Pearl's vortex near the film edge

Vladimir G. Kogan
Ames Laboratory and Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011
(Received 17 January 1994; revised manuscript received 31 March 1994)

The problem of a vortex situated near the edge of a thin superconducting film is solved. The flux associated with the vortex is shown to be less than the flux quantum in a broad domain adjacent to the film edge. The suppression of the vortex flux is strong in narrow thin-film bridges and scales with the ratio of the strip width to the film penetration depth. The magnetic moment of a vortex in a finite thin film is shown to depend on the sample size and on the vortex position.

The recently renewed interest in thin superconducting films is primarily due to the high critical current densities j_c which approach values of the depairing current $c\phi_0/16\pi^2\lambda_L^2\xi$ ($\phi_0 = hc/2|e|$ is the flux quantum, λ_L is the London penetration depth, and ξ is the coherence length).[1–4] The question of high j_c is related to the barrier for the entry of vortices at the film edges; the latter cannot be studied without precise knowledge of the magnetic structure and energy of a vortex near the film edge. Also, this knowledge is needed in studies of vortices in thin-film superconducting quantum interference devices (SQUID's) and other extended Josephson structures.[5]

As was first stressed by Pearl,[6] the situation in a thin film differs from that of a bulk since vortices in films interact mostly via the stray fields in the surrounding space. In this paper, a vortex near the edge of a thin film in zero applied field is considered. A thorough discussion of the current flow in thin films was given by Likharev[7] and recently Brandt;[8] however, the basic question of the magnetic structure of a single vortex near the film edge is still open. At first sight, the problem is similar to that of a vortex parallel to the flat surface of a bulk superconductor; the solution for a film, however, is not that simple since—as is shown below—the method of images *per se* cannot be used.

Let us consider a film of thickness $d \ll \lambda_L$ occupying the $x > 0$ part of the xy plane. For a vortex at $x = a$, $y = 0$, the London equations for the film interior read

$$\mathbf{h} + 4\pi\lambda_L^2 \mathrm{curl}\mathbf{j}/c = \phi_0 \hat{\mathbf{z}}\delta(x-a, y). \quad (1)$$

Averaging this over the thickness d, one obtains

$$h_z + 4\pi\lambda_p \mathrm{curl}_z \mathbf{g}/c = \phi_0 \delta(\mathbf{r}-\mathbf{a}), \quad (2)$$

where $\mathbf{g}(\mathbf{r})$ is the sheet current density, $\mathbf{r} = (x, y)$, $\mathbf{a} = (a, 0)$, and $\lambda_p = \lambda_L^2/d$ is the Pearl's film penetration depth. Other components of Eq. (1) turn identities after averaging. Equation (2) plays a major role in physics of thin superconducting films;[9] it is valid everywhere at the film except a narrow belt of a width ξ adjacent to the edge, where the London equation (1) no longer holds.

The distribution $\mathbf{g}(\mathbf{r})$ can be found by solving Eq. (2) combined with the continuity equation and the Biot-Savart integral which relates the field h_z to the surface current:

$$\mathrm{div}\mathbf{g} = 0\,, \quad h_z(\mathbf{r})c = \int d^2\mathbf{r}'[\mathbf{g}(\mathbf{r}') \times \mathbf{R}/R^3]_z\,; \quad (3)$$

$\mathbf{R} = \mathbf{r} - \mathbf{r}'$. The specific feature of the thin-film limit should be noted: since all transverse derivatives $\partial/\partial z$ are large relative to the tangential $\partial/\partial\mathbf{r}$, the Maxwell equation $\mathrm{curl}\mathbf{h} = 4\pi\mathbf{j}/c$ is reduced to conditions relating the sheet current to discontinuities of the tangential field:

$$2\pi g_x/c = -h_y(+0),\ 2\pi g_y/c = h_x(+0)\,. \quad (4)$$

Here, $h_{x,y}(+0) = -h_{x,y}(-0)$, and ± 0 stand for the upper and lower faces of the film. The field component perpendicular to the film, h_z, is related to currents by an integral (3), rather then by a differential equation.

