Strong correlations versus U-center pairing and fractional Aharonov-Bohm effect

F.V. Kusmartsev*
*Department of Theoretical Physics, University of Oulu, SF-90570 Oulu, Finland
and Institute for Solid State Physics, University of Tokyo, Roppongi, Minato-Ku, Tokyo 106, Japan*

J.F. Weisz† and R. Kishore
Instituto Nacional de Pesquisas Espaciaes, Caixa Postal 515, 12201-970, São José dos Campos, São Paulo, Brazil

Minoru Takahashi
Institute for Solid State Physics, University of Tokyo, Roppongi, Minato-ku, Tokyo 106, Japan
(Received 7 July 1993; revised manuscript received 28 December 1993)

The influence of the interaction between electrons on the Aharonov-Bohm effect is investigated in the framework of the Hubbard model. The repulsion between electrons associated with strong correlation is compared with the case of attraction such as U-center pairing. We apply the Bethe-ansatz method and exact numerical diagonalization to the Hubbard Hamiltonian. It is shown that the quasi-half-quantum-flux periodicity occurs for any nonzero values of U for two electrons. For a large number of sites, or strong U, the quasiperiodicity becomes an exact half-quantum-flux periodicity. The character of the state created on the ring is different in both cases. In U-center pairing the electrons are bound in pairs located on the same site. For strong correlations (large positive U) the electrons tend to be as far from each other as possible. We show by numerical solution of the Bethe-ansatz equations that for three electrons the flux periodicity of the ground-state energy is equal to $\frac{1}{3}$. The one-third periodicity may occur even for small values of the ratio U/t, for the very dilute system. It is shown analytically using the Bethe-ansatz equations for N electrons that for dilute systems with arbitrary value of the Hubbard repulsion U, the fractional Aharonov-Bohm effect occurs with period $f_T = 1/N$ in units of the elementary flux quantum. Such a period occurs when the value Nt/LU is small, where L is the number of sites. Parity effects disappear in this fractional regime. We discuss possible experiments to detect fractional Aharonov-Bohm effects.

I. INTRODUCTION

The importance of strong correlations in condensed matter physics became obvious after the discovery of the quantum Hall effect and high temperature superconductivity.[1] In fact the correlations are created by Coulomb forces which are especially important in low-dimensional systems.[2–8] In spite of rapid progress in the understanding of strongly-correlated states such as the fractional quantum Hall effect,[2] or the one-dimensional Luttinger liquid,[3] the microscopic detail of these correlations still needs to be developed further.[7,8]

In order to shed light on their nature we investigate using the Bethe ansatz as well as by direct diagonalization an exactly solvable Hubbard model describing a finite number of electrons in the magnetic field located on a ring. The filling factor is kept as a free parameter by varying the number of sites. Lieb recently pointed out that for small chains, the general "Bethe-ansatz" solution, while correct, is too complicated for numerical calculations.[9] Examining this problem, we first investigate the simplest case, when two electrons are located on the ring. In this case the solution may be given analytically for certain cases. It was Landauer[10] (see also Refs. 11–13) who recognized that the flux quantization and the persistent current may survive on the small nonsuperconducting ring, reflecting quantum mechanical properties of the charge carriers. Therefore, the Aharonov-Bohm effect may be a good tool to study the character of the electron states and electron correlations.

We use the Aharonov-Bohm effect as a tool to study the pairing of the correlated state and compare it with the U-center pairing. In the limit of large repulsion, when $U/t \to \infty$, Kusmartsev[14] found that the Aharonov-Bohm effect might be fractional. The Aharonov-Bohm period is changed from the conventional one to $1/N$, where the N is the number of electrons on the ring. Kusmartsev's result has since been confirmed in other investigations.[15–17] In the present work we show that the fractional $1/N$ Aharonov-Bohm effect may occur for an arbitrary N and for an arbitrary (even very small) ratio U/t for the very dilute system, when the filling factor drops to zero.

The numerical calculations for a given number of sites are based on the exact diagonalization of the two electron Hubbard Hamiltonian. With this method we can then analyze all the finite size effects, which may give many instances of both level crossings[18] and permanent degeneracies (as a function of U), as have been found by Heilmann and Lieb[19] in their studies of the energy levels of the benzene molecule having six sites and six electrons.

We obtain the same results from this direct diagonalization as from the Bethe-ansatz method.[20–29] This contrasts with solutions of Bethe equations obtained in the thermodynamic limit[29,25] or calculations by Woynarovich and Eckle,[24] who evaluated the asymptotics of finite size effects on the ground-state energy. In fact, finite size objects such as rings may have complicated *non-Abelian* dynamical symmetry groups which were not accounted for on the basis of the known invariance groups (spin, pseudospin, and rotational symmetries of the ring).[9] In fact we found that finite scaling finite size effects were very important. There is a scaling behavior of the ground-state energy which does not depend on size L or on U, but depends only on UL/N =const= α^{-1}, where U is measured in units of t and L is measured in units of the lattice constant. Such scaling occurs only at small values of α. Due to this scaling symmetry the fractional Aharonov-Bohm effect may arise even for small values of U, for very dilute electron systems.

Our results agree with studies by Ferretti, Kulik, and Lami[30] of the Hubbard ring with 10 or 16 sites and with 2 or 4 fermions that the quasi-half-quantum-flux periodicity appears both for the positive and the negative U values. However, we stress that the pairing of two electrons for the positive U case is different from U-center pairing. If the number of electrons is more than 2, then the fractional $1/N$ Aharonov-Bohm effect may appear, which is not related to a pairing, but rather to the bound state of N particles. Such a state is created via a competition of the kinetic and the potential energy of the particles. As a result of an interaction with the phonons the effective mass of these particles increases. Therefore, in comparison with the potential energy their kinetic energy decreases and the state may be destroyed; this was observed in Ref. 30. Note that the discussed N-particle bound state is distinct from the charge transfer pairing created on the ring described by an extended Hubbard model, where the quasi-half-quantum-flux periodicity was observed.[31] In all these works the twisted boundary conditions have been used. The twist in the boundary conditions has been applied also at the calculation of a charge stiffness and of the finite size effects on the optical conductivity.[32–34]

Note that there was another proposal of the fractional $1/N$ Aharonov-Bohm periodicity for a system of N strongly correlated parallel charge density wave (CDW) chains.[35,36] However, the physics of the fractional $1/N$ Aharonov-Bohm periodicity obtained in the present paper is distinct from $1/N$ Aharonov-Bohm periodicity obtained for the system of N strongly correlated parallel CDW chains.

The paper is organized as follows. In Sec. II we discuss the model and introduce Bethe equations for the Hubbard ring in a magnetic field. In Secs. III–V we study the ring with the repulsive Hubbard U. In Sec. III we solve analytically and by direct diagonalization (numerically) this model for the case of two electrons and make an analysis of the hidden symmetry. In Sec. IV we introduce the method for the numerical solution of Bethe equations and solve these equations for the cases of three and four electrons at different ring sizes L and values U. The investigation made in these two sections use a small parameter $\alpha = Nt/UL \ll 1$, where the behavior of the Hubbard model in a magnetic field is universal. Using this parameter in Sec. V we find an analytical solution of Bethe equations for an arbitrary number of electrons N and arbitrary sizes of the ring. In Sec. VI we discuss how the correlations destroy the parity effect. In Sec. VII we discuss the Aharonov-Bohm effect for the model with negative values U and compare the case of the repulsive positive value U. In Sec. VIII we discuss the physical interpretation of the predicted fractional Aharonov-Bohm effect as well as real experimental situations where this effect may be observed. Finally some conclusions are given in Sec. IX.

II. MODEL AND BETHE EQUATIONS

To describe the fractional Aharonov-Bohm effect foı dilute systems we study the Hubbard Hamiltonian

$$H = -t \sum_{\langle i,j\rangle,\sigma} a^{\dagger}_{i\sigma} a_{j\sigma} + U \sum_{i=1}^{L} n_{i+} n_{i-} \tag{1}$$

involving as parameters the electron hopping integral t, the on-site repulsive Coulomb potential U, and the number of sites L. The operator $a^{\dagger}_{i\sigma}(a_{i\sigma})$ creates (destroys) an electron with spin projection σ ($\sigma = +$ or $-$) at a ring site i, and $n_{i\sigma}$ is the occupation number operator $a^{\dagger}_{i\sigma}a_{i\sigma}$. The summations in Eq. (1) extend over the ring sites i or, as indicated by $\langle i,j\rangle,\sigma$, over all distinct pairs of nearest-neighbor sites, along the ring with the spin projection σ.

The effect of the transverse magnetic field is included via twisted boundary conditions, with the Bethe ansatz substitution for the wave function. It is important to note that the Zeeman energy term

$$U = -\mu_B H \sum_{i=1}^{L} (n_{i+} - n_{i-}), \tag{2}$$

where H is the magnetic field associated with the flux quantum threading the loop, cannot be simply dropped, and in some cases may give very interesting new physics. However, we will first study the ring neglecting the Zeeman term. Using an analogy between the low and high density limits we will show the implicit importance of that term.

