Penetration depth and impurity scattering in unconventional superconductors: $T=0$ results

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The behavior of the penetration depth λ at low temperatures is an important probe of the symmetry of the order parameter in superconductors. For this reason, the penetration depth in the high-T_c oxide has been the focus of much recent experimental and theoretical attention. In this paper we provide detailed results for the behavior of λ at $T=0$, as a function of impurity scattering, for an unconventional order parameter.

Experimental[1-3] and theoretical[4-6] investigations of the penetration depth in the high-T_c superconductors have recently been of great interest. The low-temperature behavior of the in-plane penetration depth $\lambda(T)$ is an important clue as to the symmetry of the superconducting order parameter $\Delta(\hat{k})$. One possibility is that $\Delta(\hat{k})$ has the full rotational symmetry of the normal-state lattice; this possibility is often labeled "*s*-wave." A contrasting scenario[7] is that $\Delta(\hat{k})$ is unconventional, having a reduced rotational symmetry; this possibility is often labeled "*d*-wave," since one simple form often considered is[8]

$$\Delta(\hat{k})=\Delta(\hat{k}_x^2-\hat{k}_y^2) \ . \tag{1}$$

Much theoretical work has been concentrated on the temperature dependence of $\lambda(T)$ at low temperature, $T \ll T_c$. Here, we focus on a different aspect of the problem,[6] an aspect of possible experimental relevance. We present results showing how the zero-temperature penetration depth, $\lambda(T=0)$, varies with impurity concentration. Our results illustrate the great sensitivity of unconventional superconductors to impurity scattering. Our results also show the recently discussed difference between impurity scattering which is in the Born limit, and scattering in the unitary limit.

We compute the superfluid density ρ_s, and then the penetration depth λ, for a two-dimensional sheet of electrons with a random array of impurities. We treat a system with a circular Fermi surface, and a gap of form (1). To do the calculation, we use a recently developed general formula for the superfluid density tensor.[9] This leads to the following formula [here, ϕ is the polar angle in the xy plane, so that $\Delta(\phi)=\Delta\cos 2\phi$]:

$$\rho_s=2N(0)v_F^2\Delta^2\pi MT \times \sum_{\epsilon}\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{\cos^2\phi\cos^2 2\phi}{[(\epsilon+ia_3)^2+\Delta^2\cos^2 2\phi]^{3/2}} \ . \tag{2}$$

Two quantities are needed as input to this formula, the order parameter Δ and the impurity self-energy $a_3(\epsilon)$. Coupled equations must be solved at this point. First, we have the weak-coupling gap equation

$$\frac{1}{g}=N(0)\pi T\sum_{\epsilon}{}'\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{\cos^2 2\phi}{[(\epsilon+ia_3)^2+\Delta^2\cos^2 2\phi]^{1/2}} \ . \tag{3}$$

Here, g is a coupling constant, and the prime on the Matsubara sum indicates that a cutoff is needed. Next, we need the equation for the impurity self-energy $a_3(\epsilon)$:

$$a_3(\epsilon)=ct_3(\epsilon) \ , \tag{4}$$

where c is the density of impurities, and $t_3(\epsilon)$ is the t matrix. It is given by

$$t_3=\frac{N(0)v^2\langle g_3\rangle}{1-(N(0)v\langle g_3\rangle)^2} \ . \tag{5}$$

Here, the impurity potential is taken to be s wave, of strength v. Small v puts us in a limit where the Born approximation is valid, while large v, $v\to\infty$, puts us in the unitarity limit. The quantity $\langle g_3\rangle$ is given by

$$\langle g_3\rangle=\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{-i\pi(\epsilon+ia_3)}{[(\epsilon+ia_3)^2+\Delta^2\cos^2 2\phi]^{1/2}} \ . \tag{6}$$

