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Haldane fractional statistics in the fractional quantum Hall effect

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We have tested Haldane's "fractional-Pauli-principle" description of excitations around the $\nu = \frac{1}{3}$ state in the fractional quantum Hall effect, using exact results for small systems of electrons. We find that Haldane's prediction $\beta = \pm 1/m$ for quasiholes and quasiparticles, respectively, describes our results well with the modification $\beta_{\rm qp} = 2 - \frac{1}{3}$ rather than $-\frac{1}{3}$. We also find that this approach enables us to better understand the *energetics* of the "daughter" states; in particular, we find good evidence, in terms of the effective interaction between quasiparticles, that the states $\nu = \frac{4}{11}$ and $\frac{4}{13}$ should not be stable.

The traditional way to define fractional statistics is in terms of the exchange phase $e^{i\alpha\pi}$ which is developed by the wave function when two identical particles are exchanged.[1] For bosons $\alpha = 0$ and for fermions $\alpha = 1$. Particles with fractional statistics (i.e., fractional α) can, in principle, exist in two dimensions. A common model of fractional statistics represents this exchange phase by adding an infinitesimal tube of magnetic flux (and a coupling charge) to ordinary fermions or bosons.

Recently Haldane[2] proposed an alternative approach to fractional statistics which applies to the case of systems with a finite Hilbert space (or subspace). Letting d_n be the size of the single-particle Hilbert space available to the nth identical particle, Haldane defined a fractional exclusion coefficient β by $\beta = -(d_{n+\Delta n} - d_n)/\Delta n$. Haldane argued for a correspondence between the exchange phase α and the generalized Pauli coefficient β for the bulk excitations in the fractional quantum Hall effect (FQHE),[3] based on the variational form of their wave functions. For quasiholes (qh) and quasiparticles (qp) near filling factor $\nu = 1/m$, with m an odd integer, Haldane obtained $\beta_{\rm qh} = 1/m$ and $\beta_{\rm qp} = -1/m$.

In this paper we test the utility of Haldane's approach for describing the low-energy excitations in the FQHE, using exact diagonalization of small systems of electrons. Our method relies on the single assumption that the number of qh and qp may be counted in the usual way. We first obtain the many-particle generalization of Haldane's definition, which is needed for analyzing the numerical results. We then find that Haldane's description works well—giving an unambiguous determination of the fractional exclusion coefficient β for a given energy scale—with one significant modification, namely, that $\beta_{\rm qp}$ is positive. Our analysis yields as a further result information about the effective interactions between the excitations, in terms of which the absence of the fractions $\nu = 4/13$ and $\nu = 4/11$ can be simply understood.

The belief that bulk excitations in the FQHE should exhibit fractional (α) statistics[4] is based upon the variational forms for their wave functions. These variational forms accurately describe the numerically obtained eigenstates for a single qh or qp (Ref. 5) or for several.[6] The variational wave functions develop phases under the motion of a qh or qp exactly as if it saw the electrons as quanta of flux.[4] This is the basis of the "hierarchy" explanation of the allowed fractions in the FQHE.[7,8] An alternative construction of the bulk excitations is given by a mapping to the integer quantum Hall effect.[9] This has recently been used by Jain and co-workers to construct other variational wave functions as an alternative explanation of the FQHE.[10]

Here we are interested in analyzing the structure of the exact low-energy spectrum (obtained numerically). In the case of qh it was first shown by Johnson and MacDonald[11] that the nominal qh states are separated by an energy gap from other states, and that these agree in number and multiplet structure with either the "hierarchy" or "integer-mapping" approach. He, Xie, and Zhang later showed a similar gap, less pronounced, for nominal qp states;[12] in this case the low-energy states can be thought of as those with an extra two zeros in their relative pseudo-wave-function. Here too the multiplet structure can be obtained either by the hierarchy[6] or integer-mapping[10] approaches. In this paper we show that the exact low-energy spectra of interacting electrons can be described simply and usefully by appealing to Haldane's formulation of fractional statistics.