For example, as shown below, the fractional Aharonov-Bohm periodicity may appear even for the model of negative U centers, provided that in that system the Zeeman energy is larger than the orbital energy (the energy of a persistent current). The Zeeman energy is proportional to $\sim \nu H$, where ν is the filling factor. The orbital energy is determined by the condition $\sim \nu H^2 R^3$. The region of fields where the intersection of levels occurs is determined when the flux $f \sim HR^2 \sim 1/N$; we assume *a priori* $1/N$ flux quantum flux periodicity. The Zeeman energy is smaller than the orbital energy, if $R \gg N$, where R is the radius of the ring.

For a large enough system, when $R \gg N$, the Zeeman energy term may be neglected. For this rough estimation we used atomic units. At first, we consider the ring when its radius is much larger in comparison with interatomic distance.

For the case of the magnetic field we will use the same form of the wave function as has been proposed in Refs. 29 and 26

$$\psi(x_1,\ldots,x_N)=\sum_P[Q,P]\exp\left[i\sum_{j=1}^{N}k_{Pj}x_{Qj}\right], \quad (3)$$

where $P=(P_1,\ldots,P_N)$ and $Q=(Q_1,\ldots,Q_N)$ are two permutations of $(1,2,\ldots,N)$ and N is the number of electrons.

The coefficients $[Q,P]$ as well as $(k_1,\ldots k_N)$ are determined from the Bethe equations which in a magnetic field are changed by the addition of the flux phase $2\pi f$,[26,34,14]

$$e^{i(k_jL-2\pi f)}=\prod_{\beta=1}^{M}\left(\frac{it\sin k_j-i\lambda_\beta-U/4}{it\sin k_j-i\lambda_\beta+U/4}\right) \quad (4)$$

and

$$-\prod_{j=1}^{N}\left(\frac{it\sin k_j-i\lambda_\alpha-U/4}{it\sin k_j-i\lambda_\alpha+U/4}\right)=\prod_{\beta=1}^{M}\left(\frac{i\lambda_\alpha-i\lambda_\beta+U/2}{i\lambda_\alpha-i\lambda_\beta-U/2}\right). \quad (5)$$

f is the flux in units of elementary quantum flux ϕ_0. The explicit form of Bethe equations in a magnetic field is[29,32,14,15,17]

$$Lk_j=2\pi I_j+2\pi f-\sum_{\beta=1}^{M}\theta(4(t\sin k_j-\lambda_\beta)/U), \quad (6)$$

$$-\sum_{j=1}^{N}\theta(4(t\sin k_j-\lambda_\beta)/U)$$

$$=2\pi J_\beta+\sum_{\lambda_\alpha=1}^{M}\theta(2(\lambda_\beta-\lambda_\alpha)/U), \quad (7)$$

where $\theta(x)=2\arctan(x)$ and the quantum numbers I_j and J_β, which are associated with charge and spin degrees of freedom, respectively, are either integers or half odd integers, depending on the parities of the numbers of down- and up-spin electrons, respectively:

$$I_j=\frac{M}{2}\ (\text{mod }1),\quad J_\beta=\frac{N-M+1}{2}\ (\text{mod }1). \quad (8)$$

The actual values (sets) of these numbers must be chosen to minimize the total energy for the given value of the flux f.

III. HALF-QUANTUM-FLUX PERIODICITY FOR TWO ELECTRONS

A. Two sites

In the case of interest, for two electrons on the ring the system of equations (6 and 7) simplify to two decoupled equations

$$Lx=2\arctan\frac{\epsilon}{\sin x}+\pi n \quad (9)$$

and

$$Ly=2\pi f+\pi m, \quad (10)$$

where $x=(k_1-k_2)/2$, $y=(k_1+k_2)/2$, and $\epsilon=U/\{4t\cos[(2\pi f+\pi m)/L]\}$. The numbers n,m may have both positive and negative values.

When $L=2$ ($U>0$), Eq. (9) may be solved immediately and the solution is valid for the range of flux $|f|<1/2$, where we must set $m=n=0$. For other values of the flux the solution must be periodically continued. The result is

$$k_{1,2}=\pm\text{arccos}(-\epsilon/2+\sqrt{\epsilon^2/4+1})+\pi\text{f}, \quad (11)$$

which shows that with the increase of U there is an increase of the difference between the k vectors of the first and second electron. The increase of this difference improves the quasi-half-quantum-flux periodicity of the Aharonov-Bohm effect. The ground-state energy (persistent current) is described by simple expression

$$E_{\text{ground-2}}=-2t(-\epsilon/2+\sqrt{\epsilon^2/4+1})\cos(\pi f), \quad (12)$$

$$j_2=-\partial E_{\text{ground-2}}/\partial f=-\frac{8\,\pi t^2\,\sin(2\,f\,\pi)}{\sqrt{[U^2+64\,t^2\,\cos(f\,\pi)]}}, \quad (13)$$

where f is limited to the region $|f|\le 1/2$. The interaction changes the value of the persistent current. At the fixed value of the flux f the current monotonically decreases as U increases. The interactions change the current-flux dependence in a way that is similar to that of disorder or temperature, as shown in our recent paper.[37,38] That is, the interactions cause the jumps in current-flux dependence, or the cuspoidal points in the energy-flux curves, to disappear. With stronger interactions these curves become gradually smoother. The energy dependence is a single-flux periodical function at any value of U. The reason for the single flux periodicity is that the two site ring with two electrons is a very special case in that it is half full with two electrons. In the limit $U\to\infty$ the current and the energy vanish, which coincides with the result obtained in Ref. 14. That is, for the half-filled cases in that limit, the persistent current equals zero.

B. Hidden symmetry

However, this single flux periodicity disappears when we go away from half-filling and instead immediately obtain quasi-half-flux periodicity. For example, for the Aharonov-Bohm effect on a ring having four sites (1/4-filling) the explicit formulas may also be written. In this case Eq. (5) simplifies to the cubic equation

$$z^3+\epsilon z^2-z-\epsilon/2=0, \quad (14)$$

where $z=\cos x$. With the aid of the Cardano formula the solution can given explicitly. In the limit of large values of ϵ it has the form

$$k_{1(2)} = +(-)\arccos\left(\frac{1}{\sqrt{2}} + \frac{1}{4\epsilon}\right) + \frac{(2\pi f + \pi m)}{4} \quad \text{when } \epsilon \gg 1, \tag{15}$$

with the flux energy dependence

$$E_{\text{ground-4}} = -4\cos\left(\frac{\pi f}{2}\right)\left(\frac{1}{\sqrt{2}} + \frac{1}{4\epsilon}\right). \tag{16}$$

A nontrivial fact here is that there is another solution, associated with singular values of λ_α, which for an arbitrary value of L has the form

$$E_{\text{ground-}L} = -4\cos\left(\frac{2\pi(f-\frac{1}{2})}{L}\right)\cos\left(\frac{\pi}{L}\right). \tag{17}$$

This formula coincides with that describing the energy-flux dependence of two noninteracting spinless fermions on a ring with L sites. For the problem under consideration this dependence also corresponds to a triplet state. This means that on the two sites ring there is a region of flux values near half-odd integer numbers in which a very surprising degeneracy between triplet and singlet states occurs. This means that at this value of flux the matrix element for interaction vanishes for the singlet state. For the singlet state there already occurs a trapping of the flux quantum, which is shared between two electrons. The trapping of the quantum flux occurs at each cuspoidal point of the energy-flux curve. On the other hand, the trapping does not occur for the triplet state. This degeneracy, reflecting some hidden symmetry, may be schematically expressed with the aid of the following formula: *singlet + flux quantum = triplet.*

C. Scaling symmetry

For the two sites ring discussed above, this solution (17) exactly equals zero, giving a single flux periodicity in the energy-flux dependence for the half-filled ring. Thus, for a non-half-filled ring the ground-state energy-flux dependence is determined by two solutions (16) and (17), which give the needed quasi-half periodicity.

From Eqs. (15) one sees that the phase increases with $U \to \infty$ from zero to $\pi/4$. The general result, which is valid for any value of L, is as follows. With the increase of U the difference between k_1 and k_2 increases, and the solution is given by the following formula:

$$k_{1,2} = (\pm\theta + 2\pi f + \pi n)/L, \tag{18}$$

where the function θ depends on ϵ and increases with U and L. For the case $L = 2$ the explicit dependence $\theta(U, f)$ is given by Eq. (11).

The ground-state energy is determined from the formula

$$E = -2t\sum_{j=1}^{N}\cos k_j, \tag{19}$$

where the Beyers-Yang theorem[27,28] (see also Refs. 26 and 18) is used to remove flux.