Equations (2) and (3) suffice for the determination of the current distribution. To this end, it is convenient to deal with a scalar function $G(\mathbf{r})$ such that $\mathbf{g} = \mathrm{curl}G\hat{\mathbf{z}}$:[8]

$$g_x = \partial_y G\,,\ g_y = -\partial_x G\,. \quad (5)$$

Then the first of Eqs. (3) is satisfied.

The kernel $\mathbf{R}/R^3$ of the Biot-Savart integral is strongly singular. To reduce the degree of singularity one can write $\mathbf{R}/R^3 = \nabla'(1/R)$ (the prime specifies $\mathbf{r}'$ as the variable of differentiation) and integrate by parts. Equation (3) then gives

$$h_z c = \int_{x'>0} \frac{d^2\mathbf{r}'}{R}\mathrm{curl}_z\mathbf{g}(\mathbf{r}') + \int_{-\infty}^{\infty} dy' \left(\frac{g_y(\mathbf{r}')}{R}\right)_0 \quad (6)$$

where the subscript 0 stands for $x' = 0$. Substituting Eqs. (5) and (6) in (2) one obtains

$$\int_{x'>0} d^2\mathbf{r}' \frac{\nabla^2 G(\mathbf{r}')}{R} + \int_{-\infty}^{\infty} dy' \left(\frac{\partial_{x'} G(\mathbf{r}')}{R}\right)_0 + 4\pi\lambda_p \nabla^2 G(\mathbf{r}) = -c\phi_0\delta(\mathbf{r}-\mathbf{a}). \quad (7)$$

This is to be solved with respect to $G(x, y)$ for $x > 0$ subject to boundary conditions $\mathbf{g}(\infty) = 0$ and the vanishing normal component of the current at the film edge $g_x(0, y) = 0$. These conditions imply a constant G at infinity and at the film edge; one can set $G = 0$ at the film

0163-1829/94/49(22)/15874(5)/$06.00

boundaries since only the derivatives of G have physical meaning.

Solving the integro-differential equation (7) on the half-plane seems difficult; one can get around the difficulty by extending formally the domain where G is defined to the whole plane. To have $G = 0$ at the edge $x = 0$, one sets $G(-x, y) = -G(x, y)$; the source (a vortex) at the right of Eq. (7) should then be complemented by a singularity of an opposite sign (antivortex) situated at ($\mathbf{r} = -\mathbf{a}$), to assure the odd solution $G(x)$.

As the next step one performs the Fourier transform (FT) of Eq. (7) with respect to the variable $\mathbf{r}$. In the integral terms of Eq. (7), only R depends on $\mathbf{r}$: $\mathrm{FT}(1/R) = (2\pi/k)e^{-i\mathbf{kr}'}$. Then,

$$\int_0^\infty dx' e^{-ik_x x'}(\partial_{x'}^2 - k_y^2)G(x', k_y) + \int_{-\infty}^\infty dy'[\partial_{x'}G(\mathbf{r}')]_0 e^{-ik_y y'} - 2\lambda_p k^3 G(\mathbf{k}) = i\frac{c\phi_0}{\pi} k \sin k_x a\,. \tag{8}$$

Had the lower limit of the first term here been $-\infty$, the integral would be just the FT of $\nabla^2 G$. To evaluate the complementary part for this FT, $\int_{-\infty}^0 dx'$, one writes Eq. (8) for $-k_x$, replaces the integration variable x' with $-x'$, and utilizes $G(-x') = -G(x')$:

$$\int_0^{-\infty} dx' e^{-ik_x x'}(\partial_{x'}^2 - k_y^2)G(x', k_y) + \int_{-\infty}^\infty dy'[\partial_{x'}G(\mathbf{r}')]_0 e^{-ik_y y'} + 2\lambda_p k^3 G(\mathbf{k}) = -i\frac{c\phi_0}{\pi} k \sin k_x a\,. \tag{9}$$

