

Crossover from the Josephson effect to bulk superconducting flow

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The crossover between ideal Josephson behavior and uniform superconducting flow is studied by solving exactly the Ginzburg-Landau equation for a one-dimensional superconductor in the presence of an effective δ -function potential of arbitrary strength. As the effective scattering is turned off, the pairs of Josephson solutions with equal current evolve into a uniform and a solitonic solution with nonzero phase offset. It is also argued that a microscopic description of the crossover must satisfy the self-consistency condition, which is shown to guarantee current conservation. The adiabatic response to an external bias is briefly described. The ac Josephson effect is shown to break down when the external voltage is applied at points which are sufficiently far from the junction.

I. INTRODUCTION

The Josephson effect between two weakly coupled superconductors and the steady flow of supercurrent in a perfect lead constitute the two main paradigms of superconducting transport. Both regimes can be viewed as the limits of a general scenario in which Cooper pairs flow coherently in the presence of a scattering obstacle of arbitrary strength. The Josephson effect corresponds to the limit in which a strongly reflecting obstacle (typically, a tunneling barrier¹ or a point contact²) reduces drastically the effective coupling between two bulk superconductors while still preserving global coherence. In the absence of an external bias, the current is given by the Josephson relation $I = I_C \sin(\Delta\varphi)$, where $\Delta\varphi$ is the phase difference between the two superconductors. The opposite limit is that of supercurrent flow in a perfect lead without appreciable scattering. In the appropriate gauge, this regime is characterized by a superconducting gap of uniform amplitude and a linearly varying phase whose gradient is proportional to the current. Specifically, in the Ginzburg-Landau limit, the current density can be written $\mathbf{j} = (e\hbar/m)|\psi|^2\nabla\varphi$, where $\psi = |\psi|e^{i\varphi}$ is the superconducting order parameter.

An adequate measure of the scattering strength is the average transmission probability T_0 for a Fermi electron passing through the barrier or contact in the normal phase,

$$T_0 \equiv (h/e^2 R_N)(2\pi/Ak_F^2), \quad (1)$$

where R_N is the device normal resistance, A is the cross section area of the semi-infinite leads, and k_F is the Fermi wave vector. "Weak" and "strong" superconductivities are then characterized by $T_0 \ll 1$ and $T_0 \simeq 1$, respectively. For a structure in which superconductivity is not weakened by one-electron reflection, such as a S - N - S junction without current concentration, a more general parameter is I_C/I_B , where I_C is the critical current of the structure and I_B is the bulk critical current of the perfect lead. It seems natural to ask what is the nature

of the superconducting flow for intermediate values of T_0 or, more generally, I_C/I_B , i.e., what is the crossover between the two extreme limits of superconducting flow. This rather fundamental question is of special current relevance, in view of the recent activity on superconducting mesoscopic structures (see, for instance, Refs. 3–5). In the case of a superconducting point contact, the intermediate regime would correspond to contact widths not much smaller than the width of the semi-infinite leads. Alternatively, in the case of tunneling barriers, the crossover could be explored by considering different degrees of transparency at the Fermi level. In the case of a S - N - S junction, the intermediate behavior would be displayed by relatively thin normal metal layers located between two superconductors.

A preliminary version of some of the results contained in this article has been briefly presented in Ref. 6.

II. SELF-CONSISTENCY AND CURRENT CONSERVATION

Theoretical studies of weak superconductivity almost invariably assume that the phase is constant within the two superconductors. This is generally a reasonable approximation, since, by definition, in this regime, $I_C \ll I_B$. As a consequence, the variation of the phase in the bulk of the superconductor displayed by current-carrying solutions can be safely neglected in a wide range of length scales. It is clear that the approximation of an asymptotically uniform phase cannot be justified if I_C becomes comparable to I_B , which will certainly be the case in structures with moderate or negligibly weakened superconductivity. The more general situation will be that of a phase which varies linearly throughout the lead except in a finite region near the scattering center where it varies faster.