So we must solve Eqs. (3)–(6) for Δ and $a_3(\epsilon)$, and then use them in Eq. (2) to compute ρ_s. We can simplify our equations slightly by performing the angular integrals. This leads to

$$\rho_s=2N(0)v_F^2MT\sum_{\epsilon}\frac{K[\Delta^2/(\tilde{\epsilon}^2+\Delta^2)]-E[\Delta^2/(\tilde{\epsilon}^2+\Delta^2)]}{\sqrt{\tilde{\epsilon}^2+\Delta^2}} \ , \tag{7}$$

$$\frac{1}{g}=\frac{2N(0)T}{\Delta^2}\sum_{\epsilon}{}'\sqrt{\tilde{\epsilon}^2+\Delta^2}\left\{E\left[\frac{\Delta^2}{\tilde{\epsilon}^2+\Delta^2}\right]-\frac{\tilde{\epsilon}^2}{\tilde{\epsilon}^2+\Delta^2}K\left[\frac{\Delta^2}{\tilde{\epsilon}^2+\Delta^2}\right]\right\} \ , \tag{8}$$

and

$$ia_3 = \frac{[2\tilde{\epsilon}N(0)cv^2/\sqrt{\tilde{\epsilon}^2+\Delta^2}]K[\Delta^2/(\tilde{\epsilon}^2+\Delta^2)]}{1+[4N(0)^2v^2\tilde{\epsilon}^2/(\tilde{\epsilon}^2+\Delta^2)]K^2[\Delta^2/(\tilde{\epsilon}^2+\Delta^2)]} . \tag{9}$$

Here, the function $\tilde{\epsilon}(\epsilon)$ is defined by

$$\tilde{\epsilon}(\epsilon)=\epsilon+ia_3(\epsilon) , \tag{10}$$

and $K(x)$ and $E(x)$ are the elliptic integrals.[10]

We define the normal-state scattering time in the usual way:

$$\frac{1}{2\tau} \equiv \frac{N(0)\pi c v^2}{1+(N(0)\pi v)^2} . \tag{11}$$

The same value of τ can correspond to different combinations of c and v, and their different combinations in general lead to different values of ρ_s and λ. However, the reduction in T_c, at least in this simple model of s-wave impurities and an order parameter given by (1), depends only on τ:[6]

$$\ln\left[\frac{T_c}{T_{c0}}\right]=\Psi(\tfrac{1}{2})-\Psi\left[\frac{1}{2}+\frac{1}{4\pi\tau T_c}\right] . \tag{12}$$

Here, $\Psi(x)$ is the digamma function and T_{c0} is the transition temperature of a pure system.

Figure 1 shows results for the dependence of $\rho_s(T=0)$ on impurity scattering. We show ρ_s as a function of $1/2\tau$, for the Born limit (small v) and the unitary limit ($v \to \infty$). We also show T_c as a function of $1/2\tau$. As emphasized by previous authors, there is a substantial difference between the Born and unitarity limits in the behavior of ρ_s. In particular, in the unitarity limit ρ_s drops off very quickly as $1/2\tau$ increases from zero, much more rapidly than T_c.

For comparison, we also consider a conventional order parameter; we take the simplest form, $\Delta(\hat{k})=\Delta$, independent of $\hat{k}$. The superfluid density is then given by[11]

$$\rho_s = N(0)v_F^2\Delta^2\pi M T \sum_{\epsilon} \frac{1}{(\epsilon^2+\Delta^2)(1/2\tau+\sqrt{\epsilon^2+\Delta^2})} . \tag{13}$$

For this completely isotropic gap, ρ_s depends only on τ, not separately on the values of c and v. Figure 1 includes a plot of this result, and we can see that for a conventional, isotropic gap, ρ_s falls off very slowly with increasing $1/2\tau$. In Figs. 2 and 3 we display our results in a different way, showing the $T=0$ value of λ^2 as a function of $1/2\tau$.

The impurity self-energy $a_3(\epsilon)$ clearly plays a key role in these calculations. In Fig. 4 we show a plot of $a_3(\epsilon)$, as a function of ϵ, for a particular value of τ. The huge difference in behavior, at small energies, between the Born and unitary limits is evident. At large energies, $ia_3(\epsilon)$ approaches its normal-state values of $1/2\tau$; for $\epsilon \gg \Delta$, superconductivity has no effect on $a_3(\epsilon)$.