From Eqs. (9) and (10) one may conclude that for the ground-state energy the momenta k_1 and k_2 must be single-quantum-flux periodical functions. Because of the difference between k_1 and k_2, due to the function θ, the ground-state energy E as a function of flux becomes a quasi-half-flux periodic function.

With the increase of U or L ($L > 2$) the shift θ increases and consequently the half-quantum-flux periodicity improves. This effect exists in two cases. Good half-flux quantum periodicity appears for a small ring with a large U or for a large ring with small U. Numerical results indicating these effects are presented in Figs. 1 and 2.

For the electrons on the Hubbard ring the repulsive potential U causes the particles to locate on opposite sides of the ring. Because of finite size effects (or alternatively the kinetic energy of electrons) this localization is not complete. The criterion for strong U to have good quasi-half-quantum-flux periodicity is then that U is much larger than the kinetic energy. This is why the half-flux periodicity improves both with larger L and larger U.

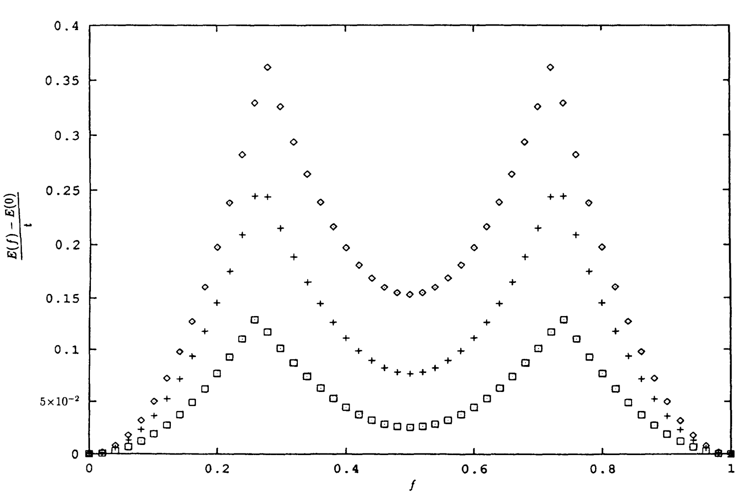

FIG. 1. The ground-state energy dependence on the flux of the external field for the Hubbard ring at the fixed value of U at the different values of L. Here $U = 50t$ and L is three, four, or six sites. ◇ is for $L = 3$, + is for $L = 4$, and ⊡ is for $L = 6$. The ground-state energy at zero flux is subtracted off so as to normalize the figures. Energies are in units of t and the flux is in units of quantum flux.

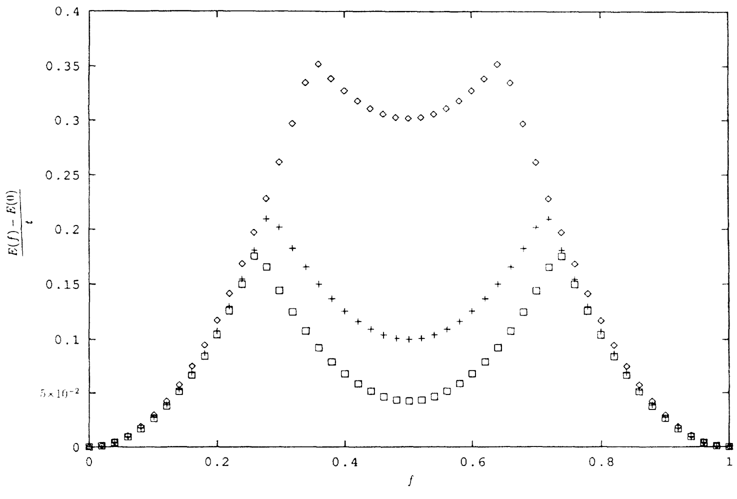

FIG. 2. The ground-state energy versus external field flux for the Hubbard ring at the fixed value of L and different values of U. Here L is five sites and $U = 5t, 20t, 50t$. $\diamond$ is for $U = 5t$, $+$ is for $U = 20t$, and $\boxdot$ is for $U = 50t$.

IV. 1/3-FLUX PERIODICITY FOR THREE ELECTRONS

In contrast to the preceding section, the three-body problem does not allow an explicit solution. As discussed by Lieb,[9] the direct numerical solution of Bethe-ansatz equations for small chains is a much more difficult problem than the case of the thermodynamic limit. However, the problem for finite chains may be solved if we give the Bethe-ansatz equations in a form convenient for numerical iteration procedure.

A. Iteration procedure

The problem is to solve these equations (6) and (7), that is, to find numerically the values of the variables k_j and λ_α. In order to find such solutions we must represent the Bethe equations in a form convenient for iteration procedure, which is usually used in numerical calculations. The first equation (6) is already in the required form if we divide both sides by L. In the second equation (7) we add to both sides the function $N\theta(4\lambda_\beta/U)$. With the aid of these tricks one reduces the second equation (7) to the form

$$\lambda_\beta = \frac{U}{4}\tan\left[\frac{N\theta\left(\frac{4\lambda_\beta}{U}\right) + 2\pi J_\beta + \sum_{\lambda_\alpha=1}^{M}\theta\left(\frac{2\lambda_\beta + 2\lambda_\alpha}{U}\right) - \sum_{j=1}^{N}\theta\left(\frac{4t\sin k_j - 4\lambda_\beta}{U}\right)}{2N}\right]. \tag{20}$$

With the substitution $\lambda_\beta = t_\beta U/4$ this equation may be simplified and we arrive at a couple of equations, convenient for the iteration procedure:

$$k_j = \frac{2\pi I_j + 2\pi f - \sum_{\beta=1}^{M}\theta(4t\sin k_j/U - t_\beta)}{L} \tag{21}$$

and

$$t_\beta = \tan\left[\frac{N\theta(t_\beta) + 2\pi J_\beta + \sum_{\alpha=1}^{M}\theta((t_\beta - t_\alpha)/2) + \sum_{j=1}^{N}\theta(4t\sin k_j/U - t_\beta)}{2N}\right]. \tag{22}$$

In what follows below we use U to represent the ratio U/t. We have solved these equations iteratively, for the case of three and four electrons on the ring and for different values of F, U, and L. The convergence depends on the value of the parameters U and L and is fast if the value of U or L/N is large.

B. Appearance of fractional periodicity

The results may be classified as follows. For a small number of sites and a small value of U, the ground-state energy dependence remains the same as that of the free particle case. The ground-state energy dependence on flux is a single-flux periodical function. The ground-state energy corresponds to the case when two particles have down spins and one particle has an up spin, or alternatively, two particles have up spins and one particle has a down spin. In that case the energy decreases monotonously when the flux increases from zero up to $f = 0.295\,167$ and then the energy increases when the flux increases up to 1/2. This behavior must be symmetrically reflected on the second half of the elementary flux unit and then periodically continued for an arbitrary value of the flux. If the value of U becomes larger than the critical value, which, for the ring of four sites, is equal to $U_c \sim 20$, there appear new minima at integer values of the flux in the energy dependence. In Fig. 3 this dependence is calculated for $U = 50$. The shape of that curve is very different from the free fermion case. Each of these parabolic curves is associated with the state characterized by a definite set of quantum numbers I_j and J_α. To show the validity of the Bethe-ansatz equations for the value $M > N/2$ we have solved these equations for two cases, associated with the values of $M = 2$ and $M = 1$. For these cases the Bethe equations have different forms, although physically the states are equivalent to each other. In fact, the energy-flux dependences for these cases are identical, although the quantum numbers are distinct. Each state associated with the parabolic curve on Fig. 3 is represented by a vertical column in Table I. One sees that for the transition of one state to the other at the value $M = 2$, the set of quantum numbers is changed drastically.

With a further increase of U the minima become more profound, transforming gradually to the curve consisting of equidistant parabolas, which is the 1/3-flux periodical function. However, one gets the same effect with the increase of the number of sites L. For example, for the value $U = 50$ of Fig. 4 the flux-energy dependence is shown for the six-site ring. With the increase of the sites number from $L = 4$ to $L = 6$ the 1/3-flux quasiperiodicity has been gradually increased. If we take a smaller value of U, for example, $U = 8$, for the same number of sites $L = 6$, two parabolic curves disappear from the ground-state energy curve, which now consists of only two parabolas (see Fig. 5). For a larger number of sites, $L = 12$, for example, one again has four parabolas for the ground-state energy and a 1/3-flux periodical dependence also appears (Fig. 6). It is clear from these calculations that the general tendency of the appearance of the 1/3-flux periodicity of the ground-state energy is either with the increase of U at the fixed L or with the increase of the value of L for fixed value of U.

