Quantum magnetotransport of a periodically modulated two-dimensional electron gas

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The magnetoresistance of two-dimensional electrons in a periodic potential is studied. The Landau levels are broadened by one-dimensional modulation, and are further split into a Hofstadter-like spectrum by two-dimensional modulation. The recently observed Weiss oscillation manifests the oscillatory bandwidth of broadened Landau levels, where the minima appear at the flat-band condition (zero bandwidth) for one-dimensional modulation. However the flat bands seem to give the maxima for two-dimensional modulation. In this paper, the conductivity for both one- and two-dimensional modulations has been obtained analytically. It is shown that the decrease of the scattering conductivity is enhanced by the band splitting, while the band conductivity is reduced significantly. Consequently, the scattering conductivity decreases faster than the increase of the band conductivity as the Landau level is broadened from the flat band. Therefore the total conductivity shows maximum at flat bands, in contrast to the one-dimensional case. Moreover a numerical study has been made by using the Thouless number method. A dramatic difference is observed between one- and two-dimensional modulations. This can be explained in the same way as the result of band splitting.

I. INTRODUCTION

The motion of two-dimensional electrons subject simultaneously to a magnetic field and a periodic potential gives rise to many interesting problems, such as the commensurability problem. The commensurability revealed in this system results from the interplay among several competing length scales, for instance the magnetic length $l=(\hbar c/eB)^{1/2}$, the lattice constant a, and the Fermi wavelength $2\pi/k_F=(2\pi/N_s)^{1/2}$, where N_s is the electron density. In the limit of strong magnetic field perturbed by a weak periodic potential, or in the limit of tight-binding electrons in a weak magnetic field, this system is well known to be described by the Hofstadter-type spectrum.[1] Recently there has been a renewal of theoretical interest, partly because of its important applications in the quantum Hall effect[2] and flux phase of high-temperature superconductivity.[3]

The modern technology of the fabrication of superlattices has made the experimental realization of the Hofstadter spectrum accessible. The superlattices are generated in the high mobility $GaAs/Al_xGa_{1-x}As$ heterostructure samples where a two-dimensional electron gas is embedded. The magnetoresistance is often used as an indirect probe to study the structures of energy spectrum; for example, the Shubnikov–de Haas oscillations indicate the Landau quantization. In addition to the usual Shubnikov–de Haas oscillations in the high fields, an unusual type of magnetoresistance oscillation, the Weiss oscillation, was observed in the low magnetic fields $B<0.5$ T in the presence of a one-dimensional periodic modulation.[4–6] This oscillation is periodic in the inverse of the magnetic field.[4,5] The periodicity is proportional to the square root of the electron density.[6] The amplitude of the oscillation has almost no temperature dependence in the range where te Shubnikov–de Haas oscillation is smeared out by the temperature.[7] The maxima and minima of the oscillation are given by[4,5]

$$2R_c=(\lambda+\varphi)a, \quad \lambda=1,2,3,\ldots, \tag{1.1}$$

where R_c is the cyclotron radius $R_c=l^2k_F$, and φ is the phase determining the maxima and minima of the oscillation. The experiments give $\varphi=0.17$ for the maxima, and $\varphi=-0.25$ for the minima. The latter is often called the flat-band condition.

Various theories are given for the explanation of the experimental results. It was pointed out by Beenakker[8] that the oscillation is classical in nature. The periodic modulation acts as an electric field which gives a nonzero drift velocity if Eq. (1.1) is satisfied. Therefore there is an additional contribution to the conductivity. Similarly, Středa and MacDonald[9] have explained that this periodic modulation makes it possible for electrons to hop from one cyclotron orbit to another with the guiding center $2\pi l^2/a$ apart. This of course results in a diffusive conductivity. However, the classical theories cannot explain the weak antiphase oscillation in the perpendicular direction of the one-dimensional modulation.[4,5] It has been argued quantum mechanically[10–12] that Landau levels are broadened because of the periodic modulation. The width of these broadened bands oscillates as a function of the Landau-level index. The Weiss oscillation stems from the oscillatory bandwidths. This is also evident from measurements of the magnetocapacitance,[13,14] which is directly related to the density of states. The contributions to the conductivity have been distinguished into the band conductivity, which is the result of band broadening, and the scattering conductivity between the Landau levels. The scattering rate is also modified due to the nonzero width of the Landau levels. This yields a reasonable explanation of the weak oscillation in the perpendicular direction, where there is no band conductivity. The theory was further generalized to an anharmonic modulation,[15] which has been previously assumed to be the

lowest harmonic. Similar results have been obtained when the system is modulated by a periodic magnetic field.[16]

The experiments for a grid potential has also been reported. Gerhardts, Weiss, and Wulf have measured the magnetoresistance of a two-dimensional electron gas modulated by a square periodic potential.[17] Magnetoresistance oscillations were also observed. However, dramatic differences of the measurements appear between two- and one-dimensional modulations. The magnetoresistance measured in the two-dimensional modulation is much weaker and has an antiphase, as observed in the perpendicular direction of the one-dimensional modulation. Fang and Stiles[18] used a hexagonal lateral periodic potential generated by latex sphere masks. The magnetoresistance measured in the deep-etched sample has an antiphase with that in the sallow-etched sample, which exhibits usual features of one-dimensional modulation. The measurement of magnetocapacitance in the grid-gate samples[14] seemed to give some fine structures, which were explained possibly to be the manifestation of the Hofstadter spectrum.