We begin by proposing a many-particle generalization of Haldane's definition.[11] Let N_n be the dimension of the many-particle Hilbert space for n identical particles. (N_n is taken finite, by assuming a finite volume and an energy cutoff.[2]) Then

$$N_n = \binom{d_n + n - 1}{n} = \binom{d_1 + (1-\beta)(n-1)}{n} . \qquad (1)$$

One can view this as a restatement of Haldane's definition of β, via the single-particle dimension d_n. An alternative derivation, which is both illustrative and appropriate to the problem at hand, may be made by considering α-anyons in the lowest Landau level,[13] confined to a disk of radius R. These have wave functions

$$\Psi' = f(z_1, z_2, \ldots, z_n) \prod_{i<j=1}^{n} (z_i - z_j)^\alpha e^{-\sum_k^n |z_k|^2/4}, \qquad (2)$$

where f is any symmetric polynomial. We can count the allowed f's by examining the related wave functions $\Phi = f e^{-\sum |z|^2/4}$, which correspond to n fictitious bosons in the lowest Landau level. For $\alpha > 0$, the most compact such state has $f = 1$. This corresponds in Φ to putting all n "bosons" in the single-particle state of angular momentum $m = 0$. We can move one boson to the edge by putting $f = \sum_i z_i^M$; then $|\Psi'|^2$ barely fits in the disk if $M \approx R^2 - \alpha(n-1)$. Moreover, any Φ in which the bosons occupy any of the single-particle states $m = 0, 1, \ldots, M$ also yields a Ψ' confined to the disk. The number of such n-boson states, and hence the number of allowed Ψ', is $\binom{M+1+n-1}{n} = \binom{R^2+1+(1-\alpha)(n-1)}{n}$. This is of the form Eq. (1), with β equal to α. Thus, for particles in the lowest Landau level, there is an intimate connection between the Haldane coefficient β and the exchange phase α (as noted by Haldane[2]), namely, $\alpha = \beta \pmod 2$.

Consider in this example the smaller subset obtained by keeping only those states with $2l$ extra relative zeros, i.e., for which the symmetric polynomial can be written with an extra Jastrow factor, $f = \prod_{i<j}(z_i - z_j)^{2l} f'$. Repeating the above argument, the subset with $2l$ extra relative zeros can be described by a Haldane coefficient $\beta + 2l$, and has a dimension

$$N_n^{2l} = \binom{d_1 + (1 - \beta - 2l)(n-1)}{n}. \quad (3)$$

This correspondence between extra zeros and the value of the Haldane β coefficient will prove to be very useful in analyzing FQHE results.

Now we use these ideas to investigate the utility of Haldane's approach to fractional statistics. To test for β statistics we work in a spherical geometry,[14] where N_e electrons move in a radially outward uniform magnetic field with $N_L - 1$ flux quanta, giving N_L states in the lowest Landau level. When $N_L = m(N_e - 1) + 1$, the filling factor in the ground state is exactly $\nu = 1/m$. When n extra flux quanta are added, we assume that n qh are created; or when n are removed, n qp are present. One can instead create qh/qp by keeping the Hamiltonian fixed (holding N_L fixed)—which is necessary in the determination of β—and instead changing N_e. This creates qh or qp in multiples of m. We use the standard decomposition[15] for the interaction between electrons in the lowest Landau level, resolving it into pseudopotential parameters $V_1, V_3, V_5, \ldots$, where V_l gives the energy of a pair of electrons in relative angular momentum (RAM) l. These fall off monotonically (if slowly) as l grows.

In this work we will mostly consider states near $\nu = 1/3$, with V_1 and V_3 nonzero, and $V_3 \ll V_1$. The fractional quantum Hall effect at $\nu = 1/3$ is related to the existence of states for $\nu \leq 1/3$ which completely avoid unit RAM, and hence pay no V_1 cost. These states do, however, typically have electron pairs in RAM 3, and hence have energies on a scale set by V_3. Thus the spectrum (for $\nu < 1/3$) has a subset of low-energy (qh) states, lying below a gap of order V_1 and separated among themselves by energies of order V_3. The qp side ($\nu \geq 1/3$) is similar but energetically harder to separate. We will begin by discussing the qh case.