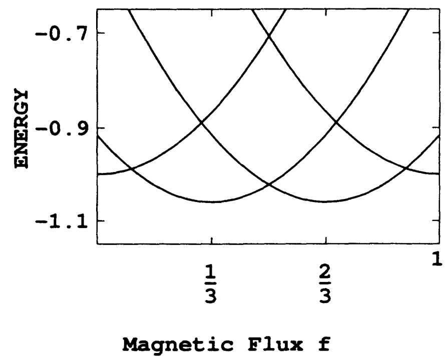

FIG. 3. The behavior of the ground-state energy and the first excited levels as a function of flux for three electrons at the values $L = 4$, $N = 3$, and $U = 50$ in the region of flux within the single fundamental flux quantum $\phi_0 = \frac{hc}{e}$. Two particles have up spins and one particle has a down spin.

The appearance of the 1/3-flux periodicity may arise also at small values of U, provided that the value of L is large. This is illustrated in Fig. 7, where the value of U is equal to 1 and the number of sites is $L = 128$. Furthermore, one can claim that the shape of the ground-state energy-flux dependence does not depend on the particular values of U and L, but depends on their product. Figure 8 shows the flux-energy dependence for the value $U = 0.5$ and the value $L = 256$ with the same product UL=128 as in the former case, given in Fig. 7. Comparison of Figs. 7 and 8, which are identical if the energy scale is neglected, allows one to conclude that there is a scaling symmetry, which states that if the product UL is constant, the shape of the flux-energy dependence is not changed. Note that to have 1/3-flux periodicity this product must have a large value. The results of the calculations we made for the case of four electrons ($N = 4$) are to a large extent the same, but with the difference that the 1/4-flux periodicity appears.

The general conclusion can be drawn that the fractional Aharonov-Bohm effect appears when the parameter $\alpha = tN/UL$ is small, but not exclusively in the case when t/U is small, as discussed in previous work by Kusmartsev[14] and by Yu and Fowler.[17] Encouraged by these numerical investigations of very small rings we investigate the general case of an arbitrary number of electrons on the ring, when α is small. These parameter values correspond to realistic situations when U/t has some fixed value but the system has a very dilute density.

TABLE I. The sets of quantum numbers I_j and J_α corresponding to the parabolic curve with the lowest energy and the values of the flux at the minimum values are given in the vertical column.

State number	1	2	3	4
$M = 2$ I_j	(-1,0,1)	(-1,0,1)	(-2,-1,0)	(-2,-1,0)
$M = 2$ J_β	(1,2)	(-1,0)	(0,1)	(1,2)
$M = 1$ I_j	(-3/2,-1/2,1/2)	(-3/2,-1/2,1/2)	(-3/2,-1/2,1/2)	(-3/2,-1/2,1/2)
$M = 1$ J_β	(3/2)	(1/2)	(-1/2)	(-3/2)
Flux $f_{\min}$	0	1/3	2/3	1

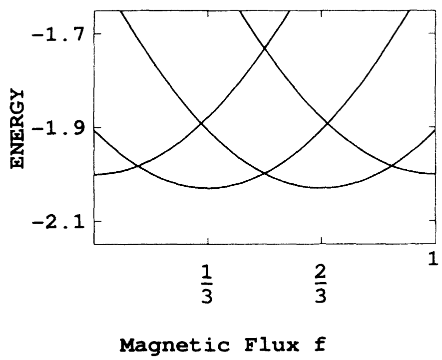

FIG. 4. The same as in Fig. 3 but at the value $L = 6$. At the cuspoidal point the flux absorption appears.

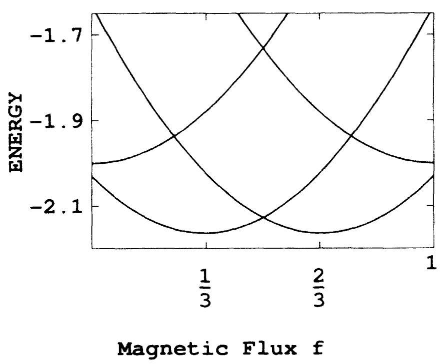

FIG. 5. The same as in Fig. 3 but at the values $L = 6$ and $U = 8$.

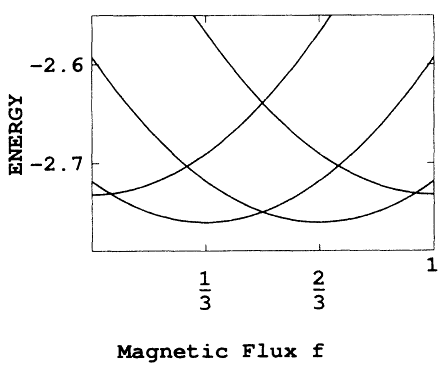

FIG. 6. The same as in Fig. 3 but at the value $L = 12$ and $U = 8$.

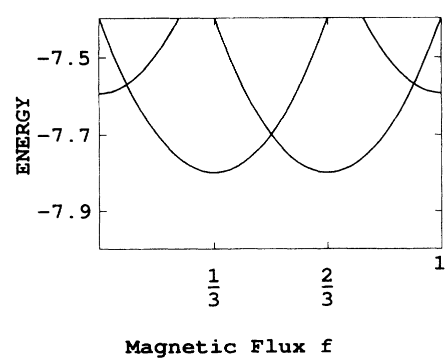

FIG. 7. The same as in Fig. 3 but at the values $L = 128$ and $U = 1$. The energy is expressed in units $t10^3$. The zero energy corresponds to $-2.99t$.

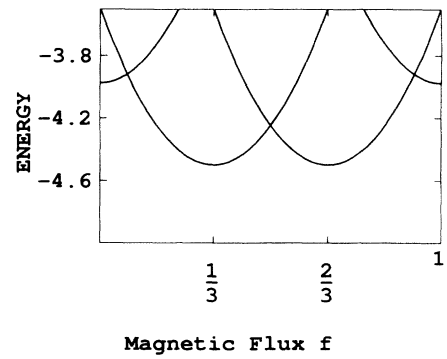

FIG. 8. The same as in Fig. 3 but at the values $L = 256$ and $U = 0.5$. One sees that the shape of this dependence is scaled to the one shown in Fig. 7. The energy is expressed in units $t10^4$. The zero energy corresponds to $-2.999t$.

V. FRACTIONAL $1/N$-FLUX PERIODICITY FOR N ELECTRONS

A. Zero approximation

Let us show that the fractional Aharonov-Bohm effect is created for an arbitrary number of electrons N on the ring and arbitrary values of U. From Eq. (21) one sees that the numerator of the right-hand side cannot be larger than $2\pi N$. This holds since for values of quantum numbers satisfying $I_j \leq N/2$, the flux $f < 1$ and $\sum_{j=1}^{M} \theta(x_j) \leq \pi M$. If $2\pi N/L \ll 1$, one has values of $k_j \ll 1$. Hence on the right-hand side of Eq. (21) the value $4 \sin k_j/U \sim 4k_j/U \sim N/UL = \alpha$ is a small parameter. Therefore in zeroth approximation, for small α, the expression $4 \sin k_j/U$ may be neglected and one gets an expression for k_j in the form

$$k_j = \frac{2\pi I_j + 2\pi f + \sum_{\beta=1}^{M} \theta(t_\beta)}{L}. \tag{23}$$

In an analogous way, from Eq. (22) one gets an equation, which does not depend on k_j:

$$N\theta(t_\beta) = 2\pi J_\beta + \sum_{\alpha=1}^{M} \theta((t_\beta - t_\alpha)/2). \tag{24}$$

This coincides with the equation obtained by Yu and Fowler,[17] in the limit of large U/t. From Eq. (24) we calculate the sum needed for the right-hand side of Eq. (23). After the substitution into that equation one gets

$$k_j = \frac{2\pi}{L}\left(I_j + f + \sum_{\beta=1}^{M} J_\beta\right). \tag{25}$$

This expression was obtained in the limit $U/t \to \infty$ by Kusmartsev[14] and then rederived by Yu and Fowler.[17]

In our derivation we have not made any assumption about the value of U. Here the ground-state energy is a $1/N$-flux periodical function, where the ground-state energy-flux dependence consists of parabolic curves with minima at the flux value $f_{\min} = p/N$, where p is the number of parabolic curves:

$$E_{\text{ground}} = -D\cos\left[\frac{2\pi}{L}\left(f - \frac{p}{N}\right)\right], \tag{26}$$

where the flux value f is changed in the region $(2p - 1)/2N < f < (2p + 1)/N$ and $D = 2\sin(\pi N/L)/\sin(\pi/L)$. Also from the above derivation it is clear that the parameter of our expansion is equal to

$$\alpha = \frac{Nt}{LU} = \rho t/U, \tag{27}$$

where $\rho = N/L$ is the filling factor. Summarizing, for any fixed value of the ratio t/U, there exists a dilute density limit associated with $\alpha \ll 1$. In this dilute limit the conventional Aharonov-Bohm effect disappears and the fractional effect takes over. Let us note that for the almost completely polarized system, for example, when the number of up spins is much larger than the number of down spins $M/N \ll 1$ Eq. (24) may also be solved analytically. In that case one assumes that the value of t_β in Eq. (24) is small. This gives

$$t_\beta = \frac{\pi}{N} J_\beta. \tag{28}$$

This looks like a spectrum of free spinless fermions on the chain with $2N$ sites.