Hirschfeld and Goldenfeld[6] have discussed the initial decrease of $\rho_s(T=0)$, as a function of c, near $c=0$, in the unitarity limit. We can enlarge on their discussion in the framework of our formalism. At $T=0$ the formula for ρ_s becomes:

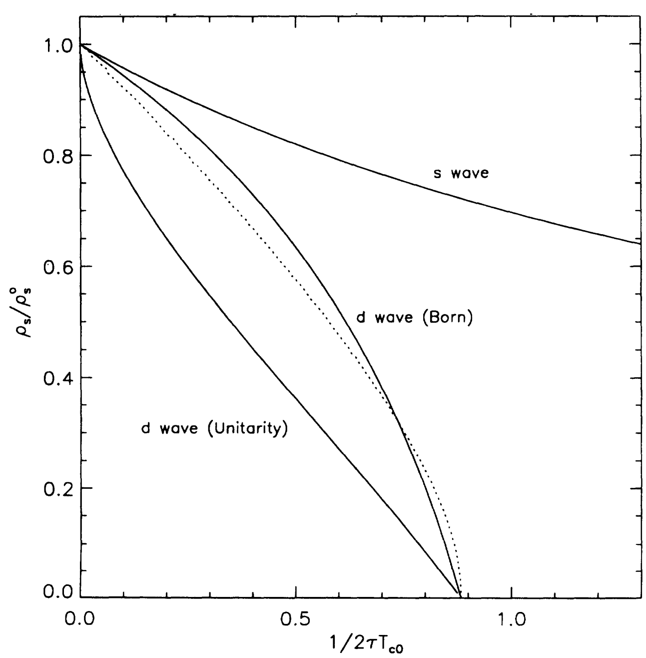

FIG. 1. Superfluid density, at $T=0$, as a function of $1/2\tau$. Solid lines show ρ_s for unitary limit d wave, Born limit d wave, and s wave. ρ_s^0 is the value of a pure system. Dotted line shows T_c/T_{c0} for the d-wave order parameter.

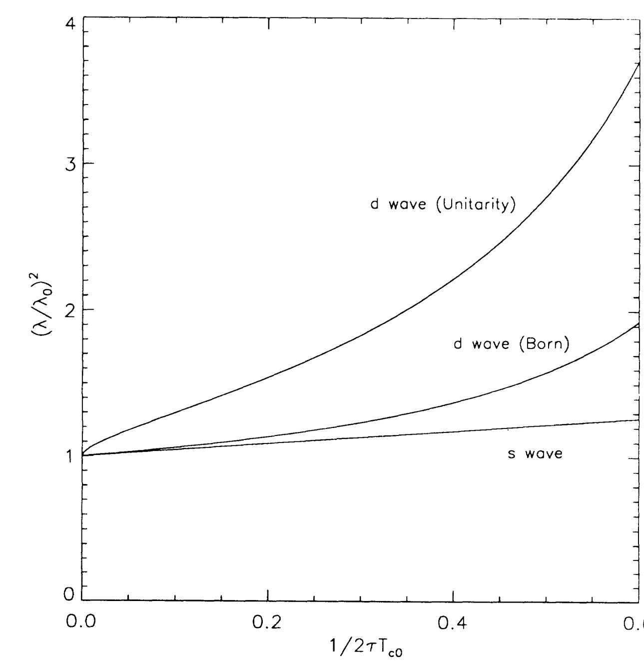

FIG. 2. Plot of λ^2, at $T=0$, as a function of $1/2\tau$. Solid lines show d-wave unitary limit, d-wave Born limit, and s wave. λ_0 is the value of the pure system.