We find numerically that when there are nominally n qh present (i.e., when $N_L = 3N_e - 2 + n$), there is a well-defined subset of states of dimension

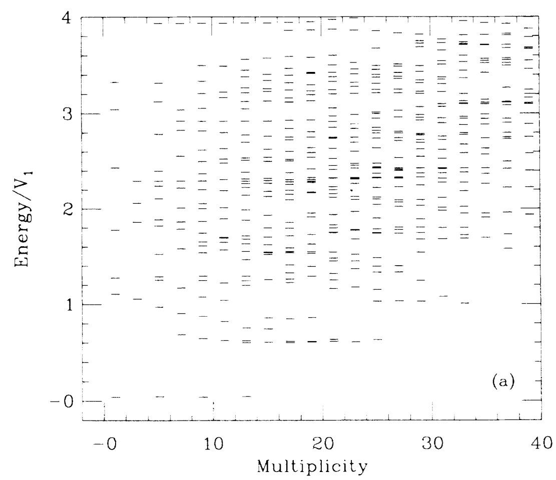
$$N_n^{\rm qh} = \binom{N_e + n}{n} \quad (4)$$

lying below a gap in energy of order V_1. This is the case for N_e from 2 to 7 and n from 1 to 6. It is illustrated for six electrons in Fig. 1(a). This dimension, and the multiplet structure, can be explained[11] equally well either by mapping to bosons as in the bosonic hierarchy approach,[8] or by mapping to fermions as in the integer-mapping approach.[10]

The beauty of Haldane's treatment is that it provides an *unambiguous* determination of what β statistics should be assigned to the excitations. Noting that $N_L = 3N_e - 2 + n$ states in the lowest Landau level give n qh for ν near but below 1/3, we can write Eq. (4) as

$$N_n = \binom{(N_L + 4)/3 + (1 - 1/3)(n-1)}{n}. \quad (5)$$

From Eq. (5) we can read off, by comparison with Eq. (1), a Haldane coefficient $\beta = 1/3$ for qh near $\nu = 1/3$. (We have also verified that $\beta_{\rm qh} = +1/5$ for qh near $\nu = 1/5$.) Hence we confirm, using exact electronic eigen-



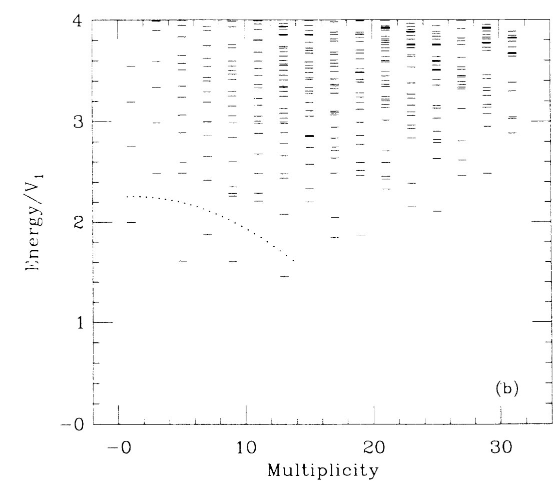


FIG. 1. (a) The low-energy spectrum for two quasiholes and six electrons on a sphere, in the truncated pseudopotential model with $V_3 = 0.01V_1$. The quasihole states lie below a gap of order V_1 and vary on an energy scale of order V_3. (b) The low-energy spectrum for three quasiparticles and eight electrons on a sphere, under the conditions of (a). The quasiparticle states lie below a less evident gap of order V_1, and vary on an energy scale which is also of order V_1.

states and standard counting of the qh, the identification $\beta_{\rm qh} = +1/m$ originally given by Haldane. This is, to our knowledge, the first nonvariational calculation of fractional statistics in the FQHE.[16]

Haldane's scheme can be used to obtain further information from the structure of the low-energy spectrum. We use as a guide pseudo-wave-functions of the form Eq. (2). Such wave functions vanish at zero separation according to the power law r^α where $r = |z_i - z_j|$. It is possible to identify smaller subsets which vanish according to a higher power law, e.g., as $r^{2l+\alpha}$. The number of such states with $2l$ extra zeros is, by Eq. (3), $\binom{N_e+n-2l(n-1)}{n}$. We find that it is in fact possible to identify subsets of two qh states with $2l$ extra zeros. This enables us to estimate effective two-body interaction energies for the excitations, which in turn allows a clear evaluation of the stability of certain hierarchy fractions.