It was pointed out by Gerhardts and co-workers[17,19] that band splitting due to two-dimensional modulations plays an important role in the calculation of the conductivity. If the band splitting is quite effective for the weak disorder, the overlap between subbands can be neglected. This reduces significantly the band conductivity. Recently the switching of dominance from band to scattering conductivities has been observed experimentally by Weiss *et al.*[20] Lorke, Kotthaus, and Ploog[21] have numerically calculated the two-dimensional classical diffusion constant. They claimed that the oscillation in the two-dimensional modulation is still of classical origin. However, unlike the one-dimensional modulation, the phase in Eq. (1.1) depends on the specific modulation potential.

Basically, the above theories for two-dimensional modulations are based on numerical calculations. The calculations[17,19] have focused on the case where band splitting dominates. It is not clear how one- and two-dimensional modulations differ from each other around flat bands. In this paper, an analytical formulation of this problem is derived, which is consistent for both one- and two-dimensional modulations. In Sec. II, the calculations are carefully repeated for one-dimensional modulations, by using the Kubo formula and the self-consistent Born approximation; however, no further assumptions have been made. In Sec. III, the conductivity for a two-dimensional modulation is carried out analytically. The results are valid whether or not the band splitting is resolved. It is shown that the decrease of the scattering conductivity is enhanced by the band splitting, whereas the band conductivity is reduced significantly. As the band is broadened from the flat band, the band conductivity increases more slowly than the decrease of the scattering conductivity. Therefore, the total conductivity shows a maximum at flat bands.

In Sec. IV, a numerical study has been made by using the Thouless number method.[22–24] The conductivity is obtained as a function of periodic modulation strength for a fixed disorder broadening factor. For one-dimensional modulation, the conductivity increases rapidly with increasing modulation strength. On the contrary, the conductivity for two-dimensional modulation decreases very slowly with the increasing modulation strength. As the modulation becomes sufficiently strong, the conductivity starts to grow. This supports the analytical calculations.

II. MAGNETORESISTANCE OSCILLATIONS

The resistance of a two-dimensional electron gas in a magnetic field exhibits the Shubnikov–de Haas oscillations. This is well known to result from the oscillatory density of states which reflects the Landau quantization. In response to the applied external electric field, the electrons hop to the nearest Landau levels. However, this needs to be assisted by the impurity scatterers. The disorders in the system broaden the discrete Landau levels into bands, and more importantly cause the overlap between the Landau levels.

A. General formulation

In the self-consistent Born approximation, for the short-range scatterers, the Green function is given by[25]

$$G_N(E)=\frac{1}{E-E_N-\Sigma^-(E)}\ , \tag{2.1}$$

where $E_N=(N+\frac{1}{2})\hbar\omega$ with the cyclotron frequency ω. The self-energy $\Sigma^-(E)$ is independent of the Landau-level index N, and can be determined self-consistently by

$$\Sigma^-(E)=\frac{\Gamma_0^2}{4}\sum_N G_N(E), \tag{2.2}$$

where Γ_0 characterizes the level broadening and is given by

$$\Gamma_0^2=\frac{2}{\pi}\hbar\omega\frac{\hbar}{\tau_f}\ , \tag{2.3}$$

and τ_f is the relaxation time in the absence of magnetic fields. τ_f is determined by the mobility of the sample, namely $\mu=e\tau_f/m^*$, and the effective mass $m^*=0.067m_e$.

The conductivity can be calculated by the Kubo-Greenwood formula

$$\sigma_{\mu\mu}(E)=\frac{e^2\hbar}{\pi^2 l^2}\sum_{N,N'}|\langle N|v_\mu|N'\rangle|^2\mathrm{Im}G_N\mathrm{Im}G_{N'}\ . \tag{2.4}$$

To calculate the velocity, the eigenfunctions are written in the form[2]

$$u_{k_1k_2}(x,y)=\exp(ik_1x+ik_2y)f_{k_1k_2}(x,y)\ . \tag{2.5}$$

For the function $f_{k_1k_2}(x,y)$ the Hamiltonian becomes **k** dependent. Therefore the velocity operators can be expressed as $1/\hbar$ times the partial derivative of this effective Hamiltonian with respect to the wave number. The expectation values can be obtained explicitly in terms of the eigenfunctions,

$$\left\langle N \left| \frac{\partial H}{\partial k_j} \right| N' \right\rangle = \frac{\partial E_N}{\partial k_j}\delta_{NN'} + (E_{N'} - E_N)\left\langle N \left| \frac{\partial}{\partial k_j} \right| N' \right\rangle . \tag{2.6}$$

The second term gives rise to the nonzero velocity between nearest Landau levels, which results in the *scattering* conductivity. If the energy is k_j dependent in the presence of periodic potentials, the first term does not vanish and the *band* conductivity can be expected.

B. Shubnikov–de Haas oscillation

In the high mobility samples where $\Gamma_0 \ll \hbar\omega$, the Green function can be solved in terms of the semielliptic function, explicitly for the energy range $-\Gamma_0 \le E - E_N \le \Gamma_0$,

$$\mathrm{Im}G_N(E) = \frac{2}{\Gamma_0}\left[1 - \frac{(E-E_N)^2}{\Gamma_0^2}\right]^{1/2} . \tag{2.7}$$

Moreover, the overlap of the densities between neighboring Landau levels is given by a factor of $\Gamma_0^2/4(\hbar\omega)^2$ in this limit.