In order to discuss some general questions related to self-consistency, we focus in this section on structures in which the decoupling between the two sides of a superconductor is due to one-electron scattering by a barrier or point contact. The conventional way of generalizing the

BCS theory to the presence of an arbitrary one-electron potential is based on the Bogoliubov-de Gennes (BdG) equations⁷

$$\begin{bmatrix} H_0 & \Delta \\ \Delta^* & -H_0^* \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} = \epsilon_n \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \quad (2)$$

where H_0 is the one-electron Hamiltonian, Δ is the gap function, and $[u_n(\mathbf{r}), v_n(\mathbf{r})]$ and $\epsilon_n > 0$ are, respectively, the normalized wave function components and the energy of the quasiparticle n . The self-consistency condition for the gap function is⁷

$$\Delta = V \sum_n u_n v_n^* (1 - 2f_n), \quad (3)$$

where V is the electron-phonon coupling constant and $f_n = [\exp(\epsilon_n/kT) + 1]^{-1}$. The BdG Hamiltonian can alternatively be written

$$H = - \sum_{n\sigma} \epsilon_n \int d\mathbf{r} |v_n(\mathbf{r})|^2 + \sum_{n\sigma} \epsilon_n \gamma_{n\sigma}^\dagger \gamma_{n\sigma}, \quad (4)$$

where $\gamma_{n\sigma}^\dagger$ creates quasiparticle n with spin σ . In terms of the quasiparticle operators, the charge and current density operators are written

$$\rho = e \left\{ \sum_{n\sigma} |v_n|^2 + \sum_{nm\sigma} (u_n^* u_m - v_n^* v_m) \gamma_{n\sigma}^\dagger \gamma_{m\sigma} + \sum_{nm\sigma} u_n v_m \sigma \gamma_{m\sigma} \gamma_{n,-\sigma} + \sum_{nm\sigma} u_n^* v_m^* \sigma \gamma_{n,-\sigma}^\dagger \gamma_{m\sigma}^\dagger \right\}, \quad (5)$$

$$\mathbf{j} = \frac{e\hbar}{2mi} \left\{ - \sum_{n\sigma} v_n^* \mathbf{D} v_n + \sum_{nm\sigma} (u_n^* \mathbf{D} u_m + v_n^* \mathbf{D} v_m) \gamma_{n\sigma}^\dagger \gamma_{m\sigma} + \sum_{nm\sigma} v_m \mathbf{D} u_n \sigma \gamma_{m\sigma} \gamma_{n,-\sigma} - \sum_{nm\sigma} v_m^* \mathbf{D} u_n^* \sigma \gamma_{n,-\sigma}^\dagger \gamma_{m\sigma}^\dagger \right\}, \quad (6)$$

where $e = -|e|$, $\sigma = \pm 1$, and \mathbf{D} is defined as $f\mathbf{D}g \equiv f(\nabla g) - (\nabla f)g$. In Eqs. (5) and (6), the contributions from the condensate and the quasiparticles have been clearly separated. The quasiparticle contribution can in turn be divided into a part which conserves the quasiparticle number and a part which does not. The non-conserving components will not contribute to the expectation values $\langle \mathbf{j} \rangle$ and $\langle \rho \rangle$ but will play an important role in the quantum fluctuations of the electronic charge and current densities.

If one attempts to solve the BdG equations (2) in a given structure subject to the boundary condition that the phase takes certain constant values on each semi-infinite lead, one generally finds from (6) a nonzero value of the total current. This general feature can be illustrated by solving exactly a specific and very important example, namely, that of a strictly one-dimensional superconductor (i.e., with only one propagating channel for the Fermi electrons) with a barrier of arbitrary transmission T_0 at the Fermi level. In this model, the phase is assumed to be uniform on each side of the barrier. A non-

self-consistent resolution of the BdG equations at zero temperature yields the current^{4,8-10}

$$I(\Delta\varphi) = \frac{e|\Delta|}{2\hbar} \frac{T_0 \sin(\Delta\varphi)}{[1 - T_0 \sin^2(\Delta\varphi/2)]^{1/2}}, \quad (7)$$

where $\Delta\varphi$ is the difference between the phases on each side of the barrier. Figure 1 shows the current $I(\Delta\varphi)$ for several values of T_0 . As the strength of the barrier decreases, the current departs from the ideal Josephson behavior and its maximum is displaced towards π . In particular, when T_0 equals unity, the current is given by the formula^{3,8}

$$I(\Delta\varphi) = (e|\Delta|/\hbar) \sin(\Delta\varphi/2), \quad (8)$$

with $-\pi < \Delta\varphi \leq \pi$ and periodicity 2π . This result is clearly not self-consistent, since a uniform phase should be associated with a vanishing equilibrium current, at least in the asymptotic region. Actually, a more detailed calculation reveals⁸ that the current (6) is localized exponentially around the barrier in a region of width $\pi\xi_0/T_0 \sin(\Delta\varphi/2)$, where $\xi_0 = \hbar v_F/\pi|\Delta|$ is the zero-temperature coherence length. This peculiar feature can be traced back to the existence of a localized, current-carrying quasiparticle at the interface.^{3,8} Thus, one finds that the equilibrium current is nonzero near the scattering center and zero in the asymptotic region. In the steady state, this situation clearly involves a violation of charge conservation. Below we show that the relation between self-consistency and current conservation is in fact a general property of the BdG equations.