We use a procedure for analyzing the multiplets which has been described previously.[11] A slight generalization of this procedure enables us to identify the multiplet structures of the subsets with $2l$ or more extra zeros. Consider, for example, $n = 2$ qh and $N_e = 6$ electrons. There are, by Eq. (1), 28 states lying below the V_1 gap; these break into multiplets with $L = 0, 2, 4, 6$. Now, by Eq. (3), there are 15 states with two or more extra zeros in the effective qh wave function; these turn out to have multiplets $L = 0, 2, 4$. In a similar way we can identify the subsets with $2l$ or more extra zeros; the results are given in Table I.[17]

Quasiholes are expected to repel one another. Thus states with extra zeros, which are further apart, are expected to be low in energy.[12] That is generally what we find (see Table I). The multiplets which by our analysis should have extra zeros usually turn out to be either the lowest-energy multiplets, or among the lowest. These results give semiquantitative information about the details of the qh interactions which can explain features of the FQHE hierarchy. To begin with, notice that a daughter of the $1/m$ level exists whenever the ground state is nondegenerate. This happens when the state with $2l$ extra zeros is a singlet ($N_n^{2l} = 1$), if that state is the ground state. If this occurs for $2l = 2$, then there is one state in which all of the qh's avoid the minimum angular momentum (i.e., in which they all have an extra pair of zeros). The resulting state is $\nu = 2/7$. The states in which all qh's have $2l = 4, 6, \ldots$ extra relative zeros are, respectively, $\nu = 4/13, 6/19, \ldots$.

This arithmetic suggests which states might exist in the hierarchy picture.[7,8] Whether or not they actually are stable is a question of energetics. We can study this in analogy to ordinary particles (bosons, say) in the lowest Landau level. When only two particles are present, the eigenstates of this sytem have the two particles in states of definite RAM ($2l = 0, 2, 4, \ldots$ for bosons). The energy of these is exactly the Haldane pseudopotential parameters V_{2l} plus the kinetic energy. The two qh states have a similar multiplet structure, and, in analogy, we assume that in these the qh are in states of definite RAM ($2l$ beyond the minimum required by statistics), with energies given by effective pseudopotential parameters $\tilde{V}_{2l}$ plus an unknown constant kinetic energy. We can then read the $\tilde{V}_{2l}$ off directly from the two qh spectra.[18] (These parameters have been estimated for qp, using a different method that is based on trial wave functions, by Béran and Morf.[19])

Our results are shown in Fig. 2. For example, for $N_e = 6$ electrons and $n = 2$ qh, the multiplets are in order $\tilde{V}_6 < \tilde{V}_2 < \tilde{V}_4 < \tilde{V}_0$; the multiplet with six extra zeros is the lowest in energy, that with no extra zeros the highest. The most interesting feature is that the $\tilde{V}_{2l}$ are nonmonotonic, unlike the V_l which describe the Coulomb interaction between electrons. In every case tested we see one feature in common: $\tilde{V}_2 < \tilde{V}_4$. This nonmonotonicity

TABLE I. Multiplicity of eigenstates for many-quasihole states near $\nu = 1/3$. N_e is the number of electrons, n the number of quasiholes, and N_L the corresponding number of states in the lowest Landau level on the sphere. N_n is the total number of quasihole states, i.e., those lying below the gap of order V_1. The remaining columns show the multiplicities of the degenerate states with no extra zeros, two or more extra zeros, etc. The multiplets are listed in order of increasing energy. Notice that states with extra zeros are typically lower in energy.

