4 interaction does not change the qualitative features of the problem. There is still no stable weak random fixed point for $d\sim2$ and the direct SF–MI transition is unlikely.

ACKNOWLEDGMENTS

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