B. First correction

With the aid of expansion with the parameter α defined above, one may find the first correction. Making an expansion in powers of the parameter α of the second Bethe equation (7), we get the form

$$N\theta(t_\beta) - \frac{8}{U}\frac{1}{(1+t_\beta^2)}\sum_{j=1}^{N}\sin(k_j) = 2\pi J_\beta + \sum_{\alpha=1}^{M}\theta((t_\beta - t_\alpha)/2), \tag{29}$$

which may be reduced to the equation

$$N\theta\left[t_\beta - \frac{4}{NU}\sum_{j=1}^{N}\sin(k_j)\right] = 2\pi J_\beta + \sum_{\alpha=1}^{M}\theta((t_\beta - t_\alpha)/2). \tag{30}$$

With the substitution $x_\beta = t_\beta - \frac{4}{NU}\sum_{j=1}^{N}\sin(k_j)$ this equation is reduced to Eq. (24), with unknown variables x_β. The equation derived for the variables x_α is independent of the flux f and the value of U. It is just the equation for an isotropic Heisenberg antiferromagnet on the ring having N sites and M down spins. The solution for x_α is independent of the flux f or the value U. However, the original variable t_β, expressed via x_α with the aid of the formula

$$t_\beta = x_\beta + \frac{4}{NU}\sum_{j=1}^{N}\sin(k_j), \tag{31}$$

does depend on both parameters: U (or α) and f. The first correction to t_β does not depend on the index β. The dependence of t_β on the flux f comes about through its explicit dependence on the momenta k_j via the second term on the right-hand side of Eq. (31). Substituting the equation for variables t_β into Eq. (23) for the momenta k_j and making an expansion using the parameter α, we get the form

$$Lk_j = 2\pi I_j + 2\pi f + \sum_{\beta=1}^{M}\theta(x_\beta) + \frac{8\sum_{l=1}^{N}(1 - N\delta_{lj})\sin k_l}{NU}\sum_{\beta=1}^{M}\frac{1}{1+t_\beta^2}. \tag{32}$$

For small parameter α (dilute density) this system of linear equations may be solved, with the result

$$k_j = \frac{2\pi I_j}{L(1+\frac{8B}{UL})} + \frac{2\pi f}{L} + \frac{2\pi}{NL}\sum_{\beta=1}^{M}J_\beta + \frac{2\pi}{NL}\frac{8B}{(UL+8B)}\sum_{l=1}^{N}I_l, \tag{33}$$

where $B = \sum_{\beta=1}^{M}\frac{1}{1+x_\beta^2}$ is a real number. For the following studies it is more convenient to represent this formula in the more symmetrical form

$$k_j = \frac{2\pi}{\tilde{L}}\left[I_j + f + \frac{1}{N}\sum_{\beta=1}^{M}J_\beta + \frac{8B}{UL}\left(\frac{\sum_{l=1}^{N}I_l}{N} + f + \frac{1}{N}\sum_{\beta=1}^{M}J_\beta\right)\right], \tag{34}$$

where $\tilde{L} = L(1 + \frac{8B}{UL})$.

Taking this solution into account, the formula for the ground-state energy takes on the form

$$E_{\text{ground}} = -\tilde{D}\cos\left\{\frac{2\pi}{\tilde{L}}\left[f - \frac{p}{N} + \frac{I_{\max} + I_{\min}}{2} + \frac{8B}{UL}\left(\frac{\sum_{l=1}^{N}I_l}{N} + f - \frac{p}{N}\right)\right]\right\}, \tag{35}$$

where $p = -\sum_{\beta=1}^{M} J_\beta, I_{\max}$ and $I_{\min}$ are maximal and minimal charge quantum numbers, and $\tilde{D} = 2\sin(\pi N/\tilde{L})/\sin(\pi/\tilde{L})$. Here the values of quantum numbers x_β do not depend on the magnetic field. Precisely speaking, they do not change their values when the flux changes within a single parabola. The ground-state energy will be associated with a new set of the quantum numbers J_β. This means that with the first correction taken into account, these parabolic curves, which the ground-state energy consists of, change their position mostly along the vertical axis and do not change in form. Thus, in this case, the quasi-$1/N$ periodicity is preserved.

To conclude, our investigation sheds light on the case in which the ring contains many electrons in the limit of very dilute electron density. In the correlated state an effective phase shift appears between the momenta of the different electrons, a shift which is associated with the repulsive interaction. Because of the shift the periodicity of the Aharonov-Bohm flux may have a fractional value. In contrast this effect is not expected to occur for the negative U-center model, where at best one will have only the half-quantum-flux periodicity. However, as we shall see below, a $1/N$ fractional Aharonov-Bohm effect may occur on a negative U-center ring, when the Zeeman energy term becomes important.

VI. PARITY EFFECTS ON A HUBBARD RING

A. Statistical flux

For spinless fermions there is a difference in responses to a magnetic field for the cases of even and odd number of particles on a ring.[18] This is the so-called parity effect. The effect is practically unchanged if there is an interaction between these spinless fermions. When the number of spinless fermions on the ring changes from odd to even, there is a statistical half-flux quantum which shifts the energy-flux dependence by exactly half of the fundamental flux quantum. Therefore, for small values of the flux and at an odd number of spinless fermions, the ring behaves like a diamagnet. When there is an even number of particles it behaves like a paramagnet. Kusmartsev obtained this result by an exact solution with the aid of Bethe ansatz in the model of interacting spinless fermions on the ring.[39,18] This was also independently qualitatively discussed by Leggett for the general case (called Leggett conjecture)[40] and was proven by Loss[41] with the aid of the bosonization method in the framework of the same model,[39,18] but for arbitrary coupling. However, taking spins into account, the situation is drastically changed.

Taking spins into account for noninteracting electrons gives the diamagnetic response only when there are $N = 4n + 2$ particles on the ring, where n is an arbitrary integer. For all other cases the response has a paramagnetic character. With finite temperature and disorder there occurs a double parity effect in which, for $N = 4n + 1$ and $N = 4n$, the response is paramagnetic; for $N = 4n + 1$ and $N = 4n + 2$ the response is diamagnetic.[37]

With the inclusion of the Hubbard interaction there appears an additional phase shift due to scattering of a given particle on the other particles, via two-particles interactions. Each scattering event gives a phase shift $\theta(x)$ in the Bethe equations. For the case of spinless fermions, the parity effect is conserved, in spite of the appearance of the new phases.[18]

However, with the Hubbard interaction, one has a different picture. The analytical solution obtained in the case of two electrons shows that the phase shift $\theta(x)$ creates a quasi-half-flux periodicity, which improves when the parameter α becomes smaller and smaller. The interaction creates an addition to the statistical flux which appears between the two electrons.

It is interesting that in the limit $U \to \infty$ this phase shift is exactly equal to a half-flux quantum. Therefore if the flux of the external magnetic field is equal to a half-flux quantum, then, with that additional statistical half-flux quantum, the total flux is equal to a unit of fundamental flux quantum. In comparison with the case of noninteracting electrons, where the periodicity is in units of flux quantum, here we already have a periodicity at half a flux quantum. For the three-electron case the phase shift is different.

The additional statistical flux arises on counting one permutation for each of the other electrons. The value of that phase can be estimated in the limit of $\alpha \to 0$ and equals

$$2\pi f_{\text{stat}} = \sum_{\beta=1}^{M} \theta(t_\beta) = \frac{2\pi}{N} \sum_{\alpha} J_\alpha. \quad (36)$$

where we get a fraction $f_{\text{stat}} = \frac{1}{N}$. In this case one may think that this flux is attached to each electron, that is, all N electrons share 1 unit of the quantum flux. Putting a new electron on the ring creates a new system, where $N + 1$ electrons will now share a unit of quantum flux. In this system, the response has a purely diamagnetic character, for any number of electrons.

B. Classification of parity effects

In general terms, the appearance of the parity effect and the conventional Aharonov-Bohm effect may be described as follows. At small values of U, or more precisely large α, we have the conventional parity effect for free electrons (see Table II). With an increase of the interaction there exists the critical value of $U = U_{\text{cr1}}$ or $\alpha = \alpha_{\text{cr1}}$, where for values $U > U_{\text{cr1}}$ or $\alpha < \alpha_{\text{cr1}}$ the parity effect looks similar to the parity effect for spinless electrons. That is, for an even number of electrons the magnetic response has a diamagnetic character and for an odd number of electrons the response is paramagnetic. Note that for spinless fermions the response is diamagnetic for an odd number of electrons. With a further increase of the coupling constant U there is a second critical value, at $U = U_2$ $(\alpha = \alpha_2)$, with the new type of parity effect. Thus for $U > U_2$ $(\alpha < \alpha_2)$ the paramagnetic response occurs only for $N = 4n + 1$ electrons. Finally, for $U > U_3$, so that $U_3 > U_2$, the parity effect disappears and the ring behaves as a diamagnetic for any number of electrons. The value of U_3 depends on

TABLE II. The classification of different regimes of the parity effect, with the change of the Hubbard interaction. The number of electrons on the ring is equal to $4n+2, 4n+1, 4n$, and $4n-1$, respectively.