N_e	N_L	n	N_n	$2l \geq 0$	$2l \geq 2$	$2l \geq 4$	$2l \geq 6$
4	12	2	15	5,9,1	5,1	1	-
	13	3	35	1,7,9,5,13	1	-	-
	14	4	70	5,9,11,5,1,13,9,17	-	-	-
5	15	2	21	7,3,11	7,3	3	-
	16	3	56	4,10,8,12,6,16	4	-	-
	17	4	126	5,9,7,13,...	-	-	-
	18	5	252	4,10,8,12,...	-	-	-
6	18	2	28	1,9,5,13	1,9,5	1,5	1
	19	3	84	7,13,3,11,15,7,9,19	7,3	-	-
	20	4	210	1,9,13,7,...	1	-	-
	21	5	462	7,15,13,9,...	-	-	-
	22	6	924	1,9,13,5,...	-	-	-
7	21	2	36	11,3,7,15	11,3,7	3,7	3
	22	3	120	10,6,16,8,...	10,6,4	-	-

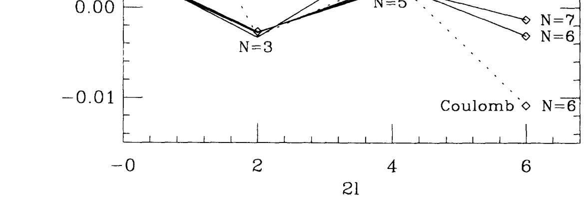


FIG. 2. Effective pseudopotential parameters for quasiholes near $\nu = 1/3$, as extracted from two-quasihole spectra. $\tilde{V}_{2l}$ is the energy of a pair of quasiholes in a state with relative angular momentum $2l$ more than the minimum required by statistics. Notice the nonmonotonicity which can explain the absence of certain FQHE fractions (see text). The solid lines are for the truncated pseudopotential model, with $V_3 = 0.01V_1$. The dashed line is the Coulomb result (in units of e^2/ℓ, where ℓ is the magnetic length). We find similar results for quasiparticles.

can explain the absence of certain fractions. The hierarchy construction predicts that $\nu = 4/13$ is a daughter of $1/3$ in which the qh avoid paying $\tilde{V}_2$, i.e., every pair of qh has at least four extra zeros. By doing so, however, they pay a $\tilde{V}_4$ cost. If $\tilde{V}_4 > \tilde{V}_2$, as we find, the $\nu = 4/13$ state is not stable—a conclusion apparently consistent with experiment.[20]

We also note that, in contrast to the estimates of Béran and Morf[19] for qp, our estimates for $\tilde{V}_{2l}$ are not always positive. These results are puzzling, but cannot be viewed as an artifact of our method: they simply mean that there are some two qh states which are lower in energy than twice the lowest one qh energy. Further study is needed to test the significance of this result.

The detailed analysis we have performed of the qh spectrum can be repeated for the qp case, with the important difference[21] that the gap separating the low-lying states is typically of the same magnitude as the splitting among these states—i.e., of order V_1 [compare Figs. 1(a) and 1(b)]. However, we can in most cases identify (in agreement with Ref. 12) a low-energy subspace with which to test Haldane counting.

Haldane's argument suggests that qp should have statistics coefficient $\beta = -1/m$. Note that qp see electrons as flux in the opposite direction of that seen by qh, and so the appropriate pseudo-wave-functions to guide analysis of the qp spectrum involve factors of the form[7] $(z_i^* - z_j^*)^{-1/m}$. However, such a behavior for qp at short distances is energetically costly.[12] Hence we expect not to see a subspace with negative β, but rather only subsets with $\beta = 2l - (1/m)$ for nonzero $2l$.

This expectation is borne out in our results. Repeating the procedure applied to qh states, we find a Haldane coefficient $\beta = 2 - 1/3$ for qp states near $\nu = 1/3$. For general $\nu = 1/m$, this approach gives $\beta = 2 - 1/m$. As argued above, the states in the low-energy subspace can be viewed as those of $\beta = -1/3$ particles with two or more extra zeros. In fact, either the states with no extra zeros do not exist at all, or they are hidden in the higher-energy states. Hence it is better to view $\beta = 2 - 1/3$ as the fundamental statistics coefficient for the qp, to be compared with $\beta = 1/3$ for qh.