The time derivative of the charge density operator can be computed by applying (4) and (5) to the relation $\dot{\rho} = (1/i\hbar)[\rho, H]$. The result is

$$\dot{\rho} = \frac{e}{i\hbar} \sum_{nm\sigma} \{ (\epsilon_n - \epsilon_m) (v_n^* v_m - u_n^* u_m) \gamma_{n\sigma}^\dagger \gamma_{m\sigma} + (\epsilon_n + \epsilon_m) \sigma [u_n v_m \gamma_{m\sigma} \gamma_{n,-\sigma} + u_n^* v_m^* \gamma_{n,-\sigma}^\dagger \gamma_{m\sigma}^\dagger] \}, \quad (9)$$

which obviously yields $\langle \dot{\rho} \rangle = 0$, as expected from a stationary scattering description (we have used the properties $\langle \gamma_{n\sigma}^\dagger \gamma_{n'\sigma'} \rangle = f_n \delta_{nn'} \delta_{\sigma\sigma'}$ and $\langle \gamma_{n\sigma} \gamma_{n'\sigma'} \rangle = 0$). Combining (6), (9), and (2), we obtain for the continuity equation^{6,8}

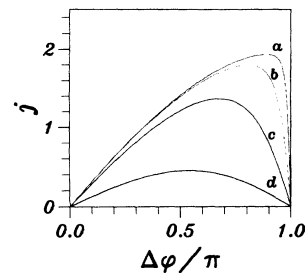


FIG. 1. Current j as function of the phase offset $\Delta\varphi$ for a non-self-consistent solution. The curves labeled a , b , c , and d are for the cases $T_0 = 0.999, 0.99, 0.9$, and 0.4 , respectively.

$$\langle \nabla \cdot \mathbf{j} \rangle + \left\langle \frac{\partial \rho}{\partial t} \right\rangle = \frac{2e}{\hbar} \sum_{n\sigma} \text{Im} \{ \Delta^* u_n v_n^* (1 - 2f_n) \}. \quad (10)$$

By comparing this result with Eq. (3), it becomes clear that charge conservation is only guaranteed when the self-consistency condition is satisfied. In the language of Ref. 11, the BCS-BdG theory is a conserving approximation only for solutions that satisfy the mean-field equations. It is interesting to note that the condensate and quasiparticle contributions to the electric charge are not conserved separately, but only the sum of the two, and if the description is fully self-consistent. The relation between self-consistency and current conservation has also been noticed by Furusaki and Tsukada,¹² who have derived an equation similar to (10), albeit in a more compact operator form that does not separate condensate from quasiparticle contributions. The absence of a clear distinction between condensate and quasiparticle components of the current may lead to an oversimplified picture. It is suggested in Refs. 12 and 13 that current conservation is preserved by the conversion of quasiparticle current into condensate current. While this mechanism may provide an acceptable explanation for simple systems like the N - S boundary at low temperatures and voltages¹³ (the current in the bulk of the N and S sides is carried only by quasiparticles and condensate, respectively), it cannot be considered of general validity. As clearly indicated in Eq. (10), preservation of current is not achieved in general by merely converting quasiparticle current into condensate current, but by truly implementing global self-consistency. In fact, within a non-self-consistent scheme, the source term in Eq. (10) is generally nonzero even at zero temperature and voltage, when no quasiparticles exist.

Before we proceed further a few additional remarks are appropriate. In one dimension, Eq. (7) is incorrect when T_0 is not much smaller than unity. In particular, Eq. (8) is clearly wrong, since no bound quasiparticle should exist in the absence of a barrier. Of course, the main inconsistency lies in the very assumption of an existing phase difference, which cannot be maintained without a scattering obstacle (an abrupt change in the phase cannot survive the implementation of self-consistency). It will be seen in the following section that the appropriate generalization of the concept of phase difference to structures with arbitrary transparency is the phase offset, in terms of which the transparent limit will be quite different from (8). In studies of superconducting quantum point contacts, equations which generalize (7) (Ref. 4) and (8) (Refs. 3 and 5) to the presence of many transverse modes can be found. In these cases, the lack of formal self-consistency is justified. The localized nonzero current corresponds to the current in the vicinity of the point contact and the vanishing of the asymptotic current describes the widening of the contact into the reservoir. Therefore, Eqs. (7) and (8), as well as their multimode generalizations,³⁻⁵ are correct as long as $I_C \ll I_B$. This is the case when the number of propagating modes in the contact is much smaller than the number of modes in the wide leads.

