Exact solution for the charge soliton in a one-dimensional array of small tunnel junctions

G. Y. Hu and R. F. O'Connell
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001
(Received 18 January 1994)

An exact solution for the single charge soliton, in a one-dimensional array of N gated junctions with equal junction capacitances C and equal gate capacitances C_g, is presented. Analytical expressions for the total energy, as well as the injection threshold voltage of a charge soliton in a biased array, are derived. Based on the exact solution, we analyze the effects of N and C_g/C on the charge soliton, to provide an understanding of the existing experiments.

Recently, the study of the charge soliton in a one-dimensional (1D) array of small tunnel junctions has attracted much attention.[1–9] In a single small tunnel junction, having capacitance C_J such that the charging energy $e^2/2C_J$ exceeds the characteristic energy k_BT of thermal fluctuations, it is found that Coulomb blockade, a suppression of single charge tunneling, dramatically reduces the current at voltages $V<e/2C_J$. A 1D array of small tunnel junctions consists of many such small tunnel junctions, fabricated in series, with the regions (the islands) between them being controlled by gate voltages through gate capacitances. These devices[1] have the advantage of minimizing environmental effects, i.e., each junction inside the array is effectively decoupled from the parasitic capacitance and conductance of the leads by its high-resistance neighbors. Also, it is a very useful system to study the time and space correlations between tunneling events in small tunnel junctions. As a result of these unusual properties, it is predicted that a charge soliton,[2–8] a core of electronic charge on one of the islands, could be formed inside the 1D array.

Originally, the charge soliton solution was deduced by solving the electrostatic problem for single-electron tunneling in 1D arrays. For a 1D array of N junctions, one needs to solve a set of $2N-1$ linear equations for the corresponding voltages (or equivalently, the charges) on the N junctions and $N-1$ gate capacitors. In the literature, there appear two different approaches, one of which uses the infinite chain approximation,[1–6] and the other which is a numerical approach.[7,8] Our purpose here is to provide an exact analytic approach.

In the infinite array[1–6] approach, one deals with a simple 1D array, where the junction capacitances $C_1=C_2=\cdots=C_N=C$ and the gate capacitances $C_g^{(1)}=C_g^{(2)}=\cdots C_g^{(N-1)}=C_g$. Concentrating on the potential φ_i $(i=1,2,\ldots,N-1)$ on each of the individual $N-1$ islands, and assuming an infinitely long chain so to simplify the $N-1$ electrostatic equations into one recursion relation, one obtains an analytical expression for $\{\varphi_i\}$. Explicitly, the potential of an arbitrary island j as a function of the "distance" j-k from the kth island, in the case where there is an excess electron on the kth island, takes the form[1–6]

$$\varphi_j^\infty=-\frac{e}{C_{\text{eff}}}e^{-\lambda|j-k|}\,, \tag{1a}$$

where the symbol ∞ emphasizes that it is a result for an infinite array, and

$$\lambda=\ln\left[\frac{C_{\text{eff}}+C_g}{C_{\text{eff}}-C_g}\right]\,,\quad C_{\text{eff}}=\sqrt{C_g^2+4CC_g}\ . \tag{1b}$$

When Eqs. (1a) and (1b) are applied to a finite array,[1–6] it is assumed that near the edge of the array the charge soliton will induce an image (the antisoliton). Such an interpretation extends (1a) into an expression of the potential[2]

$$\varphi_j^\infty(k,\lambda)=\frac{-e}{C_{\text{eff}}}(e^{-\lambda|j-k|}-e^{-\lambda|j+k|})\ . \tag{1c}$$

We note that (1) is known to be correct in the $N\lambda\gg1$ limit,[1–6] but it is not clear what kind of error it will produce when applied to the case of a finite array not satisfying the condition $N\lambda\gg1$. In the other approach,[7,8] one solves the $2N-1$ linear equations numerically without any presumptions. In both approaches, the charge soliton profile for the electrostatic potential $\{\varphi_i\}$ was identified. Nevertheless, the validity of (1) is not easily checked directly by a detailed study of the numerical solution of the $2N-1$ linear equations. In fact, (1) has been widely used as the foundation for understanding the electrostatic problem for single-electron tunneling in 1D arrays of junctions without discussion of its validity.