The velocity operator connects the Landau levels with their nearest neighbors. This yields the conductivity

$$\sigma_{xx}(E) = \frac{e^2}{\hbar\pi^2}\sum_N (N+\tfrac{1}{2})\frac{\Gamma_0^2}{4}[\mathrm{Im}G_N(E)]^2 . \tag{2.8}$$

At the band center, the conductivity is quantized as half integers, namely

$$\sigma_{xx}(E_N) = \frac{e^2}{\hbar\pi^2}(N+\tfrac{1}{2}) . \tag{2.9}$$

C. One-dimensional modulation

Let us then switch on a weak one-dimensional periodic modulation given by

$$V(x) = V_0\cos(2\pi x/a) . \tag{2.10}$$

The first-order perturbation gives the correction to the energy,[10–12]

$$E_N(k_2) = E_N + 2V_N\cos(2\pi l^2 k_2/a) , \tag{2.11}$$

where $V_N = 0.5V_0\exp(-\pi^2 l^2/a^2)L_N(2\pi l^2/a^2)$, and L_N is the Laguerre polynomial. One needs to replace E_N in Eq. (2.1) by $E_N(k_2)$, and correspondingly an average over the wave number k_2 in Eqs. (2.2) and (2.4) is necessary. Equation (2.2) can be evaluated explicitly, and it gives a self-consistent equation for the self-energy:

$$\Sigma^-(E) = \frac{\Gamma_0^2}{4}\sum_N \frac{1}{\sqrt{[E - E_N - \Sigma^-(E)]^2 - 4V_N^2}} . \tag{2.12}$$

For weak disorder, this gives two peaks at the band edges reflecting the Van Hove singularity for one-dimensional periodic systems.

Similarly the conductivity can be calculated by Eq. (2.4), where the average over k_2 should be included. The analytical results for both scattering and band conductivities as a function of energy can be obtained explicitly. Since the Landau levels are broadened by the periodic modulation, the correction to the scattering conductivity needs to be taken into account. For wide bands, the conductivity remains a constant in a quite wide region of energy around the band center. There are two peaks close to band edges $E_{\mathrm{edge}} \approx 2V_N$. As the bandwidth decreases, the conductivity at central region will increase according to Eq. (2.14), and the width of this region will shrink. Therefore two peaks at the edges move toward the center. At the center of the band, the calculation is simple and illuminating. It describes exactly the envelope of Shubnikov–de Haas oscillations, and consequently the Weiss oscillations. The real part of the Green function vanishes, while the imaginary part is given by Eq. (2.12), which can be solved explicitly:

$$\mathrm{Im}\Sigma^-(E_N) = \frac{1}{\sqrt{2}}\left[\left[16V_N^4 + \frac{\Gamma_0^4}{4}\right]^{1/2} - 4V_N^2\right]^{1/2} . \tag{2.13}$$

The conductivity is given by Eq. (2.9), modified by a factor γ_1,

$$\gamma_1 = \frac{\Gamma_0^2}{4}\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{[4V_N^2\cos^2 k + [\mathrm{Im}\Sigma^-(E_N)]^2]^2} \times [\mathrm{Im}\Sigma^-(E_N)]^2 dk$$
$$= \left[1 + \frac{64V_N^4}{\Gamma_0^4}\right]^{1/2}\left[\left[1 + \frac{64V_N^4}{\Gamma_0^4}\right]^{1/2} - \frac{8V_N^2}{\Gamma_0^2}\right] . \tag{2.14}$$

For a given degree of disorder Γ_0, as the bandwidth V_N is increased γ_1 decreases rapidly but saturates at a value of 0.5 for wide bands.

The results for the band conductivity shows a peak at the band center, which spans the whole band. The peak value of the conductivity is given by

$$\Delta\sigma_{yy}(E_N) = \frac{e^2\hbar}{\pi^2 l^2}\left[\frac{2\pi l^2}{\hbar a}\right]^2 \frac{1}{2\pi}\int_0^{2\pi} \frac{4V_N^2\sin^2 k_2}{\{4V_N^2\cos^2 k_2 + [\mathrm{Im}\Sigma^-(E_N)]^2\}^2}[\mathrm{Im}\Sigma^-(E_N)]^2 dk_2$$
$$= \frac{e^2}{\hbar}\frac{8l^2}{a^2}\frac{4V_N^2}{\Gamma_0^2} \approx \frac{e^2}{h}\frac{V_0^2\tau_f}{\hbar^2\omega}\frac{8}{ak_F}\cos^2\left[2\pi\frac{R_c}{a} - \frac{\pi}{4}\right] , \tag{2.15}$$

where, in the last step, V_N is expanded for high Landau levels. R_c is the classical radius of cyclotron orbits. The minima of $\Delta\sigma_{yy}$ are certainly given by Eq. (1.1). This is in agreement with those obtained previously by other authors.[8–12] Furthermore, this band conductivity varies quadratically with respect to the ratio V_N/Γ_0. Though the factor l^2/a^2 is a considerably small parameter for experiments, Eq. (2.15) contributes significantly for wide

bands where V_N/Γ_0 is large. For lower Landau levels, this band conductivity increases faster than the decrease of the scattering conductivity for small V_N. Therefore the total conductivity shows a minimum at the flat-band condition $V_N=0$.

III. EFFECT OF BAND SPLITTING

In this section, the magnetoresistance in a two-dimensional periodic potential is considered. There have been two experiments reported. Though the geometry of the lattices, the square[17] and hexagonal[18] lattices, respectively, is different, similar patterns have been measured. For simplicity, only the square lattice is studied, and the possible generalization to others such as rectangular and hexagonal lattices will be discussed. The main results are contained in Sec. III D, where it is shown that the Weiss oscillation in the two-dimensional modulation has an antiphase with that in the one-dimensional modulation.