Particle Number	$U < U_{c1}$	$U_{c1} < U < U_{c2}$	$U_{c2} < U < U_{c3}$	$U_{c3} < U$
$4n+2$	diamagnet	diamagnet	diamagnet	diamagnet
$4n+1$	paramagnet	paramagnet	paramagnet	diamagnet
$4n$	paramagnet	diamagnet	diamagnet	diamagnet
$4n-1$	paramagnet	paramagnet	diamagnet	diamagnet

the electron density. As discussed above, the parameter α is what matters. From our investigation we may conclude that the critical value for the disappearance of the parity effect equals $\alpha_3 \sim 0.02$. With the disappearance of the parity effect the quasi-$1/N$-fractional Aharonov-Bohm periodicity will appear. The classification of the different parity regimes is shown in Table II.

The parity effect is preserved with disorder or finite temperature. However, the change of this classification with the change of temperature is nontrivial and will be discussed in a forthcoming paper.

VII. AHARONOV-BOHM EFFECT FOR U-CENTER PAIRING MODEL

A. Half-flux periodicity

It is interesting to compare the results with the Aharonov-Bohm effect in the case when there exists a pairing of the electrons induced by a negative U potential (pairing due to a negative U center). In a one-dimensional system with negative U pairing one expects that the two electrons will tend to pair together on the same site. On the ring these electrons will also have the tendency to move in pairs. The kinetic energy, due to the finite size, will try to destroy the pairs.

Therefore we may have in this case the approximate half-quantum-flux periodicity. For the same reason as in the correlated state discussed above in Sec. III, the half-quantum-flux periodicity is improved with the increase of $|U|$ and L. However, the character of this state is different from the correlated one. For large negative U one needs an activation energy (spectral gap) to destroy the localized pair.

For illustration we show the ground-state energy dependence on the flux in two cases: at a constant value of U with an increase in the number of sites and at a constant value of L with the decrease of the negative value of U. We see in Figs. 9 and 10 that in both cases half-quantum-flux periodicity improves as both $|U|$ and L increase. The change in slope of the ground-state energy as a function of flux will correspond to the change in the direction of persistent current. In ring superconductors this current keeps flux quantized in units of half a quantum flux, for which the negative U case is a plausible model. The magnetization also behaves similarly to the current.

It is noted that here there is also a parity effect: in the cases of an even and an odd number of particles the Aharonov-Bohm effects are qualitatively distinct. For an odd number of electrons on the ring there is an unpaired electron with energy above the spectral gap. Therefore its contribution to persistent current is important, disrupting the half-flux periodicity created in the case of an even number of particles on the ring.

B. Low versus high density limit

There is, however, a correspondence between the states associated with the positive and negative U values. The dilute density limit of electrons on the ring described by the Hubbard Hamiltonian with positive values of U cor-

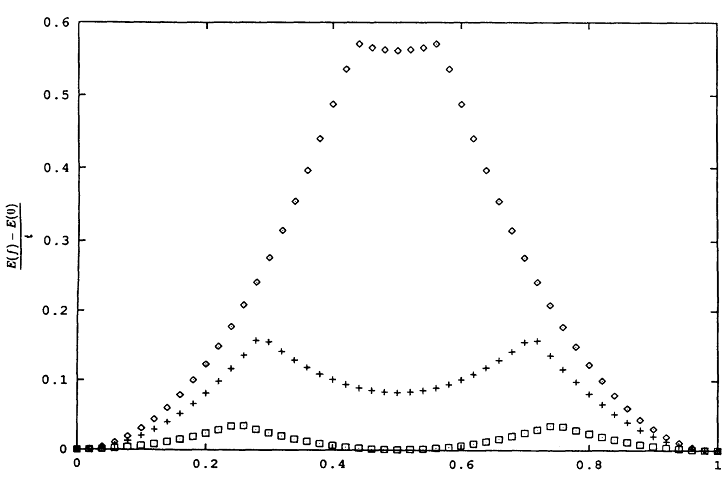

FIG. 9. The same as in Fig. 2 but for the case of U-center pairing. Here L is kept fixed at five sites, but $U = -1t, -5t$ or $-20t$. $\diamond$ is for $U = -t$, $+$ is for $U = -5t$, and $\boxdot$ is for $U = -20t$.

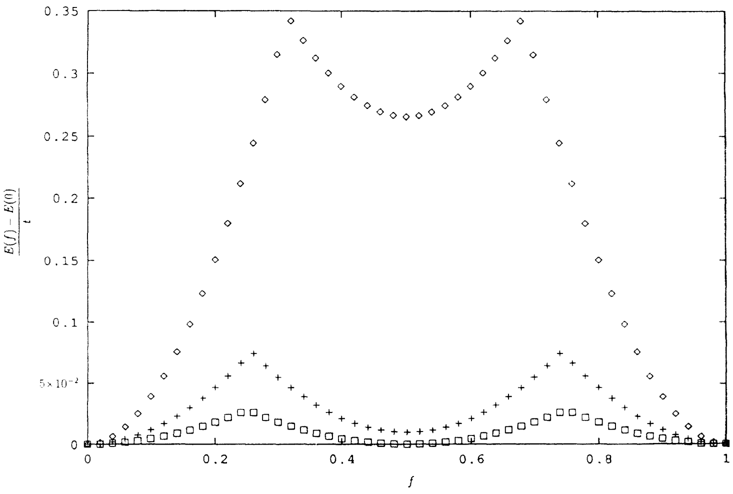

FIG. 10. The same as in Fig. 1, but for the case of U-center pairing. Energies are in units of t. Here $U = -10t$ and L takes on values of three, five, and eight sites. ◇ is for three sites, + is for five sites, and ⊡ is for eight sites. The ground-state energy at zero flux has been subtracted to normalize the curves. The flux is in units of the quantum flux.

responds to the high electron density limit, described by the same Hubbard model with negative values of U, with the aid of the transformations

$$c_{i,+} \to a_{i,+} \tag{37}$$

and

$$c_{i,-} \to a_{i,-}^{+}. \tag{38}$$

The Hamiltonian (1) transforms into

$$H = -t\sum_i (a_{i,+}^{\dagger} a_{i+1,+} - a_{i,-}^{\dagger} a_{i+1,-}) + U\sum_i n_{i,+} - U\sum n_{i,+}n_{i,-}. \tag{39}$$

Introducing an auxiliary field acting only on the spin-down electrons with a flux through the ring of $\Phi_- = \pi L$ the Hamiltonian (39) may be transformed to the form

$$H = -t\sum_{i,\sigma} a_{i,6}^{\dagger} a_{i+1,6} + U\sum_i n_{i,+} - U\sum n_{i,+}n_{i,-}. \tag{40}$$

This is a negative U-center model in a magnetic field of strength U, creating the Zeeman term. Note that with this transformation the number of spin-up particles is not changed $N_{+,\text{new}} = N_+$, but the number of spin-down particles is equal to $N_{-,\text{new}} = L - N_-$. In other words, spin-down particles are equivalent to holes in the original Hamiltonian. Thus the spin-up particles are moving in a field of flux $\Phi_+ = f$ and spin-down particles are moving in a field of flux $\Phi_- = f + L\pi$. Therefore the systems with an odd and an even number of sites will have different energy-flux dependence. One of them will be transformed into the other one with a shift of a half-flux quantum. Thus there is a parity effect here. Let us now discuss the ring with an even number of sites. We have shown that electrons on the ring, described by the Hubbard Hamiltonian with positive values of U in low density limit, behave in the same way as the high density electron case, described by the Hubbard Hamiltonian with the negative values of U, provided that the system is highly polarized.

This comparison shows that the Zeeman energy term gives very interesting new physics, i.e., fractional Aharonov-Bohm periodicity for the model of negative U centers. In the high density electron case, we again have the fractional flux periodicity of the ground-state energy and the persistent current with the period equal to $1/N$. This number N is equal to the sum of the number of spin-up particles and the number of holes. Physically it is not clear why the fractional $1/N$ periodicity would occur in that model. Since the fractional $1/N$ periodicity in the repulsive Hubbard model indicates strongly correlated state on N electrons, the same fractional periodicity in the attractive Hubbard model might mean the appearance of some strongly correlated (bound) state of P holes and N_+ spin-up particles, where $P + N_+ = N$. We postpone the detailed investigation of the implicit influence of the Zeeman energy on the Aharonov-Bohm effect to a forthcoming paper,[42] assuming for now that this energy is small (i.e., the ring has a very large radius). In that paper[42] we found that with the corrections with the parameter α the discussed fractional $1/N$ AB effect may coexist with integer or half flux quantum AB effects. The latter depends on the Zeeman energy . With the increase of the Zeeman energy the half-flux quantum periodicity transforms continuously to the integer flux quantum Aharonov-Bohm periodicity.