The work presented in this paper demonstrates that the electrostatic problem for single-electron tunneling in 1D arrays of N junctions can be solved exactly and analytically. This enables us to identify the range of validity for the simple soliton solution (1).

Consider a 1D array of N small junctions, with capacitances $C_1,C_2,\ldots,C_N$, and tunnel resistances $R_1,R_2,\ldots,R_N$, biased with a voltage V. The islands (total number $N-1$) between N junctions are connected through capacitors $C_g^{(1)},C_g^{(2)},\ldots,C_g^{(N-1)}$, biased with gate voltages $U_1,U_2,\ldots,U_{N-1}$. We assume that $R_i\gg h/e^2$, which ensures that the wave function of an excess electron on an island is localized there.

We adopt the semiclassical model[1–9] to describe the 1D array. In this model the voltage V_j across the jth junction (or capacitor) is a classical variable calculated by $V_j=C_jQ_j$, where Q_j is the charge on the jth junction. Existing approaches to the problem develop a set of $N-1$ linear equations for the voltages $\{V_i\}$. The key to our approach is to rewrite these equations as equations for the island potentials $\{\varphi_i\}$; this enables us to obtain an exact analytic result. Thus we describe the state of the system at a given time by a set of $2N-1$ variables $\{V,\varphi_i,\ldots,\varphi_{N-1},n_1,n_2,\ldots,n_{N-1}\}$, where n_j is the number of excess electrons on the jth island. These variables obey $2N-1$ linear equations resulting from the charge conservation law and Kirchhoff's laws.[7-9] For the simple case, where $C_1=C_2=\cdots=C_N=C$ and $C_g^{(1)}=C_g^{(2)}=\cdots=C_g^{(N-1)}=C_g$, and where one is not interested in the relationship between the gate potential φ and gate voltages $\{U_i\}$, the latter become hidden variables, and the $2N-1$ equations can simply be reduced to $N-1$ linear equations in the following form:

$$-(C_g+2C)\varphi_1+C\varphi_2=n_1e-CV , \tag{2a}$$

$$C\varphi_{i-1}-(C_g+2C)\varphi_i+C\varphi_{i+1}=n_ie \quad (i=2,3,\ldots,N-2) , \tag{2b}$$

$$C\varphi_{N-2}-(C_g+2C)\varphi_{N-1}=n_{N-1}e . \tag{2c}$$

We note that in the infinite array approach,[1-6] one neglects the boundary equations (2a) and (2c) and only (2b) is retained. On the other hand, in the numerical approach, Ben-Jacob, Mullen and Amman[7] and Ammen, Ben-Jacob, and Mullen[8] studied the linear equations for the $\{V_i\}$ instead of for the $\{\varphi_i\}$.

To facilitate our discussion, (2) is written in its matrix form

$$\overline{\overline{M}}\,\overline{\varphi}=\frac{e}{C}\overline{n} , \tag{3}$$

where we use a double bar for the matrix, and a single bar for the column. Explicitly, (3) stands for

$$\begin{bmatrix} D & 1 & 0 & & 0 & 0 & 0 \\ 1 & D & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & D & & 0 & 0 & 0 \\ & \cdots & & \vdots & & \cdots & \\ 0 & 0 & 0 & & D & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & D & 1 \\ 0 & 0 & 0 & & 0 & 1 & D \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{N-3} \\ \varphi_{N-2} \\ \varphi_{N-1} \end{bmatrix} = \begin{bmatrix} n_1e/C-V \\ n_2e/C \\ n_3e/C \\ \vdots \\ n_{N-3}e/C \\ n_{N-2}e/C \\ n_{N-1}e/C \end{bmatrix} , \tag{4}$$

where $D=-2-C_g/C$.

We find that (4) can be solved analytically, and the result is

$$\overline{\varphi}=\overline{\overline{M}}^{\,-1}\overline{n}e/C\equiv-\overline{\overline{R}}\,\overline{n}e/C , \tag{5}$$

where the element of the symmetric matrix $\overline{\overline{R}}$ is given by

$$R_{ij}=\frac{\cosh(N-|j-i|)\lambda-\cosh(N-i-j)\lambda}{2\sinh\lambda\sinh N\lambda} , \tag{6}$$

where λ is given by (1b).