Subtracting now Eq. (9) from (8), one obtains

$$G(\mathbf{k}) = \frac{2c\phi_0}{i\pi}\frac{\sin k_x a}{k(1+4\lambda_p k)}. \tag{10}$$

It is worth noting that for a vortex-antivortex pair at a distance $2a$ in an infinite film, the result is different:

$$G_0(\mathbf{k}) = \frac{c\phi_0}{i\pi}\frac{\sin k_x a}{k(1+2\lambda_p k)}. \tag{11}$$

Thus, the "plain" method of images cannot be applied for the problem at hand. Still, at short distances ($k\lambda_p \gg 1$) the current distribution described by Eq. (10) is the same as that given by Eq. (11).

In the real space, the solution (10) reads

$$G(\mathbf{r}) = \frac{c\phi_0}{16\pi\lambda_p}\left[\Phi_0\left(\frac{|\mathbf{r}-\mathbf{a}|}{4\lambda_p}\right) - \Phi_0\left(\frac{|\mathbf{r}+\mathbf{a}|}{4\lambda_p}\right)\right], \tag{12}$$

where $\Phi_0 = Y_0 - \mathbf{H}_0$, and $\mathbf{H}_0$ and Y_0 are the Struve and the second kind Bessel functions in the notation of Ref. 12. Lines of the current $\mathbf{g} = \mathrm{curl} G\hat{z}$ are contours of constant $G(x, y)$. Far from the vortex core,

$$G(\mathbf{r}) = \frac{c\phi_0}{\pi^2}\frac{ax}{r^3}, \tag{13}$$

where $r^2 = x^2 + y^2$. In the vicinity of the core,

$$G = \frac{c\phi_0}{16\pi\lambda_p}\left[\frac{2}{\pi}\ln\frac{8\lambda_p}{e^\gamma \rho} + \Phi_0 - \left(\Phi_1 + \frac{2}{\pi}\right)\frac{x-a}{2\lambda_p}\right]; \tag{14}$$

here $\rho = \sqrt{(x-a)^2 + y^2}$ is the distance from the core, $\gamma = 0.577...$, functions Φ are taken at $a/2\lambda_p$, and the terms of the order ρ^2/λ_p^2 are neglected.

In some applications the Fourier representation is more convenient. According to (5) and (10), the current is given by

$$\mathbf{g}(x > 0, y) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}(i\mathbf{k}\times\hat{\mathbf{z}})G(\mathbf{k})e^{i\mathbf{kr}},$$
$$\mathbf{g}(x < 0, y) = 0\,. \tag{15}$$

The function $\mathbf{g}(x, y)$ is discontinuous at $x = 0$; evaluating its FT one finds

$$g_x(\mathbf{k}) = k_y \int_{-\infty}^\infty \frac{dq_x}{2\pi}\frac{G(q_x, k_y)}{k_x - q_x - i\delta},$$
$$g_y(\mathbf{k}) = -\int_{-\infty}^\infty \frac{dq_x}{2\pi}\frac{q_x G(q_x, k_y)}{k_x - q_x - i\delta}, \tag{16}$$

where $\delta \to +0$ indicates how the pole at $q_x = k_x$ should be treated. The field $\mathbf{h}(\mathbf{r}, z)$ can now be evaluated using the Biot-Savart law:

$$\mathbf{h}(\mathbf{Q}) = \frac{4\pi}{icQ^2}[\mathbf{g}(\mathbf{k})\times\mathbf{Q}], \quad \mathbf{Q} = (\mathbf{k}, k_z)\,. \tag{17}$$