III. STUDY OF THE CROSSOVER

From the discussion in the previous section, it is clear that, in order to achieve a unified view of the crossover from weak to strong superconductivity, one must deal with self-consistent, current-conserving solutions of the BdG equations in which a nonzero current is associated with a linearly varying asymptotic phase, and allow for arbitrary critical currents $I_C \leq I_B$. Unfortunately, the self-consistent resolution of the BdG equations for arbitrary currents is in general a demanding numerical task. By contrast, the formalism of Ginzburg-Landau (GL) provides a relatively simple method to learn about the global properties of those self-consistent solutions. Therefore, our goal in this section is to study the solutions of the GL equation for a one-dimensional superconductor in the presence of a δ -function potential of arbitrary strength. Specifically, we wish to analyze the stationary solutions of the free-energy functional

$$F = \int dx \{ |\nabla \psi|^2 / \kappa^2 - [1 - V_0 \delta(x)] |\psi|^2 + |\psi|^4 / 2 \}, \quad (11)$$

where $\kappa = \lambda / \xi$ [$\lambda(T)$ is the penetration depth and $\xi(T)$ is the temperature-dependent coherence length] and Abrikosov units are used. In these units, $\lambda(T)$ is the unit of length, the order parameter ψ is measured in units of ψ_∞ (absolute value of the bulk order parameter at zero current), and $(\hbar/m)(\psi_\infty^2 / \xi(T))$ is the unit for current. The complete crossover between weak and strong superconductivity will be explored by considering *all values of the scattering strength* $g \equiv \kappa V_0$ ranging from g very large (ideal Josephson behavior) to $g = 0$ (uniform superconductor). In Eq. (11), F must be understood as the free energy per unit area. This model should give a fairly adequate picture of a quasi-one-dimensional superconductor (of width $w \ll \lambda, \xi$) in which a (narrower) point contact⁴ or a normal metal island has been inserted. A clean point contact at low temperature could not be described by (11), since, in the weak superconductivity limit, this structure yields a current-phase relation of the type (8) (Refs. 2, 3, and 14) instead of the usual $\sin(\Delta\varphi)$ behavior. On the other hand, the model (11) is not appropriate for a quantitative description of tunneling barriers because, in the limit of large g , the repulsive potential $V_0 \delta(x)$ yields hard-wall boundary conditions, which do not correspond to a GL description of the metal-insulator interface.⁷ A similar model, with the δ function replaced by a square barrier, was studied by Jacobson,¹⁵ who however focused on the low current limit. Volkov¹⁶ also used a δ function to describe a S - N - S junction but only analyzed the small current case.

If we factorize $\psi(x) = R(x)e^{i\varphi(x)}$, the GL equations take the form

$$\begin{aligned} \kappa^{-2} d^2 R / dx^2 + [1 - V_0 \delta(x)] R - j^2 / R^3 - R^3 &= 0, \\ d\varphi / dx &= \kappa j / R^2, \end{aligned} \quad (12)$$

where the current density j is a conserved number ($I = jA$). We are interested in solutions which satisfy the boundary conditions

$$\begin{aligned} dR(x)/dx &= 0, \\ \varphi(x) &= qx \pm \Delta\varphi/2, \quad \text{for } x \rightarrow \pm\infty. \end{aligned} \quad (13)$$

Current conservation requires the product $R^2\varphi' = \kappa j$ to be constant, which can only be achieved with a *nonzero* $q \equiv \kappa j/R_\infty^2$ in the asymptotic solution. The general solutions for R and φ are of the form

$$\begin{aligned} R^2(x) &= a + b \tanh^2[\kappa u(x_0 + |x|)], \\ \varphi(x) &= qx + \text{sgn}(x) (\arctan\{\beta \tanh[\kappa u(x_0 + |x|)]\} \\ &\quad - \arctan[\beta \tanh(\kappa u x_0)]). \end{aligned} \quad (14)$$