Equation (5), supplemented by (6), is a key result of this paper. Once a charge profile $\{n_i\}$ is known, we can use (5) to determine the potential profile $\{\varphi_i\}$. In the following, we analyze the *single charge soliton* case, where there is no charge on any of the islands except that a single charge appears on the kth island, i.e., $n_i=\delta_{ik}$. In this case for a 1D array of N junctions with bias voltage V, (5) reduces to a simple form for the potential $\varphi_j^N(k,V)$ of an arbitrary electrode j as a function of the distance from the kth electrode,

$$\varphi_j^N(k,V)=-\frac{e}{C}R_{jk}-VR_{j1} , \tag{7}$$

where R_{jk} is given by (6). Equation (7) is an *exact* solution for a single charge soliton in a biased 1D array of N small junctions with equal junction capacitances C and equal gate capacitances C_g. Some comments are in order in the following.

First, using (6) we rewrite the $V=0$ case of (7), in a form suitable for comparison with the existing theory underlying (1), as

$$\begin{aligned}\varphi_j^N(k)&=-\frac{e}{C_{\text{eff}}}\frac{\cosh(N-|j-k|)\lambda-\cosh(N-j-k)\lambda}{\sinh N\lambda}\\ &=\varphi_j^\infty(k,\lambda)+\frac{e^{-2N\lambda}}{1-e^{-2N\lambda}}[\varphi_j^\infty(k,\lambda)+\varphi_j^\infty(k,-\lambda)] ,\end{aligned} \tag{8}$$

with C_{eff} and $\varphi_j^\infty(k)$ given by (1b) and (1c), respectively. For an array satisfying $N\lambda\gg1$, it is a good approximation to neglect the second term on the right-hand side of (8), and it reduces to the *infinite array* result (1c), i.e., $\varphi_j^N(k)$ does not depend on the number N of junctions in the array. Also, the result given by (8) is consistent with the results of Ref. 2(b) but it is also in a form which is easier to use. Second, for a 1D *finite array* which does not satisfy $N\lambda\gg1$, $\varphi_j^N(k)$ can no longer be approximated by the form of (1c). In fact, a direct comparison of (8) and (1c) in the $C_g\ll C$ limit shows that the smaller the value of C_g/C, the larger the differences between the two forms. Third, (8) represents a potential profile for a charge soliton with the peak value ($j=k$) as

$$\varphi_k^{(N)}(k) = -\frac{e}{C_{\rm eff}} \frac{\cosh N\lambda - \cosh(N-2k)\lambda}{\sinh N\lambda} \ . \tag{9}$$

From (9), it is straightforward to observe that $\varphi_k^N(k) = \varphi_{N-k}^N(N-k)$, and $\varphi_k^N(k)$ increases with increasing k and reaches a maximum value $-(e/C_{\rm eff})\tanh(N\lambda/2)$ at $k=N/2$ for N even; for N odd, there are two equivalent maximum values $-(e/C_{\rm eff})\{(\cosh N\lambda - \cosh\lambda)/\sinh N\lambda\}$ at $k=(N\pm 1)/2$. The width of the charge soliton, defined by the $j-k$ value at which (8) is reduced to the half of (9), can be directly evaluated from (8) and (9) as

$$\Delta_k = N - \frac{1}{\lambda}\{\sinh^{-1}[\tfrac{1}{2}\sinh(N-k)\lambda] + \sinh^{-1}[\tfrac{1}{2}\sinh k\lambda]\} \ . \tag{10}$$

From (10), it is straightforward to show that $\Delta_k = \Delta_{N-k}$, $\Delta_k \to 2\ln 2/\lambda$ for $N\lambda \gg 1$, and $\Delta_k \to 0$ for $\lambda \to 0$. As an example, in Fig. 1 we illustrate the peak potential (9) and the peak width (10), as a function of N for a charge soliton in a 1D array with $k=1$ and $C_g/C = 0.001$, 0.01, and 0.1. The figure shows that for large values of N (the specific number depends on the value of C_g/C), the peak potential and the peak width have weak dependence on N, and the infinite junction approximation is good. In general, the smaller value of C_g/C, the larger the dependence of the peak potential and of the width on N.