VIII. POSSIBLE EXPERIMENTS

A. Quantum dot arrays

It is worth noting that there is a possible practical realization where the above model may be applicable; that is the case of a ring consisting of quantum dots placed in succession. These structures may be built and investigated using modern technology.[43] The single quantum dot will act as a potential well for the electrons. If the radius R of the quantum dot decreases, the charging energy e^2/R increases and a case occurs in which no more than two electrons with opposite spin can be accommodated on a single dot. The system of quantum dots is to be described with a Hubbard Hamiltonian with $U = e^2/R$.

Therefore in a ring consisting of quantum dots, with high charging energy, one may observe the destruction of the simple Aharonov-Bohm effect and the appearance of the fractional period. Note that since the reason for the effect is a local charging energy of the dot, the round shape of the ring is not important. Its shape, for example, may be square or even triangular.

The second system where the proposed effect might be observed is a two-dimensional array of quantum dots. If we put such an array in the magnetic field, an edge current flowing through the dots at the edge occurs. The flow covers a maximal cross-sectional area and therefore needs the lowest field to have Aharonov-Bohm period. This current is similar to the creation of edge currents in the fractional quantum Hall effect.

If we assume that the size of the quantum dot is about 50 Å, then the charging energy may be estimated as about $U = E_c = 0.1$ eV. In this case if the tunneling integral is about $t = 0.1$ eV, then for a circumference of the array (or the ring) of about $L_R \geq 10\,000$ Å one may already see the fractions 1/3 or 1/4 or even 1/10 of the Aharonov-Bohm period. This depends on the number of electrons in the edge current.

The last remarks need to be extended. The issue is to observe an edge or surface current in small semiconducting samples. The current is assumed to flow through edge states; the Aharonov-Bohm effect in such systems is then due to transport via the edge states. It has been shown that the periodic magnetoconductance oscillations can occur in singly connected geometry, such as a point contact[44,45] or as a disk shaped region in a two-dimensional electron gas.[46,47] One of these systems might be, for example, a single quantum dot of the radius about $R = 1000$ Å or an array of such dots at large distances from each other. The quantum rings may be also created from a two-dimensional electron gas by applying concentric gates.[48] By applying such a gate voltage it is possible to deplete a central region and to form a quantum ring. The electron concentration on such a ring may also be changed and controlled with the gate voltage. In such a way it is possible to create the conditions where the fractional Aharonov-Bohm effect may be observed. The importance of the Coulomb interaction contribution in such quantum rings has been estimated recently via a Hartree-Fock approximation in Ref. 49 and via a variational method in Ref. 50. However, in both of these publications the fractional Aharonov-Bohm effect has not been recognized.

Sivan and Imry[47] have shown that edge states of such a quantum dot make the geometry effectively doubly connected. In this case the quantum dot traps flux quanta equally well as a ring and the orbital current flows through the edge states. With the change in the gate voltage one may change the number of electrons in the dot. Measuring the Aharonov-Bohm period at different gate voltages with the aid of fractional Aharonov-Bohm effect one may see how the number of electrons in the edge states increases or decreases. In this case again the predicted $1/N$ fractional Aharonov-Bohm effect might be useful, to measure the number of electrons in the edge states of the single dot.

B. Semiconductor single ring

It is worth mentioning here a recent experiment on a semiconductor single loop in the GaAs/GaAlAs system,[51] for which single-flux periodicity has been detected. It was estimated that this loop has only a few electron channels (equal to 4). The result of this experiment has been explained in our previous publication,[37] i.e., the single-flux periodicity is due to disorder.

By doping or by applying a gate voltage the number of electrons may be decreased. In that case the role of the electron-electron interaction increases. Here again we may have a physical situation when the kinetic energy of electrons (an energy of size quartization) will be equal to the energy of electron-electron interaction, i.e., the situation of the fractional Aharonov-Bohm effect. That is, we argue that even in that experiment the fractional Aharonov-Bohm effect might be observed in the very low density limit.

C. Carbon clusters

One of the possible systems where one may look for the fractional Aharonov-Bohm effect is carbon clusters, where recently considerable progress has been made. The complicated polymeric molecules, different macromolecules of ring-chain shape, have been synthesized.

One example is the molecule C_{60}. When intercalated with alkali metal atoms to form the fullerides A_xC_{60}, that material with $x = 3$ becomes superconducting below temperatures 18 and 28 K for A =K (Ref. 52) and for A =Rb (Ref. 53), respectively. These discoveries stimulated next progress in the construction of carbon nanotubules, created by rolling up a graphene sheet and having a cylinder shape of an arbitrary radius.[54] A short review of these investigations may be found in Ref. 55. These systems are especially attractive for an investigation of the fractional Aharonov-Bohm effect, since the concentration of the current carried on a single tubule may be easily changed by doping without the destruction of the tubule structure.

Attention must also be given to the clusters from C_2 to C_{200}, which are important from another interesting aspect, namely, aromaticity. It is known that with a large number of atoms such monocyclic structures are more stable than the corresponding linear forms.[56] We suggest some mention of short chain polymers which can be obtained by cutting long chain polymers such as polyethylene where the Hubbard model is known to apply.

D. Raman scattering

It is worth noting that the findings might be applicable to the description of the frequency changes of phonons on the ring in magnetic field (see Ref. 57). That is, as was shown in Ref. 57, that the phonon or vibron frequency changes for the atomic oscillation along the orbital current repeats the Aharonov-Bohm periodicity of the loop. Therefore the studies by high precision Raman scattering of vibron frequency changes on the carbon clusters in the transverse magnetic field may be helpful to detect the predicted effect.

The effect of the fractional or $1/N$ periodicity is directly related to the phenomenon of the Coulomb blockade. The feature of both these phenomena is due to many-body effects associated with the interacting current carriers. That is, the motion or a change of state of a single electron changes the states of all other electrons.

If we put the ring of quantum dots in a transverse magnetic field single electrons will have the tendency to move along the ring. However, hops onto a dot already containing two electrons are not allowed, since this will cost energy equal to the charging energy. One may have incoherent or independent hops. However, there is another possibility. This is a process, which will not cost the charging energy, in which all electrons on the ring make coherent hops (cotunneling). In other words, this is the simultaneous motion of all electrons on the ring. It is clear that if we move all electrons together the charging energy is not important (U may be arbitrarily large) and the total change in the phase of the many-body wave function will be equal to $2\pi f N$.

From the gauge invariance of the ground state of the ring we conclude that the equivalent state to $f = 0$ is $Nf = 1$. Hence the period of interacting electrons on the ring is $f = 1/N$, in agreement with the results obtained in this paper.

IX. CONCLUSIONS

In the present work we have studied the effects of the electron-electron correlations on the Aharonov-Bohm effect in a quantum ring. We have shown that the correlations result in a fractional Aharonov-Bohm effect, which appears when the parameter $\alpha = Nt/LU$ is small. This case may occur when U/t is large or in low density limit when the filling N/L is small. The conclusion that the low density limit of the Hubbard model is equivalent to a strong coupling limit $U/t \gg 1$ coincides with one obtained by Shulz,[58] a work which describes Luttinger liquid properties in the framework of the bosonization approach. Using the Aharonov-Bohm effect we proved this theorem far beyond the low-frequency limit of that theory.

We have found also a very interesting scaling symmetry, hold for the low density limit, namely, that the shape of the ground and excited energy-flux dependences depend only on the parameter $\alpha = \frac{tN}{UL}$. In other words, when $\alpha \ll 1$ the energy-flux dependences obtained for the different values U and L, provided that the parameter α =const is fixed, can be transformed one into another by a scaling transformation of the energy scale. This confirms the Lieb suggestion[9] that the Hubbard Hamiltonian describing a system of the finite size has some very nontrivial internal symmetries.

The fractional effect is not expected to occur for the negative U model if the Zeeman term is neglected. If the Zeeman term is not neglected the fractional effect may occur even for negative U.

The density of electrons may be well controlled in many experimental situations, for example, by doping, or by applying the gate voltage to change of the position of the chemical potential as it is used in quantum wells. Therefore the predicted fractional Aharonov-Bohm effect is a good challenge for experimentalists.