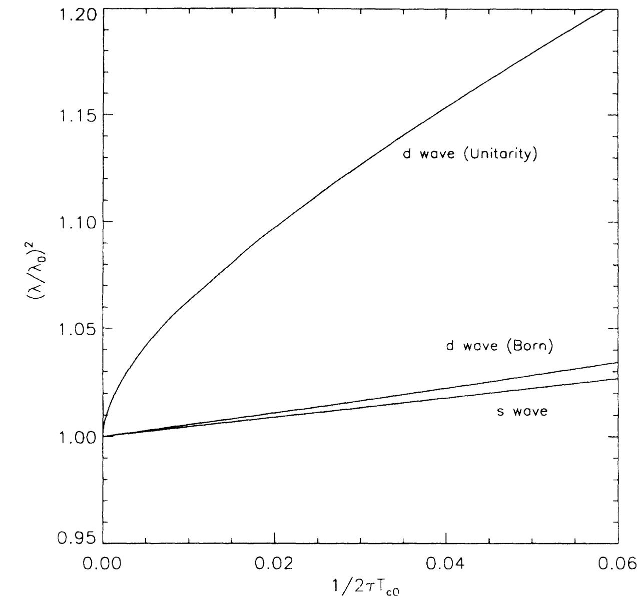

FIG. 3. Same plot as Fig. 2, on expanded scales. This shows more clearly the behavior in the vicinity of $1/2\tau=0$.

$$\rho_s=\frac{2N(0)v_F^2M}{\pi}\times\int_0^\infty d\epsilon\frac{K[\Delta^2/(\tilde{\epsilon}^2+\Delta^2)]-E[\Delta^2/(\tilde{\epsilon}^2+\Delta^2)]}{\sqrt{\tilde{\epsilon}^2+\Delta^2}} . \quad (14)$$

The elliptic function $K(x)$ has a logarithmic singularity as $x\rightarrow 1$; thus, if the density of impurities c is small, this divergence comes into play near $\epsilon=0$.

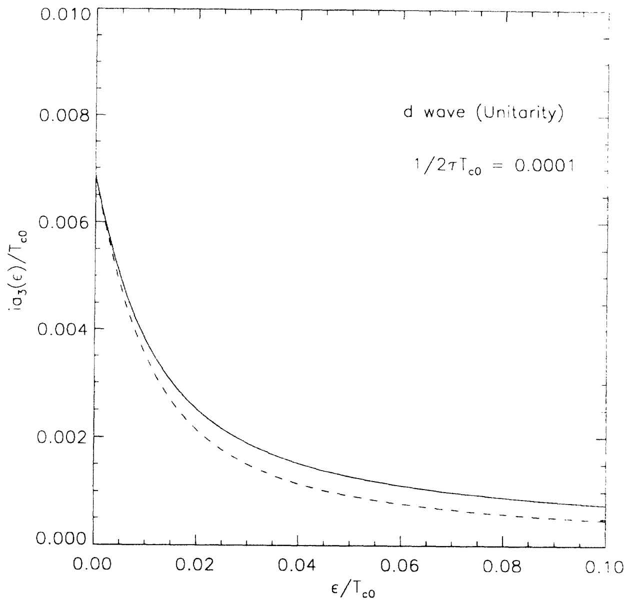

FIG. 5. Impurity self-energy, $ia_3(\epsilon)$ as a function of ϵ, at $T=0$ for the unitary limit. We chose $1/2\tau T_{c0}=0.0001$. The solid line shows the actual, computed value, while the dashed line plots the approximate formula $ia_3=-\epsilon+\sqrt{\epsilon^2+4\gamma^2}/2$.

We also need to understand the small c, unitary limit, behavior of $a_3(\epsilon)$. From Eq. (9), it is easy to see that in this limit we have

$$\gamma^2\ln\left[\frac{4\Delta}{\gamma}\right]=\frac{c\Delta}{2N(0)} , \quad (15)$$