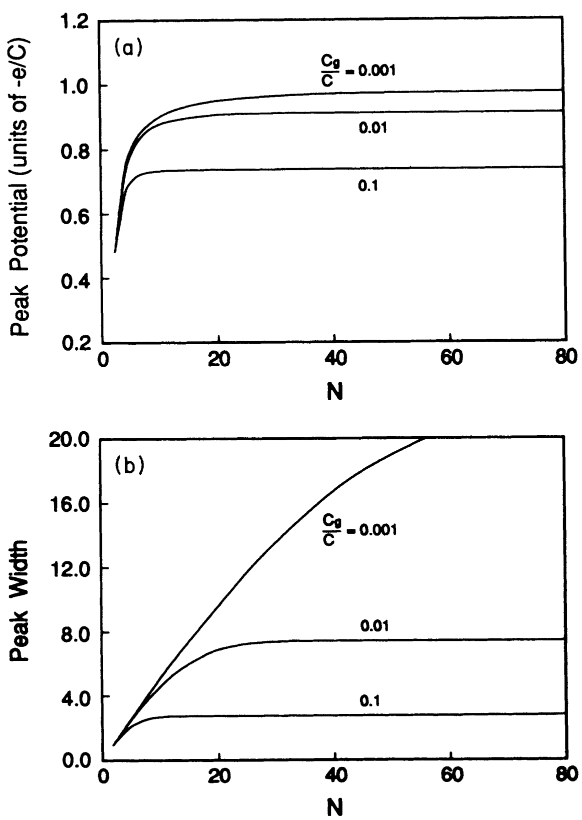

FIG. 1. The characteristics of a single charge soliton in a 1D array of N small junctions, with the extra charge on the first island; (a) peak potential ($\varphi_j^{(k)}$ at $j=k$, in units of e/C); and (b) peak width, as a function of N. From top to bottom, $C_g/C = 0.001$, 0.01, and 0.01, where C is the junction capacitance and C_g is the gate capacitance.

We are now in a position to evaluate the free energy of the biased 1D array with a charge soliton, by means of the exact solution (5). The free energy of the biased 1D array with a charge soliton at the kth electrode takes the form

$$F(k) = \frac{C_g}{2}\sum_{j=1}^{N-1}[\varphi_j^N(k,V)]^2 + \frac{C}{2}\sum_{j=1}^{N}[\varphi_j^N(k,V) - \varphi_{j-1}^N(k,V)]^2 - eV \ , \tag{11}$$

where the first term on the right-hand side is the total charging energy for the gate capacitors, the second term is the total charging energy for the junctions, and the last term is the work done by the bias voltage in transferring an electron. Using (8), after some lengthy algebra, we obtain from (11)

$$F(k) = \frac{e^2}{C_{\rm eff}}\frac{\sinh(N-k)\lambda \sinh k\lambda}{\sinh N\lambda} - eV\left[1 - \frac{\sinh(N-k)\lambda}{\sinh N\lambda}\right] + \frac{CV^2}{2}\left[1 - \frac{\sinh(N-1)\lambda}{\sinh N\lambda}\right] \ . \tag{12}$$

One can directly observe from (12) that $F(k) = F(N-k)$ for $V=0$, and this symmetric property disappears once $V \neq 0$.

The injection of a charge soliton from the voltage source to the kth island of the array is energy favorable when $F(k) - F(0)$ is less than zero, and vice versa. Thus, the threshold energy V_t for the injection of a charge soliton onto the kth island can be obtained by equating $F(k) - F(0) = 0$, with the result

$$V_t = \frac{e}{2C_{\rm eff}} \frac{\sinh(N-3k/2)\lambda + \sinh(N-k/2)\lambda}{\cosh(N-k/2)\lambda} \ . \tag{13}$$

Equation (13) is an interesting result. By using (13) one immediately finds that $dV_t/dN > 0$ and $dV_t/dk < 0$. This implies two facts: (1) for fixed values of C and C_g, an array of larger N will generally have larger V_t; and (2) once a charge soliton is injected into the array, it will have no difficulty in traveling through (with increasing k) the array. Furthermore, in the $N\lambda \gg 1$ limit, (13) reduces to

$$V_t(N\lambda \gg 1) = \frac{e}{2C_{\rm eff}}(1 + e^{-k\lambda}) \ , \tag{14}$$

which is previously known in the literature[1–6] for the $k=1$ case. Apart from the fact that our expression (14) is more general, we also know that (14) is an upper limit to the value of V_t, since we have shown already that $dV_t/dN > 0$. In Fig. 2, we plot V_t vs N for $k=1$ case for several arrays with $C_g/C = 0.001$, 0.01, and 0.1.