A. Properties of Hofstadter spectrum

The two-dimensional periodic modulation is described by

$$V(x,y)=\tilde{V}_1\cos(2\pi x/a)+\tilde{V}_2\cos(2\pi y/b) , \tag{3.1}$$

where a and b are the periods in each direction. The flux through each unit cell play an important role, which we write as

$$\phi=Bab=\frac{p}{q}\phi_0 , \tag{3.2}$$

where $\phi_0=h/e$ is the flux quantum, and p and q are mutually primed integers. The relation between the period and the magnetic length is shown as

$$\frac{2\pi l^2}{ab}=\frac{q}{p} . \tag{3.3}$$

When the modulation is weak compared to the Landau-level spacing, the perturbation theory is applicable. The basis wave function is constructed as the superposition of those describing the Landau level with different guiding centers, explicitly[2]

$$u_{k_1k_2}=\exp(ik_1x+ik_2y)\sum_n d_n^{\alpha}(k_1,k_2)f_{k_1k_2} , \tag{3.4}$$

where

$$f_{k_1k_2}=\sum_{t=-\infty}^{\infty}\chi_N\left[x-l^2k_2-tqa-\frac{nqa}{p}\right]\times\exp\left[-ik_1\left[x-tqa-\frac{nqa}{p}\right]+2\pi iy\frac{(tp+n)}{b}\right] , \tag{3.5}$$

and α is the subband index. The function $f_{k_1k_2}$ is independent of the subband index α. $\chi_N(x)$ is the usual wave function of Landau levels. In the absence of disorder, the coefficient d_n satisfies

$$V\exp(-iqak_1/p)d_{n-1}+2V'\cos(qbk_2/p+2\pi nq/p)d_n+V\exp(iqak_1/p)d_{n+1}=Ed_n . \tag{3.6}$$

This is well known as the Harper equation.[26] For the square lattices, we have $\tilde{V}_1=\tilde{V}_2=V_0$ and $a=b$, thus $V=V'=V_N$, whose explicit form can be found in Sec. II. The coefficient d_n does not depend on the Landau-level index, since the energy can properly be scaled and is proportional to V_N.

It has been shown the eigenvalues of Eq. (3.6) can be obtained by solving

$$P(E/V_N)=2\cos(2\pi l^2pk_1/a)+2\cos(2\pi l^2pk_2/a) , \tag{3.7}$$

where $P(x)$ is a pth-order polynomial independent of the wave numbers k_1 and k_2. This gives a Hofstadter spectrum where there are p subbands separated by gaps. The integrated density of states for each subband is equal, therefore is $1/p$. In general, the explicit form of the dispersion formula for each subband, $E_N^{\alpha}(k_1,k_2)$, is difficult to obtain. It is possible to expand $P(x)$ around some special energies, such as the critical energies[27] where the real part of the Green function vanishes. Specifically, the critical energy is defined when the right side of Eq. (3.7) is zero, explicitly, when k_1 and k_2 satisfy

$$2\pi l^2(k_1\pm k_2)/a=\pm\pi/p,\mp\pi/p . \tag{3.8}$$

One can show that at these energies, the density of states diverges logarithmically. When Eq. (3.7) is expanded around the critical energy E_c^{α}, this yields

$$E_N^{\alpha}(k_1,k_2)\approx E_c^{\alpha}+2g_{\alpha}V_N[\cos(2\pi l^2pk_1/a)+\cos(2\pi l^2pk_2/a)] , \tag{3.9}$$

where the coefficient g_{α} is

$$g_{\alpha}=[P'(E_c^{\alpha})]^{-1}=b_{\alpha}p^{-1} . \tag{3.10}$$

The constant b_{α} is determined by the structure of the polynomial $P(x)$, and $|b_{\alpha}|$ is less than 1, as is evident from the sum rule,[28]

$$\sum_{\nu=1}^{p}\frac{1}{|P'(E_c^{\nu})|}=\frac{1}{p} . \tag{3.11}$$

B. Self-energy and density overlap

As disorder is introduced, the self-consistent Born approximation is used.[25] The Green function can be expressed as

$$G_{N\alpha}(E,\mathbf{k})=\frac{1}{p}\frac{1}{E-E_N^{\alpha}(k_1,k_2)-\Sigma^{-}(E)} . \tag{3.12}$$

The self-energy $\Sigma^{-}(E)$, independent of N, α, and $\mathbf{k}$, is given self-consistently by

$$\Sigma^{-}(E)=\frac{\Gamma_0^2}{4p}\sum_{N,\alpha}\frac{1}{4\pi^2}\int_0^{2\pi}dk_1\int_0^{2\pi}dk_2 G_{N\alpha}(E,\mathbf{k}) . \quad (3.13)$$

For the two-dimensional modulations, the conductivity appears to be maximum at the critical energies. Therefore the approximation in Eq. (3.9) gives accurate descriptions of the Weiss oscillation. Assume that the Fermi energy resides at the critical energy E_c^α. Equation (3.8) can be approximated as

$$\mathrm{Im}\Sigma^{-}(E_c^\alpha)\approx\frac{\Gamma_0^2}{4p}\frac{1}{4\pi^2}\int_0^{2\pi}dk_1\int_0^{2\pi}dk_2\frac{\mathrm{Im}\Sigma^{-}(E_c^\alpha)}{4g_\alpha^2V_N^2[\cos(k_1)+\cos(k_2)]^2+[\mathrm{Im}\Sigma^{-}(E_c^\alpha)]^2}+\frac{\Gamma_0^2}{4p}\sum_{i=1}^{p-1}\frac{\mathrm{Im}\Sigma^{-}(E_c^\alpha)}{[C_iV_N]^2+[\mathrm{Im}\Sigma^{-}(E_c^\alpha)]^2} , \quad (3.14)$$

where C_i is a constant of order unity determined by the band structure, and C_iV_N measures the distance between E_c^α and the critical energy at the ith subband. The first term on the right side can be evaluated explicitly:

$$\frac{\Gamma_0^2}{4p}\frac{2}{\pi\sqrt{[4g_\alpha V_N]^2+[\mathrm{Im}\Sigma^{-}(E_c^\alpha)]^2}}\times K\left[\frac{4g_\alpha V_N}{\sqrt{[4g_\alpha V_N]^2+[\mathrm{Im}\Sigma^{-}(E_c^\alpha)]^2}}\right] , \quad (3.15)$$

where $K(x)$ is the elliptic integral of the first kind. For the narrow band where disorder dominates, each subband contributes equally. The self-energy is given, to leading order, by

$$\mathrm{Im}\Sigma^{-}(E_c^\alpha)\approx\frac{\Gamma_0}{2}\left[1-\frac{2}{p}\sum_{i=1}^{p-1}C_i^2\frac{V_N^2}{\Gamma_0^2}-\frac{8g_\alpha^2}{p}\frac{V_N^2}{\Gamma_0^2}\right] . \quad (3.16)$$

The last term is much smaller than the second one, which represents the effect of band splitting. Thus the decrease is dominated by the band splitting, while, for the wide band, the second term on the right side of Eq. (3.14) can be neglected because the overlap between subbands is small. This yields

$$\mathrm{Im}\Sigma^{-}(E_c^\alpha)\approx-\frac{\Gamma_0^2}{4p}\frac{1}{2\pi g_\alpha V_N}\ln\left[\frac{\mathrm{Im}\Sigma^{-}(E_c^\alpha)}{16g_\alpha V_N}\right]\approx\frac{\Gamma_0^2}{4\pi p g_\alpha V_N}\ln\left[\frac{8\sqrt{\pi p}\,g_\alpha V_N}{\Gamma_0}\right] . \quad (3.17)$$

This differs from that in the one-dimensional modulation by a logarithmic factor.

The overlap between subbands can be obtained readily when the band is split. The density of states of subband $\alpha+1$ in the energy range of subband α is

$$D_{N,\alpha+1}(E)=\frac{\Gamma_0^2}{4}\left[-\frac{d}{dE}G^0_{N,\alpha+1}(E)\right]D_{N,\alpha}(E) , \quad (3.18)$$

where $G^0_{N,\alpha+1}(E)$ is $[E-E_N^{\alpha+1}(k_1,k_2)]^{-1}$ averaged over $\mathbf{k}$. For the energy outside the subband $\alpha+1$, there is only the real part left for $G^0_{N,\alpha+1}(E)$, which is divergent logarithmically at the band edge of subband $\alpha+1$. Actually it is in term of the elliptic integral modified by E^{-1}; as E moves far away from the edge, the elliptic integral goes to a constant. This allows expansion of $G^0_{N,\alpha+1}(E)$ in powers of E^{-1} when E is in subband α. For sufficiently weak disorder, Eq. (3.18) reduces to

$$D_{N,\alpha+1}(E)\approx\frac{\Gamma_0^2}{4C_\alpha^2V_N^2}D_{N,\alpha}(E) . \quad (3.19)$$

C. Velocity operator and conductivity

The velocities can be calculated by using Eq. (2.6), where for *inter-Landau-levels,* explicitly,

$$\langle N\alpha|v_j|N'\beta\rangle=(E_{N'}^\beta-E_N^\alpha)\delta_{\alpha\beta}\frac{1}{\hbar}\left\langle f_N^*\frac{\partial f_{N'}}{\partial k_j}\right\rangle_{\mathbf{r}} , \quad (3.20)$$

with $\langle\ \rangle_{\mathbf{r}}$ denoting the spatial integration. The velocity between Landau levels is diagonal in subband index α, and is independent of the wave number $\mathbf{k}$. This gives the *scattering* conductivity. Similarly the correction factor to Eq. (2.9) can be found to be

$$\gamma_2=\frac{\Gamma_0^2}{4p^2}x_1^2+\frac{2}{p^2}\mathrm{Im}\Sigma^{-}(E_c^\alpha)x_1+\frac{1}{p^2}\left[\frac{1}{2}+\frac{1}{2}\frac{E(x)}{K(x)}\right] , \quad (3.21)$$

where $E(x)$ is the elliptic integral of the second kind. x and x_1 are given by

$$x=\frac{4g_\alpha V_N}{\sqrt{[4g_\alpha V_N]^2+[\mathrm{Im}\Sigma^{-}(E_c^\alpha)]^2}} , \quad (3.22a)$$

and

$$x_1=\sum_{i=1}^{p-1}\frac{\mathrm{Im}\Sigma^{-}(E_c^\alpha)}{[C_iV_N]^2+[\mathrm{Im}\Sigma^{-}(E_c^\alpha)]^2} . \quad (3.22b)$$

This factor γ_2 decreases very rapidly with increasing V_N.

For *intra-Landau-levels,* the nonzero velocity gives the *band* conductivity, which can also be estimated. The velocity within the same subband is given by

$$\langle N\alpha|v_j|N\alpha\rangle=\frac{1}{\hbar}\frac{\partial E_N^\alpha}{\partial k_j}=-2g_\alpha V_N\frac{p}{\hbar}\frac{2\pi l^2}{a}\sin(2\pi l^2pk_j/a) . \quad (3.23)$$

Therefore the band conductivity $\Delta\sigma_2$ can be found in the

form

$$\frac{e^2}{\hbar}\frac{4l^2}{a^2}\left[2x_1^2 g_\alpha^2 V_N^2 + x_1\left[\frac{2E(x)}{\pi x}4g_\alpha V_N - \mathrm{Im}\Sigma^-(E_c^\alpha)\right] - \frac{[\mathrm{Im}\Sigma^-(E_c^\alpha)]^2}{\Gamma_0^2} + \frac{E(x)}{2\pi x}\frac{4g_\alpha V_N}{\mathrm{Im}\Sigma^-(E_c^\alpha)}\right], \quad (3.24)$$

where x and x_1 are given by Eq. (3.22).