The multiplet structures can be also understood as in the qh case, with similar results. We again find a consistently nonmonotonic behavior of the $\tilde{V}_{2l}$ ($\tilde{V}_4 > \tilde{V}_2$ always), with $\tilde{V}_2$ and $\tilde{V}_6$ typically negative. Hence we find, in close analogy to our qh result, that the 4/11 state should not be stable—again in agreement with experiment[20] and with previous work.[22,19]

In summary, we have tested the application of Haldane's generalization of the Pauli principle to charged excitations of the FQHE, using exact spectra for small numbers of electrons, and assuming only that we know how to count the excitations. At the largest energy scale at which the daughter states are defined, we find fractional exclusion coefficients $\beta_{\rm qh} = 1/3$ and $\beta_{\rm qp} = 2 - 1/3$. Our results thus confirm Haldane's suggestion that, in the FQHE, fractional α is accompanied by fractional β—i.e., a fractional exclusion principle. Our confirmation then strongly suggests that models of anyons involving flux attached to bosons or fermions (fractional α, integral β) may not be well justified for application to real condensed-matter systems.

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[1] See, e.g., *Fractional Statistics and Anyon Superconductivity*, edited by F. Wilczek (World Scientific, Singapore, 1990), and references therein.
[2] F.D.M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).
[3] For an introduction see A.H. MacDonald, *A Perspective on the Quantum Hall Effect* (Kluwer, Boston, 1989).
[4] Daniel Arovas, J.R. Schrieffer, and Frank Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[5] R.B. Laughlin, in *The Quantum Hall Effect*, 2nd ed., edited by R.E. Prange and S.M. Girvin (Springer-Verlag, New York, 1990).
[6] J. Yang and W.P. Su, Phys. Rev. Lett. **70**, 1163 (1993).
[7] B.I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[8] F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[9] A.H. MacDonald and D.B. Murray, Phys. Rev. B **32**, 2707 (1985); Michael Stone, *ibid.* **42**, 8399 (1990).
[10] J.K. Jain, Phys. Rev. Lett. **63**, 199 (1989); G. Dev and J.K. Jain, *ibid.* **69**, 2843 (1992).
[11] M.D. Johnson and A.H. MacDonald, in *Physical Phenomena at High Magnetic Fields*, edited by E. Manousakis *et al.* (Addison-Wesley, Redwood, 1992), p. 102.
[12] S. He *et al.*, Phys. Rev. Lett. **68**, 3460 (1990).
[13] M.D. Johnson and G.S. Canright, Phys. Rev. B **41**, 6870 (1990).
[14] See T. Chakraborty and P. Pietilainen *The Fractional Quantum Hall Effect* (Springer-Verlag, Berlin, 1988).
[15] F.D.M. Haldane, in *The Quantum Hall Effect* (Ref. 5), Chap. 8.
[16] Given our identification of β with α for the lowest Landau level (which is appropriate here), our extraction of fractional β-statistics also implies fractional α-statistics.
[17] We note that only for 2 qh does each multiplet correspond to states with a definite number of extra zeros. For example, with 3 qh and 6 electrons, there are 84 states below the V_1 gap, in 8 different multiplets. Of these, two multiplets ($L = 1, 3$) have two extra zeros.
[18] We take $\tilde{V}_{2l} = (E_{2l} - E_0) - 2(E_1 - E_0)$, where E_{2l} is the energy of the relevant 2-qh multiplet and $E_{0,1}$ is the lowest-energy state with 0 or 1 qh present (Ref. 19).
[19] P. Béran and R. Morf, Phys. Rev. B **43**, 12 654 (1991).
[20] See, for example, V.J. Goldman *et al.*, Phys. Rev. Lett. **61**, 881 (1988), and references in Refs. 21 and 22.
[21] F.C. Zhang and X.C. Xie, Phys. Rev. B **40**, 11 449 (1989).
[22] Claudius Gros and A.H. MacDonald, Phys. Rev. B **42**, 9514 (1990); A.H. MacDonald, Surf. Sci. **229**, 1 (1990).