where

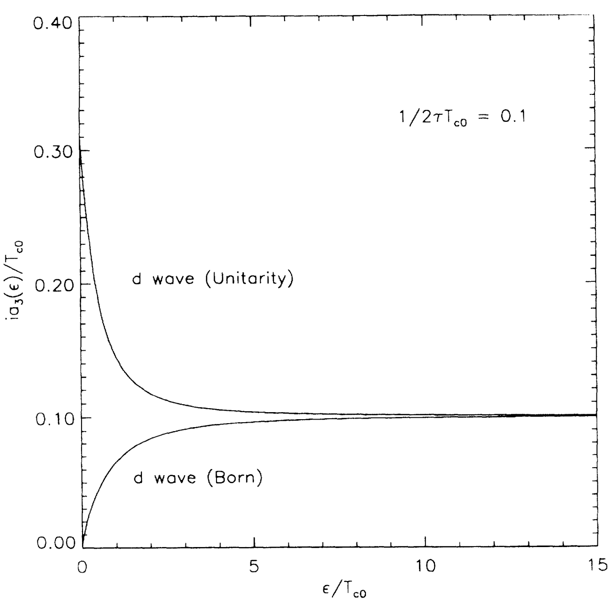

FIG. 4. Impurity self-energy, $ia_3(\epsilon)$, as a function of ϵ, at $T=0$, for the d-wave order parameter. We close $1/2\tau T_{c0}=0.1$, and show results for the Born and unitarity limits. In the Born limit, $ia_3(\epsilon)$ does not vanish at $\epsilon=0$; it approaches a value too small to resolve on this graph.

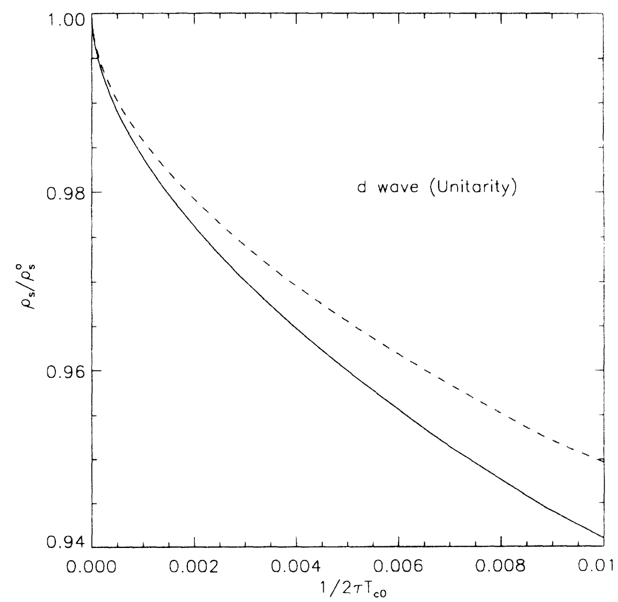

FIG. 6. Superfluid density at $T=0$ in the unitarity limit for the d-wave order parameter. The solid line shows the actual computed value, while the dashed line plots the approximate formula $\rho_s=\rho_s^0-[4N(0)Mv_F^2/\pi\Delta_0]\gamma$. The value of γ is obtained by numerical solution of Eq. (15).

$$\gamma = ia_3(0) \ . \tag{16}$$

So, as Hirschfeld and Goldenfeld[6] note, $\gamma \propto \sqrt{c}$, up to logarithmic corrections. Furthermore, in this same limit (small c, unitary) it is easy to see that

$$ia_3'(\epsilon=0) = -\tfrac{1}{2} \ . \tag{17}$$

Results (16) and (17) are both embodied in the following formula:

$$ia_3(\epsilon) = \frac{-\epsilon + \sqrt{\epsilon^2 + 4\gamma^2}}{2} \ , \tag{18}$$

which is asymptotically correct for small values of ϵ. Figure 5 compares the function $ia_3(\epsilon)$ with the approximation (18). If we use (18) in the integral for ρ_s, Eq. (14), we can extract the leading correction to $\rho_s(T=0,c)$ in the small c unitarity limit:

$$\delta\rho_s = \rho_s(T=0,c) - \rho_s(T=0,c=0) = -\frac{4N(0)Mv_F^2}{\pi\Delta_0}\gamma \ . \tag{19}$$

Here, Δ_0 is the pure value. So, Eq. (19), together with the transcendental equation for γ, Eq. (15), gives the leading correction to ρ_s in the unitary limit. Figure 6 compares the approximation (19) with the true answer. We can rewrite (19) in terms of the penetration depth as follows:

$$\frac{\lambda(T=0,c) - \lambda(T=0,c=0)}{\lambda(T=0,c=0)} = \frac{2\gamma}{\pi\Delta_0} \ . \tag{20}$$

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