In Eq. (14), $a(2-a)^2 = 8j^2$, with $0 \leq a \leq 2/3$, $b = 1 - 3a/2$, $u = \sqrt{b/2}$, $\beta = \sqrt{b/a}$, and x_0 is obtained from the matching condition at the site of the δ potential, which gives rise to the cubic equation

$$\sqrt{2b}\beta^2 y_0(1 - y_0^2) - g(1 + \beta^2 y_0^2) = 0, \quad (15)$$

where $y_0 \equiv \tanh(\kappa u x_0)$ and thus only the solutions satisfying $0 \leq y_0 \leq 1$ are of interest. The solutions turn out to be uniquely parametrized by the phase offset $\Delta\varphi$, whose general expression is

$$\Delta\varphi = 2[\arctan(\beta) - \arctan(\beta y_0)]. \quad (16)$$

The resulting curve j - $\Delta\varphi$ is displayed in Fig. 2. The inset shows the critical current as a function of g . The Josephson limit is well achieved for $g > 8$ while j_C saturates to $j_B = 2/3\sqrt{3} = 0.385$ as $g \rightarrow 0$. For large g , one finds the ideal Josephson behavior, $j = j_C \sin(\Delta\varphi)$, with $j_C = 1/2g$ taking small values. For $g = 0$, two types of solutions are obtained. One of them is entirely expected: for $\Delta\varphi = 0$, all currents are possible ranging from $j = 0$ to $j = j_B$. These are the solutions of the uniform superconductor in which φ' and R take constant values. The second type of solutions are⁶ *the solitons of the ψ^4 theory defined by (11) for arbitrary values of the current j* . These kinks separate two domains in which the phase varies linearly,

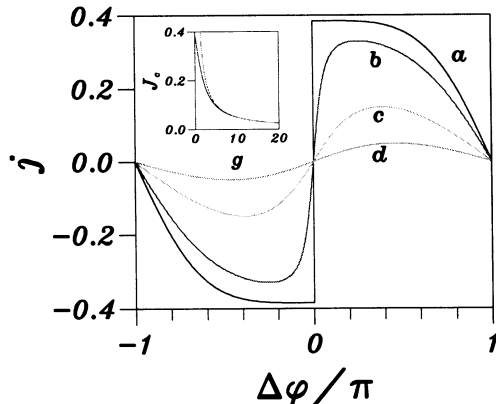


FIG. 2. Current j as function of the phase offset $\Delta\varphi$. The curves are labeled a , b , c , and d for the cases $g = 0, 0.5, 3$, and 10 , respectively. Inset: critical current j_C versus scattering strength g ; solid line gives the the exact result and dotted line corresponds to the Josephson limit $1/2g$.

$$\varphi(x) = qx + \arctan[\beta \tanh(\kappa u x)] \quad (17)$$

with a total phase offset of $\Delta\varphi = 2 \arctan(\beta)$. It is interesting to note that, unlike in the $j = 0$ case, the phase offset (which here plays the role of the soliton charge) can be different from π . These solitonic solutions are equivalent to the saddle-point configurations which were considered by Langer and Ambegaokar¹⁷ in their study of the resistive behavior of one-dimensional superconductors.

Figure 3 shows the behavior of $R(x)$ and $\varphi(x)$ for $g = 0.2$ and two different values of the current. In Fig. 3(a), it is clearly seen that, for $j = 0.01$, the solitonic solution almost vanishes at $x = 0$. For the same kink-like solution, Fig. 3(b) shows that the spatial variation of $\varphi(x)$ is almost negligible except for a steplike feature at $x = 0$ (the phase can be shown to vary in a length scale j/κ if j is small). For $j = 0.35$ (close to j_B), the phase displays a linear increase with x with an offset due to a faster variation in the vicinity of $x = 0$.

The foregoing analysis of Figs. 2 and 3 reveals that, for a given value of the current $j < j_C$, there are two solutions with different values for the phase offset. As the effective barrier strength is turned off (g becomes small), the solutions with smaller (larger) offset [and, correspondingly, a weaker (stronger) depression of the order parameter at the barrier] evolve continuously into the uniform (solitonic) solutions of the scattering-free case.