The velocity between subbands can be written explicitly in terms of solutions of Eq. (3.6):

$$\langle N\alpha|v_j|N\beta\rangle = (E_N^\beta - E_N^\alpha)\sum_n (d_n^\alpha)^* \frac{1}{\hbar}\frac{\partial d_n^\beta}{\partial k_j} . \quad (3.25)$$

D. Phase of Weiss oscillation

As mentioned in Sec. III C for one-dimensional modulation flat bands give the minima of the Weiss oscillation, while the maxima appear for wide bands. Let us consider two-dimensional modulation. Close to the flat bands where V_N/Γ_0 is small, Eq. (3.21) varies in the following manner:

$$\gamma_2 \approx 1 - \frac{4(p^2+2p-1)}{p^3}\sum_{i=1}^{p-1} C_i^2 \frac{V_N^2}{\Gamma_0^2} + \frac{16(p^2-3p+1)}{p^3} g_\alpha^2 \frac{V_N^2}{\Gamma_0^2} , \quad (3.26)$$

where only the leading term in the power of V_N^2/Γ_0^2 is kept. The last term is much smaller than the second one, so it can be ignored. It can be shown that $\sum_{i=1}^{p-1} C_i^2 \geq 4p$.[29] Therefore the scattering conductivity is reduced more rapidly than that in the one-dimensional modulation given by Eq. (2.14). For the band conductivity, the asymptotic expansion of Eq. (3.24) gives

$$\Delta\sigma_2 \approx \frac{e^2}{\hbar}\frac{4l^2}{a^2} 28 g_\alpha^2 \frac{V_N^2}{\Gamma_0^2} . \quad (3.27)$$

Compared with Eq. (3.26), this term grows much more slowly, and can be ignored. Therefore the total conductivity decrease as V_N is increased, in sharp contrast to the one-dimensional case.

As V_N/Γ_0 is considerably large, γ_2 is dominated by the last term of Eq. (3.21), and can be approximated by

$$\gamma_2 \approx \frac{1}{2p^2} + \frac{1}{4p}\left[\ln\frac{8\sqrt{\pi p}\, g_\alpha V_N}{\Gamma_0}\right]^{-1} . \quad (3.28)$$

It converges rather slowly to its limit. The approximation of Eq. (3.24) gives the band conductivity

$$\Delta\sigma_2 \approx \frac{e^2}{\hbar}\frac{4l^2}{a^2} 8p g_\alpha^2 \left[\ln\frac{8\sqrt{\pi p}\, g_\alpha V_N}{\Gamma_0}\right]^{-1} \frac{V_N^2}{\Gamma_0^2} . \quad (3.29)$$

It continues to grow and becomes dominant with increasing V_N/Γ_0, though the coefficient of the quadratic term decreases logarithmically. Since $p g_\alpha^2 \sim p^{-1}$, the conductivity is smaller than that in one-dimensional modulation.[17,19]

For wide bands where V_N/Γ_0 is large, the *intersubband* coupling also contributes to the band conductivity. It is difficult to obtain an explicit form of Eq. (3.25). However, the upper bound of this contribution can be estimated, roughly, as

$$\Delta\sigma_2' \leq \frac{e^2}{\hbar}\frac{4l^2}{a^2}\sum_{i=1}^{p-1}\frac{\Gamma_0^2}{4C_i^2 V_N^2}[g_\alpha' C_i V_N]^2 \frac{4\gamma_2}{\Gamma_0^2} \approx \frac{e^2}{\hbar}\frac{4l^2}{a^2}\frac{p-1}{2p^2}[g_\alpha']^2 , \quad (3.30)$$

where the constant g_α' is of the same order as g_α. This may be overestimated, since some subbands may contribute a term proportional to $[\ln V_N]^{-1}$. In general, for large V_N/Γ_0, Eq. (3.30) is negligible.

It seems that the flat band always produces a maximum. This is certainly true for high magnetic fields. However, as the magnetic field is lowered, V_N decreases much more slowly than Γ_0. Therefore the maxima are described by Eq. (3.29) instead. The flat bands provide only some local maxima which may be smeared out by other factors. This explains the experimental results of Weiss *et al.*[20]

In summary, it is shown that the conductivity appears to be at a maximum at the flat-band condition, in contrast to the one-dimensional modulation. This results from the band splitting, which reduces the scattering conductivity faster than the band conductivity can compensate. Though this calculation is carried out for the square lattice, it does not rely on this geometry. Rather, it is valid for any *two-dimensional* lattices whenever the band is split. It is plausible that Eq. (3.7) holds for any two-dimensional lattices, though the dependence of its right side on wave numbers is different. At least, this has been shown to be true for rectangular, triangular, and hexagonal lattices.[3] The existence of critical energies and the corresponding logarithmic divergence in the density of states reflect the Van Hove singularities in two dimensions. The modulation given by Eq. (3.1) is the simplest, where only one parameter (V_N) determines the bandwidth. However, for other types of lattices, even the square lattice with some cross terms or higher-order harmonics, there may be more than one parameter which controls the bandwidth. This may change the phase factor significantly.[21] Therefore, in order to observe the effect of band splitting, there must be the corresponding one-dimensional modulation to compare with.[17,18]