Note added in proof: Recently we have become aware that in AuIn rings prepared by e-beam lithography, the fractional 1/4 effect has been observed.[59] The effect has been observed in the region of a superconducting transition, which is very broad, i.e., where the phase separation into superconducting droplets might occur. Therefore, it seems that a system reminiscent of the quantum dot chain ring is created. If the phase coherence between the superconducting droplets is broken, the unpaired electrons, which are created due to the parity effect in some droplets and which number very few, may play an important role in the creation of the fractional Aharonov-Bohm effect, i.e., our discovery of the fractional $1/N$ effect may be relevant.

ACKNOWLEDGMENTS

F.V.K. and J.F.W. thank INPE and CNPq for financial support, during their stay at INPE Campos, São Paulo, Brazil. The work by F.V.K. has been supported by Faculty of Natural Science of Oulu University and by Ministery of Education, Science and Culture of Japan. We thank also T. C. Choy for the pointing out on the importance of Zeeman energy term, stimulating us to write Sec. VI B and to find the fractional Aharonov-Bohm effect in the U-center model for high density polarized electrons.

*Permanent address: L.D. Landau Institute for Theoretical Physics, Moscow 117940, GSP-1, Kosygina 2, V-334, Russia.

†Permanent address: INTEC(UNL-CONICET), Guemes 3450, 3000 Santa Fe, Republic of Argentina.

[1] P.W. Anderson, Phys. Rev. Lett. **67**, 3844 (1991); **64**, 1839 (1990); **65**, 2306 (1990).

[2] T. Chakraborty and P. Pietilainen, in *Fractional Quantum Hall Effects*, edited by K.v. Klitzing (Springer, Berlin, 1988), and references therein

[3] F.D.M. Haldane, Phys. Lett. **81A** 153 (1981).

[4] P.W. Anderson, Phys. Scr. **T27**, 60 (1989).

[5] P. Wiegmann, Phys. Scr. **T27**, 160 (1989).

[6] Y. Hasegawa, P. Lederer, T.M. Rice, and P.B. Wiegmann, Phys. Rev. Lett. **63**, 907 (1989).

[7] F.V. Kusmartsev, A. Luther, and A.A. Nersesyan, Pis'ma Zh. Eksp. Teor. Fiz. **55**, 692 (1992) [JETP Lett. **55**, 724 (1992)].

[8] A.A. Nersesyan, A. Luther, and F.V. Kusmartsev, Phys. Lett. A **176** 363 (1993).

[9] Elliott Lieb, *Advances in Dynamical Systems and Quantum Physics, Capri, 1993* (World Scientific, Singapore, in press).

[10] Rolf Landauer (unpublished).

[11] Igor O. Kulik, Pis'ma Zh. Eksp. Teor. Fiz. [JETP Lett. **11**,

275 (1970)].

[12]M. Büttiker, Y. Imry, and Rolf Landauer, Phys. Lett. **96A**, 365 (1983).

[13]Rolf Landauer and M. Büttiker, Phys. Rev. Lett. **54**, 2049 (1985).

[14]F.V. Kusmartsev, J. Phys. Condens. Matter **3**, 3199 (1991).

[15]R.M. Fye, M.J. Martins, D.J. Scalapino, J. Wagner, and H. Hanke, Phys. Rev. B **44**, 6909 (1991).

[16]A.J. Schofield, J.M. Wheatley, and T. Xiang, Phys. Rev. B **44** 8349 (1991).

[17]N. Yu and M. Fowler, Phys. Rev. B **45**, 11 795 (1992).

[18]F.V. Kusmartsev, Pis'ma Zh. Eksp. Teor. Fiz. **53**, 27 (1991) [JETP Lett. **53**, 28 (1991)]; Phys. Lett. A **161**, 433 (1992).

[19]O.J. Heilmann and E.H. Lieb, Trans. N.Y. Acad. Sci. **33**, 116 (1970); also in Ann. N.Y. Acad. Sci. **172**, 583 (1971).

[20]H. Bethe, Z. Phys. **71**, 205 (1931).

[21]C.N. Yang and C.P. Yang, Phys. Rev. **147**, 303 (1966); **150**, 321 (1966); **150**, 321 (1966); **150**, 327 (1966); **151**, 258 (1966).

[22]J. des Cloizeaux and M. Gaudin, J. Math. Phys. **7**, 1384 (1966).

[23]C.J. Hamer, G.R.W. Quispel and M.T. Batchelor, J. Phys. A **20**, 5677 (1987).

[24]F. Woynarovich and H.P. Eckle, J. Phys. A **20**, L97 (1987).

[25]F. Woynarovich, Phys. Rev. Lett. **59**, 259 (1987); **59**, 1264(E) (1987).

[26]B. Sutherland and B.S. Shastry, Phys. Rev. Lett. **65**, 1833 (1990).

[27]N. Bayers, and C.N. Yang, Phys. Rev. Lett. **46**, 7 (1961).

[28]C.N. Yang, Rev. Mod. Phys. **34**, 694 (1962).

[29]E. Lieb, F. Wu, Phys. Rev. Lett. **20**, 1445 (1968).

[30]A. Ferretti, Igor O. Kulik, and A. Lami, Phys. Rev. B **45**, 5486 (1992).

[31]A. Sudba, C.M. Varma, T. Giamarchi, E.B. Stechel, and R. Scalettar, Phys. Rev. Lett. **70**, 978 (1993).

[32]B.S. Shastry and Bill Sutherland, Phys. Rev. Lett. **65**, 243 (1990).

[33]C.A. Stafford, A.J. Millis, and B.S. Shastry, Phys. Rev. B **43**, 13 660 (1991).

[34]N. Kawakami and S.K. Yang, Phys. Rev. Lett. **65**, 3063 (1990).

[35]E.N. Bogachek, I.V. Krive, I.O. Kulik, and A.S. Rozhavsky, Phys. Rev. B **42**, 7614 (1990).

[36]I.V. Krive, A.S. Rozhavsky, Int. J. Mod. Phys. B **6**, 1255 (1992).

[37]J. Weisz, R. Kishore, and F.V. Kusmartsev, Phys. Rev. B **49**, 8126 (1994).

[38]A. Gogolin and N. Prokof'ev (unpublished).

[39]F.V. Kusmartsev (unpublished); in *High-Temperature Superconductivity: Physical Properties, Microscopic Theory and Mechanisms,* edited by J. Ashkenasi, S. Barnes, Fulin Zuo, G.C. Vezzoli, and Barry M. Klein (Plenum, New York, 1992), pp. 77-89.

[40]A.J. Leggett, in *Granular Nanoelectronics*, Vol. 251 of *NATO Advanced Study Institute, Series B: Physics,* edited by D.K. Ferry, J.R. Barker, and C. Jacoboni (Plenum, New York, 1991), p. 297.

[41]D. Loss, Phys. Rev. Lett. **69**, 343 (1992).

[42]F.V. Kusmartsev and M. Takahashi (unpublished)

[43]T. Demel, D. Heitmann, P. Grambow, and K. Ploog, Phys. Rev. Lett. **64**, 788 (1990).

[44]P.H.M. van Loosdrecht, C.W.J. Beenakker, H. van Houten, J.G. Williamson, B.J. van Wees, J.E. Mooij, C. T. Foxon, and J.J. Harris, Phys. Rev. B **38**, 10 162 (1988).

[45]B.J. van Wees, L.P. Kouwenhoven, C.J.P.M. Harmans, J.G. Williamson, C.E. Timmerling, M.R.I. Broekaart, C. T. Foxon, and J.J. Harris, Phys. Rev. Lett. **62**, 2523 (1989).

[46]U. Sivan, Y. Imry and C. Hartzstein, Phys. Rev B **39**, 1242 (1989).

[47]U. Sivan, and Y. Imry, Phys. Rev. Lett. **61**, 1001 (1988).

[48]C.W. Beenakker, H. van Houten, and A.A.M. Staring, Phys. Rev. B **44**, 1657 (1991).

[49]U. Eckern and A. Schmid, Europhys. Lett. **18**, 457 (1992).

[50]P. Pietilainen and T. Chakaborty, Solid State Commun. **89**, 809 (1993).

[51]D. Mailly, C. Chapelier, and A. Benoit, Phys. Rev. Lett. **70**, 2020 (1993).

[52]A.F. Hebard *et al.*, Nature **350**, 600 (1991).

[53]M.J. Rosseinsky *et al.*, Phys. Rev. Lett. **66**, 2830 (1991).

[54]M.S. Dresselhaus, G. Dresselhaus, and R. Saito, Phys. Rev. B **45**, 6234 (1992).

[55]P.M. Ajayan and S. Iijima, Nature (London) **361**, 333 (1993).

[56]W. Weltner, Jr. and R.J. Van Zee, Chem. Rev. **89**, 1713 (1989).

[57]F.V. Kusmartsev, Phys. Rev. B **46**, 7674 (1992).

[58]H.J Shulz, Int. J. Mod. Phys. B **5**, 57 (1991).

[59]Y. Liu, John C. Price, and XiaXian Zhang, Bull. Am. Phys. Soc. **V39**, 850 (1994).