An interesting feature of the j - $\Delta\varphi$ curves which can be clearly observed in Fig. 2 is that, as the scattering is turned off, the maximum current is displaced towards lower values of $\Delta\varphi$. This is in sharp contrast with the behavior shown in Fig. 1 for the non-self-consistent solutions. It has already been noticed that a superconducting

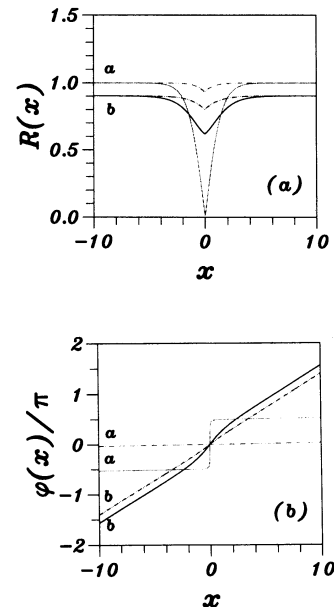


FIG. 3. The amplitude (a) and the phase (b) of the order parameter plotted as a function of position in the $g = 0.2$ case ($\kappa = 1$), for values of the current $j = 0.01$ (curves labeled a) and $j = 0.35$ (curves labeled b).

point contact displays the same behavior as its propagating channels evolve from low to high transmission.⁴ There is of course no contradiction between our results and those obtained for point contacts, since the latter apply only in the limit $I_C \ll I_B$, while the low g curves in Fig. 2 are only relevant in the $I_C \sim I_B$ case.

Baratoff *et al.*¹⁸ considered a S - S' - S structure in which S and S' are two dirty superconductors of differing properties. As a function of the similarity between S and S' , they obtained results which qualitatively resemble those obtained by us. However, their focus was not in the crossover from weak to strong superconductivity, but rather in the qualitative modeling of weak links. In particular, they did not consider the $S = S'$ case and, although the relation to Ref. 17 is noticed, no association is made between the branch to the left of the maximum in the j - $\Delta\varphi$ curve and the trivial solutions of the uniform case. More recently, Kupriyanov¹⁹ has studied the properties of a S - I - S structure by means of the Usadel equations, which apply in the dirty limit. He considers several values of the barrier transparency and obtains results which, after a nontrivial scale transformation (the phase change across the barrier instead of the phase offset is used as a parameter), can be shown to be qualitatively similar to those displayed in Fig. 2. However, in the transparent limit, no mention is made in Ref. 19 of the relation to the solitonic solutions of Ref. 17 nor to the uniform solutions, as discussed here by us.

IV. CROSSOVER IN LONG BRIDGES: BREAKDOWN OF THE AC JOSEPHSON EFFECT

So far we have focused on the relation between the current j and the phase offset $\Delta\varphi$, which uniquely parametrizes the solutions of the GL equations (12). However, it is also convenient to plot the current as a function of the total phase difference χ between two reference points. These two points can be, for instance, the extremes of a superconductor of length L with an effective barrier in its center. A typical case would be that of a narrow bridge connecting two wide reservoirs through smooth contacts beyond which the phase gradient can be safely neglected.² For a given length L , one can compute χ from the relation $\chi = \varphi(L/2) - \varphi(-L/2)$. If $L \gg (\kappa u)^{-1}$, we can approximate

$$\chi \simeq qL + \Delta\varphi. \quad (18)$$

Since $u \rightarrow 0$ as $j \rightarrow j_B$, there is a threshold current $j_{\text{th}}(L)$ above which Eq. (18) does not apply. For $\kappa L \gg 1$, $j_{\text{th}} \simeq j_B(1 - 27/16\kappa^2 L^2)$. In Fig. 4, the resulting curve $j(\chi)$ is plotted for $\kappa L = 10$. In such a case $(j_B - j_{\text{th}})/j_B \simeq 0.016$. It can be observed that, for large g , the ideal Josephson behavior is displayed, while, for sufficiently small g , the current becomes a multivalued function of the total phase χ . The pattern shown in Fig. 4 is actually repeated periodically with a period of 2π . In the case of small g it becomes clear from the comparison with Fig. 2 that the upper branch corresponds to

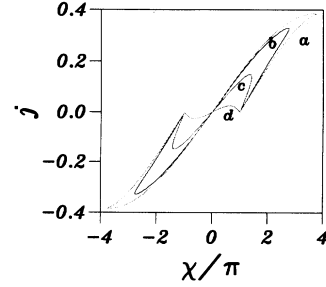


FIG. 4. Same as Fig. 2, for the total phase difference χ between the extremes of a superconductor of length $L = 10$.

solutions with a linearly varying phase ($\Delta\varphi \simeq 0$), while the lower branch is given by the solitonic solutions^{6,17} with a nonzero phase offset. The existence of soliton solutions has later been confirmed by Martin-Rodero *et al.*,²⁰ who have computed numerically the self-consistent solutions of the BdG equations for a linear chain coupled to two Bethe lattices at zero temperature. The discontinuity in the derivative at the top of the $g = 0$ curve in Fig. 4 reflects the discontinuous transition from the uniform to the solitonic shown in Fig. 2. However, this cusp cannot be observed in bridges of finite length since it always lies above the threshold of validity of Eq. (18).