IV. NUMERICAL RESULTS

It has been shown that a small change in boundary conditions can be related directly to the conductivity.[22,23] This provides a very useful way to calculate the conductivity numerically. This method was later generalized to the system in a magnetic field where the velocity operators can be replaced by derivatives of the guiding center coordinates.[24] Explicitly, for a sample of size $L\times L$ the conductivity can be written in the form

$$\sigma = \frac{1}{4}\frac{e^2}{\hbar} g(E,L) , \quad (4.1)$$

where the Thouless number $g(E,L)$ is defined as

$$g(E,L)=L^2D(E)\overline{\Delta E} \ . \tag{4.2}$$

$D(E)$ is the density of states normalized to $1/2\pi l^2$ for each Landau level. $[L^2D(E)]^{-1}$ measures the level spacing, and $\overline{\Delta E}$ is the shift of a specific energy level due to the change in boundary conditions. The localization length can be estimated from Eq. (4.2), because it decays exponentially with the sample size in the localization regime. For extended states, it is independent of the sample size for two-dimensional systems. Therefore small samples are suitable to obtaining some reasonable results.

The generalized boundary conditions can be obtained by magnetic translation operators. In particular, from the wave function Eqs. (3.4) and (3.5), we have[2]

$$\begin{aligned} u_{k_1k_2}(x+L_1,y)&=\exp(iyL_1/l^2+ik_1L_1)u_{k_1k_2}(x,y) \ , \\ u_{k_1k_2}(x,y+L_2)&=\exp(ik_2L_2)u_{k_1k_2}(x,y) \ . \end{aligned} \tag{4.3}$$

Therefore $k_1=0$ keeps the periodic boundary condition in the x direction, while $k_2=0$ and π/L_2 give periodic and antiperiodic boundary conditions, respectively, in the y direction.

Numerical calculations are made by direct diagonalization of the Hamiltonian, which is represented by a finite matrix. In the limit of strong magnetic fields, the Hamil-

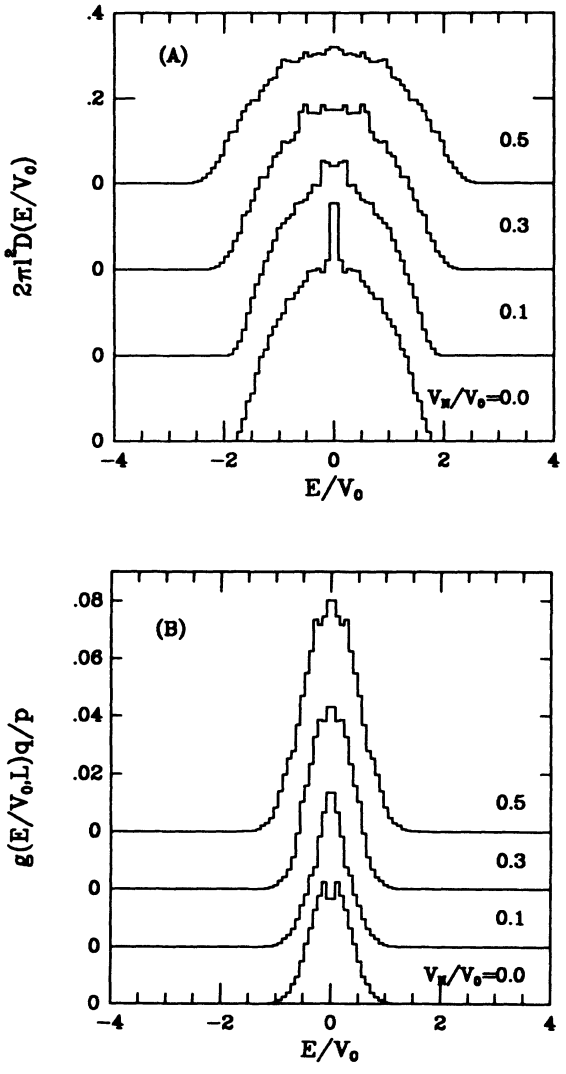

FIG. 1. The density of states and the Thouless number for one-dimensional periodic modulation are plotted in (a) and (b), respectively, as a function for energy. The sample size is $L=6a$, where a is the period of the modulation. The magnetic field is given by $B=(p/q)\phi_0/a^2=3\phi_0/a^2$, where $\phi_0=h/e$ is the flux quantum.