As an example, let us evaluate the flux $\phi_z(a)$ through the film due to the vortex at $\mathbf{a}$, i.e., integrate Eq. (2) over the film at $x > 0$:

$$\phi_z(a) = \phi_0 + \frac{4\pi\lambda_p}{c}\int_{-\infty}^\infty dy\, g_y(x = \varepsilon, y)\,, \tag{18}$$

where $\varepsilon \to +0$ (the integral is over the *superconducting* side of the edge). Writing here $g_y(\varepsilon, y)$ as a Fourier integral with $g_y(\mathbf{k})$ given in Eqs. (16) and (10), one has for the last term in Eq. (18)

$$\frac{2i\phi_0\lambda_p}{\pi^2}\int_{-\infty}^\infty \frac{dq_x q_x \sin q_x a}{|q_x|(1+4\lambda_p|q_x|)}\int_{-\infty}^\infty \frac{dk_x e^{ik_x\varepsilon}}{k_x - q_x - i\delta}. \tag{19}$$

The last integral is taken by closing the integration path in the upper half-plane of the complex k_x; it is $2\pi i$. One then obtains[11]

$$\frac{\phi_z(a)}{\phi_0} = 1 - \frac{2}{\pi}f\left(\frac{a}{4\lambda_p}\right), \tag{20}$$

where the monotonic function $f(\eta) = \mathrm{Ci}(\eta)\sin\eta - \mathrm{si}(\eta)\cos\eta$ (the notation of Ref. 12) has the following properties: $f(\eta \ll 1) \approx \pi/2 + \eta(\ln\eta + \gamma - 1)$ and $f(\eta \gg 1) \approx 1/\eta$. Thus, the flux $\phi_z(a)$ goes to zero when the vortex approaches the edge and to ϕ_0 at large distances, as expected. The reduction of the flux carried by vortices in restricted geometries has been discussed in the literature.[13] However, unlike the bulk where the full value of ϕ_0 is reached (exponentially fast) on distances about the London length λ_L from the surface, in thin films the change from 0 to ϕ_0 is quite slow: $\phi_z(a)$ reaches 90% of ϕ_0 at about $a \approx 24\lambda_p$. Since λ_p can be 10 or 100 micron large in sufficiently thin films,[3] the width of the domain where the vortex flux is substantially less than the flux quantum ϕ_0 can be *macroscopic*. Physically, this happens because the short-range exponential screening of the bulk superconductors is replaced in thin films by the long-range Coulomb interaction via vacuum.

In applications one might be interested in the vortex flux ϕ_z^L which crosses the half-plane $x < 0$ in the free space to the left of the film edge:

$$\phi_z^L = \int_{-\infty}^{\infty} dy\, A_y(x = -\varepsilon, y, z = 0) = A_y(x = -\varepsilon, k_y = 0, z = 0)\,, \qquad (21)$$

where the vector potential $\mathbf{A}(\mathbf{Q}) = 4\pi\mathbf{g}(\mathbf{k})/cQ^2$ and the arguments of $A_y(x, k_y, z)$ indicate in which variable the FT is performed. Calculation similar to that leading to Eq. (20) results in $\phi_z^L = -\phi_z$, i.e., the whole flux ϕ_z crossing the film from the lower half-space to the upper one, returns back round the edge. Moreover, the flux $\phi_z(z)$ through any plane z =const is zero. Indeed, using Eq. (17) we have

$$\phi_z(z) = \int d^2\mathbf{r}\, h_z(\mathbf{r}, z) = \frac{2\pi}{ic}\int d^2\mathbf{k}\frac{g_x(\mathbf{k})k_y - g_y(\mathbf{k})k_x}{k}\, e^{-kz}\delta(\mathbf{k}), \qquad (22)$$

where integrations over $\mathbf{r}$ and k_z yielded $\delta(\mathbf{k})$ and $\pi\, e^{-kz}/k$ (for $z > 0$), respectively. With the help of Eqs. (16) and (10), the integral at the right-hand side (RHS) of (22) is further transformed:

$$\int_{-\infty}^{\infty}\frac{dq_x}{2\pi}\left[\frac{(k_y^2 + k_x q_x)G(q_x, k_y)}{k(k_x - q_x - i\delta)}\right]_{\mathbf{k}\to 0} = -\frac{2c\phi_0}{i\pi}\left(\frac{k_x}{k}\right)_{\mathbf{k}\to 0}\int_{-\infty}^{\infty}\frac{dq_x}{2\pi}\frac{\sin q_x a}{|q_x|(1 + 4\lambda_p|q_x|)} = 0. \qquad (23)$$

This contrasts the case of a vortex in an infinite film where $\phi_z = \int d^2\mathbf{r} h_z(\mathbf{r}, z) = \phi_0$ for any z.