In Fig. 5 we display the phase of the order parameter as a function of the position and the phase offset for $g = 10$ (Josephson limit). When $\Delta\varphi = \pm\pi$, the current is zero. This requires an abrupt jump of $\pm\pi$ at $x = 0$, which is possible because $R(0) = 0$ in these solutions, as can be proven quite generally. These are the phase-slip configurations which permit the existence of the ac Josephson effect. As an external driving voltage is applied between two points on different sides of the junction, the phase is forced to vary at a constant rate

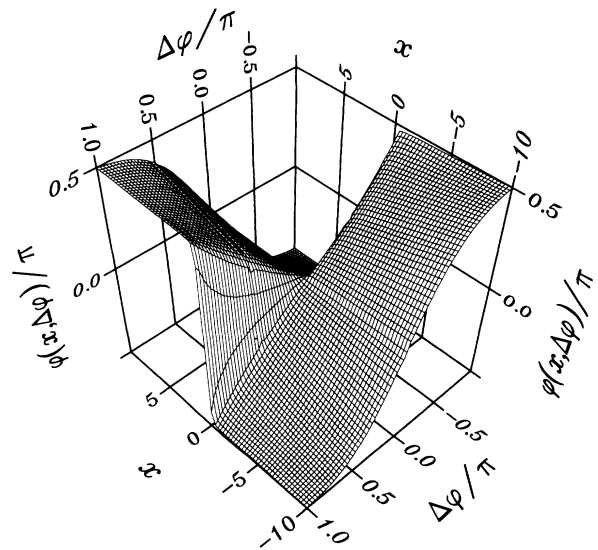


FIG. 5. The phase of the order parameter is plotted as a function of the position x ($\kappa = 1$) and the phase offset $\Delta\varphi$ for the case $g = 10$.

and the whole system responds adiabatically by evolving along the continuous set of stationary solutions. The existence of these step function solutions makes it topologically possible for the phase at every point to increase both monotonically and continuously with time. Since $R(0) = 0$, the two superconductors are completely decoupled and $\Delta\varphi = \pi$ is equivalent to $\Delta\varphi = -\pi$. As the system is driven by the external bias through the different values of $\Delta\varphi$ and reaches the value $\Delta\varphi = \pi$, it automatically reenters through the topologically equivalent configuration with $\Delta\varphi = -\pi$ and the phase at the boundary can continue to increase monotonically. Thus the existence of the ac Josephson effect relies on the ability of the system to undergo adiabatic phase slips under the action of an external bias. It is interesting to note that, at the particular value $\Delta\varphi = \pm\pi$, the configuration of the order parameter is independent of g , since then $R(0) = 0$. In particular, it is identical to the phase-slip configuration in the absence of a barrier, as studied by other authors (see, for example, Refs. 21 and 22).

At zero current, all points on one side of the barrier have the same phase. In particular, $\varphi(x > 0) = \pm\pi/2$ for $\Delta\varphi = \pm\pi$. By contrast, the solutions with nonzero current have an asymptotic phase which grows linearly with position, as shown in Eq. (17). Thus, for sufficiently large x , it is not possible to have $\varphi(x)$ increasing monotonically as $\Delta\varphi$ varies between the two equivalent configurations with $\Delta\varphi = \pm\pi$. As a consequence, the system cannot respond adiabatically to a constant voltage being applied at points that are sufficiently distant from the junction. The only choice for the system will be to undergo nonadiabatic, fluctuating processes of the type studied by Langer and Ambegaokar¹⁷ (albeit with $g \neq 0$), which will originate a resistive behavior. The threshold for this type of response is given by the condition

$$\left. \frac{\partial\varphi(x_b)}{\partial\Delta\varphi} \right|_{\Delta\varphi=\pi} = 0 \quad \text{at } \Delta\varphi = \pi. \quad (19)$$

If one identifies $x_b = L/2$ and $\varphi(x_b) = \chi/2$, this is also the condition for the onset of bivaluedness in the $j(\chi)$ curve of Fig. 4, which requires $\partial\chi/\partial j = 0$ at $\chi = \pi$ (note that $\partial j/\partial\Delta\varphi \neq 0$ at $\Delta\varphi = \pi$). Thus, if the electrodes are applied at points $|x| > x_b$, there is a breakdown of the ac Josephson effect due to the fundamental inability of the system to respond adiabatically to that particular type of external constraint.