Experimentally, data for the threshold voltage V_t for 14 different arrays with N ranging from 15 to 53, are

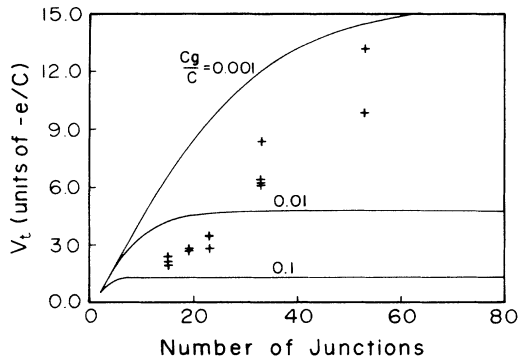

FIG. 2. The theoretical values of the threshold voltage V_t (in units of $-e/C$) for injecting a charge soliton into the first island of a 1D array of N small junctions, as a function of N. From top to bottom, $C_g/C=0.001$, 0.01, and 0.1; C is the junction capacitance; and C_g is the gate capacitance. Symbols are experimental data taken from Table I of Ref. 6.

presented in Table I of Ref. 6. The data show that in general the arrays with larger numbers of N tend to have larger values of V_t. This behavior is consistent qualitatively with what we have identified from our exact result (13) as discussed above. Obviously, at this stage it is not possible to make a quantitative comparison between our theoretical result (13) and the experiments, since the exact values of C_g of the samples are unknown and they may have some kind of inhomogeneities.[6] On the other hand, the present theory may provide a useful estimate for the values of C_g if one assumes that the samples are ideal 1D array of junctions with equal C and C_g. For this purpose, in Fig. 2 we plot the data of Ref. 6 by dark dots. The figure shows that to fit the ideal model of 1D array junctions having equal values for C and C_g, for those junctions having small values of N (from 15 to 23), the ratio of C_g/C should be in the range of 0.01–0.1, while for large N (from 33 to 53) arrays the ratio should be in the range 0.001–0.01. In other words, if the ratio C_g/C falls in these regions, then the experimental data of Ref. 6 can directly be understood by the present theory.

In summary, in this paper we have presented an exact solution (5) for the potential profiles of a biased 1D array of N gated junctions with equal junction capacitance and equal gate capacitance. Based on (5), an analytical expression (7) for a single charge soliton in a biased 1D array is derived. In addition, we have analyzed the peak potential (9), the peak width (10), and the threshold voltage (13) for the single charge soliton as a function of the number of junctions N in the array and the capacitance ratio C_g/C. It is also shown that the commonly used expression (14) for the threshold voltage is an upper limit. The qualitative behavior that, in general, the arrays with larger number of N tend to have larger value of V_t shown in the experimental data of Ref. 6, is consistent with what we have identified from our exact result (13).

This work was supported in part by the U.S. Office of Naval Research under Grant No. N00014-90-J-1124.

[1]P. Delsing, in *Single Charge Tunneling, NATO Advanced Study Institute Series B: Physics*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1991), p. 249.

[2](a) N. S. Bakhvalov, G. S. Kazacha, K. K. Likharev, and S. I. Serdyukova, Zh. Eksp. Teor. Fiz. **95**, 1010 (1989) [Sov. Phys. JETP **68**, 581 (1989)]; (b) Physica B **173**, 319 (1991).

[3]K. K. Likharev, N. S. Bakhvalov, G. S. Kazacha, and S. I. Serdyukova, IEEE Trans. Magn. **25**, 1436 (1989).

[4]L. S. Kuzmin, P. Delsing, T. Claeson, and K. K. Likharev, Phys. Rev. Lett. **62**, 2539 (1989).

[5]P. Delsing, K. K. Likharev, L. S. Kuzmin, and T. Claeson, Phys. Rev. Lett. **63**, 1180 (1989).

[6]P. Delsing, T. Claeson, K. K. Likharev, and L. S. Kuzmin, Phys. Rev. B **42**, 7439 (1990).

[7]E. Ben-Jacob, K. Mullen, and M. Amman, Phys. Lett. A **135**, 390 (1989).

[8]M. Amman, E. Ben-Jacob, and K. Mullen, Phys. Lett. A **142**, 431 (1989).

[9]G. Y. Hu and R. F. O'Connell, J. Phys. Condens. Matter **5**, 7259 (1993).