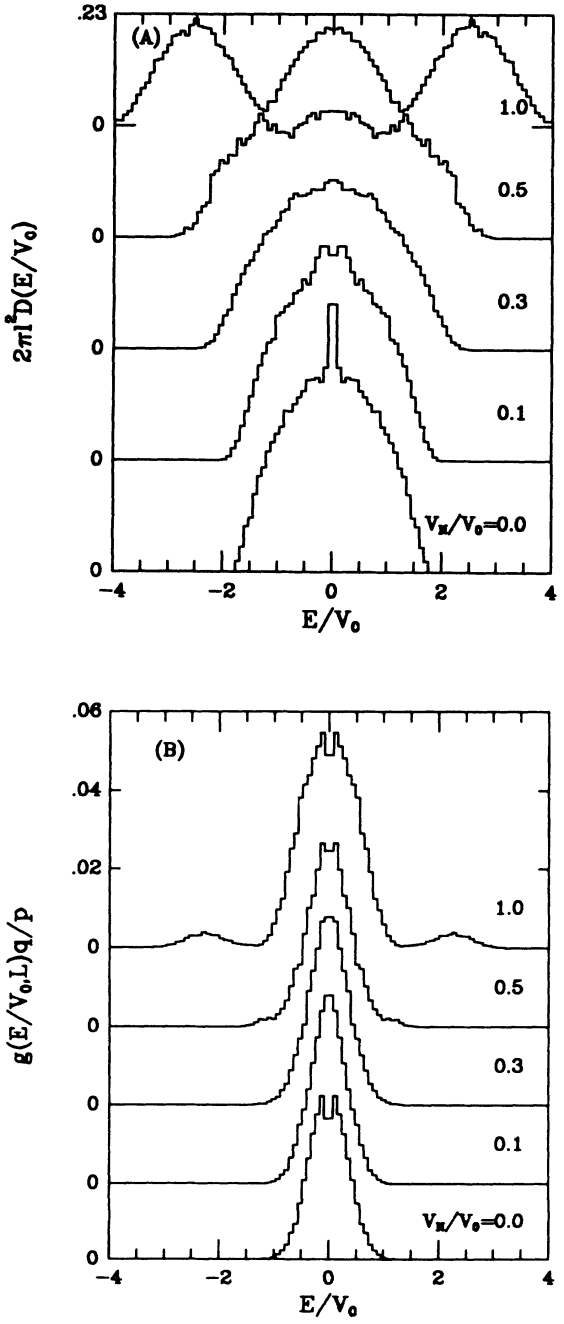

FIG. 2. The density of states and the Thouless number for two-dimensional periodic modulation are plotted against energy in (a) and (b), respectively.

tonian can be projected onto the Landau levels. Specifically, for a fixed degree of disorder, the conductivity is obtained to show how it changes with the strength of the periodic modulation, namely V_N. In this calculation, the disorder is introduced by randomly placed short-range scatterers.[24] For simplicity, scatterers of δ potential have been chosen, whose strengths are $\pm V_1$. There are equal number of positive and negative scatterers. The broadening factor has shown to be given by[25]

$$\Gamma_0^2 = 4\frac{V_1^2}{2\pi l^2}x_i \ , \tag{4.4}$$

where x_i is the concentration of the scatterer, and x_i is set to be 5 in order to be effective.

The finite size of the sample needs to be taken into account; the construction of the basis wave function can be found elsewhere.[30] Moreover, the sample average is necessary to reduce the fluctuation. Normally this is done by calculating the geometric average of the energy shift ΔE. The sample size we have calculated is $6a \times 6a$, while the magnetic field is characterized by $p/q = a^2/2\pi l^2 = 3$. $V_1 = 0.2V_0$ gives a broadening factor $\Gamma_0 \approx 1.55V_0$. Typically, 100 samples have been averaged.

Figure 1 shows the results of one-dimensional modulation for four values of V_N/V_0, namely 0, 0.1, 0.3, and 0.5. The density of states is plotted in Fig. 1(a). There seems to be a singularity at the band center when the modulation is absent. The Thouless number is shown in Fig. 1(b). The conductivity increases very rapidly as the band becomes wider. However, for two-dimensional modulation, the conductivity barely changes with V_N when the modulation is weak. The Thouless number is calculated for $V_N/V_0 = 0$, 0.1, 0.3, 0.5, and 1.0. As shown in Fig. 2(b), the peak values of the conductivity decrease very slightly as V_N/V_0 varies from 0.1 to 0.5. But these values are slightly higher than that with $V_N/V_0 = 0$, which is strongly affected by the singularity in the density of states. When the modulation is further increased, the conductivity starts to grow, though very slowly. As V_N/V_0 is as large as 1.0 when the Hofstadter spectrum is resolved, the peak is increased by only about 20%. Though a rapid decrease in conductivity is not observed as a two-dimensional modulation is introduced, the dramatic difference between one- and two-dimensional modulations certainly exists. The two-dimensional modulation gives some extra off-diagonal matrix elements in addition to the matrix of the one-dimensional modulation. From the energy spectrum point of view, this causes the band to split. The difference in conductivity is certainly attributed to the band splitting.

V. CONCLUSIONS

In this paper the magnetoresistance of two-dimensional electrons in a periodic potential is studied. In Sec. II, the general formulation of the problem is given, based mainly on the Kubo formula and the self-consistent Born approximation. This has been used to give the well-known results of Shubnikov–de Haas oscillations. In the limit of weak periodic modulations, perturbation theory is applied to modify the energy spectrum. When a weak one-dimensional periodic modulation is introduced, each Landau level is broadened into an energy band whose bandwidth is an oscillatory function of the Landau-level index. This level broadening gives rise to the band conductivity which dominates the scattering conductivity. The Weiss oscillation can be explained to result from the oscillatory bandwidth. The minima of the oscillation occur at the flat bands whose widths are zero. It is also shown that the scattering conductivity is reduced due to the level broadening. This effect is reflected in a weaker oscillation in the direction perpendicular to the periodic modulation. In this case, flat bands give the maxima of the oscillation.

In Sec. III the conductivity for a two-dimensional periodic modulation is carried out analytically. This is usually known as the Hofstadter spectrum, where the broadened Landau band is further split into subbands. It is found that the decrease of the scattering conductivity is enhanced by the band splitting, whereas the band conductivity is reduced significantly. As a result, the band conductivity grows more slowly than the decrease of the scattering conductivity as the band is broadened from the flat band. Therefore, the total conductivity shows a maximum at flat bands. This is in sharp contrast to one-dimensional modulations. There is also a discussion of the recent experimental observation of the switching in dominance between band and scattering conductivities as a function of magnetic fields.

In Sec. IV a numerical study has been made by using the Thouless number method. The conductivity is calculated as a function of periodic modulation strength for fixed disorder broadening factor. For one-dimensional modulation, the conductivity increases rather rapidly with increasing modulation strength. However, a dramatic difference is observed for two-dimensional modulation. The conductivity decreases very slowly with increasing modulation strength. As the modulation becomes sufficiently strong, the conductivity starts to grow slowly. This can be explained similarly as the result of band splitting.

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