Let us estimate the breakdown length x_b . For large g , one can show that $x_b \simeq g/\kappa$. We notice at this point that the value of the parameter g can be adjusted to a realistic setup by exploiting the relation

$$g = 1.30(j_B/j_C), \quad (20)$$

which applies in the Josephson regime, and noting that $j_B/j_C = I_B/I_C$. We have considered explicitly four types of structures which are known to display a standard $\sin(\Delta\varphi)$ behavior in the Josephson limit for T close to T_c : (a) a tunnel junction with average transmission T_0 for the Fermi electrons, (b) a clean point contact with average transmission T_0 , (c) a narrow bridge between

two superconductors made of a dirty normal metal of length L and coherence length ξ_N at $T \simeq T_c$, and (d) a S - N - S structure without current concentration (N and S have the same width). Cases (a) and (b) fall within the same category in the GL limit, with an expression $I_C = \pi\Delta^2(T)/4eR_Nk_B T$ for the critical current.² Noting that, for T close to T_c , the gap function and the order parameter are related by²³ $\psi = 0.326\sqrt{n}\Delta/k_B T_c$, where n is the electron number density, we arrive at

$$g^{-1} \simeq 2.0 T_0 [\xi(T)/\xi_0]. \quad (21)$$

For case (c), the critical current is² $I_C = [4\Delta^2(T)/\pi e R_N k_B T](L/\xi_N) \exp(-L/\xi_N)$, if $L \gg \xi_N$. Thus one obtains

$$g^{-1} \simeq 3.23 T_0 [\xi(T)/\xi_0](L/\xi_N) \exp(-L/\xi_N). \quad (22)$$

For a S - N - S structure without current concentration, the critical current is²⁴ $I_C = A(e\hbar n/2m)(|T - T_c|/T_c)[\xi_N/\xi^2(T)] \exp(-L/\xi_N)$. As a consequence,

$$g^{-1} \simeq 10.6 [\xi_N/\xi(T)] \exp(-L/\xi_N). \quad (23)$$

Shifting to real units, we arrive at the relations

$$\begin{aligned} x_b &\simeq 0.50 \xi_0/T_0, & \text{(a), (b),} \\ x_b &\simeq 0.31 (\xi_0 \xi_N/T_0 L) e^{L/\xi_N}, & \text{(c),} \\ x_b &\simeq 0.094 [\xi^2(T)/\xi_N] e^{L/\xi_N}, & \text{(d),} \end{aligned} \quad (24)$$

for the maximum distance at which a constant voltage can be applied in order to observe the ac Josephson effect.

V. CONCLUSIONS

We have studied the nature of the crossover from ideal Josephson behavior between two weakly coupled superconductors to bulk superconducting flow in a perfect superconducting lead. We have argued that a self-consistent resolution of the BdG equations is mandatory in a microscopic study of the crossover and have proved that charge conservation is only guaranteed when the requirement of self-consistency is satisfied. We have performed a study of the crossover by solving exactly the Ginzburg-Landau equation for a one-dimensional superconductor in the presence of a δ -function potential of arbitrary strength. The pairs of Josephson solutions with equal current have their scattering-free counterparts in the pairs formed by a uniform and a solitonic solution. This relation has allowed us to understand some aspects of the multivalued current-phase relation in narrow bridges. The complete knowledge of the set of stationary solutions for different values of the scattering strength g has helped us to gain a more detailed understanding of the adiabatic response to a constant external bias, which has been shown to rely on the feasibility of adiabatic phase slips. If a voltage is applied at points which are sufficiently far from the junction, there is a breakdown of the Josephson effect due to the intrinsic impossibility of changing adiabatically the phase at a distant point in a continuous and monotonic manner.

Note added. Bagwell²⁵ has independently derived Eq. (10). He also presents an alternative, nonconserving expression for the electric current. We wish to emphasize here the unambiguous character of Eqs. (4), (5), (6), (9), and (10), which are derived from Eq. (2) and from a careful introduction of the quasiparticle field in the exact pairing interaction. The onset of irreversibility in the ac Josephson effect which is discussed in Sec. IV has also been pointed out by Goldbart,²⁶ who in addition has predicted the existence of a spin current analog in punctured magnetic films. Zapata²⁷ proved recently that the δ -function model (11) provides a rigorous quantita-

tive description of a S - N - S structure with $g(T)$ given by Eq. (23).

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