Another quantity of interest is the vortex magnetic moment $\mu = \int(\mathbf{r}\times\mathbf{g})d^2\mathbf{r}/2c$. In an infinite film, μ diverges due to the slow decay of vortex currents ($g \propto r^{-2}$) at distances $r \gg \lambda_p$.[6] For the half-plane film, $2c\mu_z = -\int_{x>0}\mathbf{r}\cdot\nabla G\, d^2\mathbf{r}$ can easily be rearranged to

$$2c\mu_z = -[x\, G(x, k_y = 0)]_0^\infty + 2\int_0^\infty dx\, G(x, k_y = 0), \qquad (24)$$

where G is evaluated using the FT, Eq. (10):

$$\frac{\pi^2}{c\phi_0}G(x, k_y = 0) = \ln\frac{x + a}{|x - a|} - g\left(\frac{|x - a|}{4\lambda_p}\right) + g\left(\frac{x + a}{4\lambda_p}\right) \qquad (25)$$

Here $g(\eta) = -df/d\eta$ with $f(\eta)$ defined above [see Eq. (20), the notation of Ref. 12]. The first term at the RHS of Eq. (24) is then $-2c\phi_0 a/\pi^2$. The integral in the second term is logarithmically divergent since $\ln[(x + a)/(x - a)] \simeq 2a/x$ at $x \gg a$. One therefore should introduce a finite sample size L to cut off the divergence:

$$\mu = \frac{2\phi_0}{\pi^2}a\ln\frac{L\sqrt{e}}{a} - \frac{4\phi_0}{\pi}\lambda_p\left[1 - \frac{2}{\pi}f\left(\frac{a}{4\lambda_p}\right)\right]. \qquad (26)$$

Thus, the vortex magnetic moment depends logarithmically on the sample size and changes with the distance a from the edge. Near the edge, the moment is proportional to a:

$$\mu(a \ll 4\lambda_p) = \frac{2\phi_0}{\pi^2}a\left(\ln\frac{L}{4\lambda_p} + \gamma - 0.5\right). \qquad (27)$$

This dependence is somewhat slower at intermediate distances:

$$\mu(4\lambda_p \ll a \ll L) = \frac{2\phi_0}{\pi^2}a\ln\frac{L\sqrt{e}}{a}. \qquad (28)$$

At large distances of the order L, $\mu \approx \phi_0 L/\pi^2$.

Let us now turn to the question of energy, starting with the general situation of a vortex in a *finite* bulk sample. The energy consists of the London energy (magnetic + kinetic) inside the sample, $\epsilon^{(i)} = \int[h^2 + (4\pi\lambda_L j/c)^2]dV/8\pi$ and the magnetic energy outside, $\epsilon^{(a)} = \int h^2 dV/8\pi$. Since $\mathrm{curl}\mathbf{h} = \mathrm{div}\mathbf{h} = 0$ outside, one can introduce a potential so that $\mathbf{h} = \nabla\varphi$ and $\nabla^2\varphi = 0$. Then, for the potential gauged to zero at ∞ (which is possible in zero applied field) one has

$$8\pi\epsilon^{(a)} = \oint \varphi\mathbf{h}\cdot d\mathbf{S}\,, \qquad (29)$$

where the integral is over the sample surface with $d\mathbf{S}$ directed inward the material. The London part is transformed integrating by parts the kinetic term: $8\pi\epsilon^{(i)} = (4\pi\lambda_L^2/c)\oint(\mathbf{j}\times\mathbf{h})\cdot d\mathbf{S}$ where the integral is over the sample surface and the surface of the vortex core. The integral over the sample surface is further transformed: $\oint d\mathbf{S}\cdot(\mathbf{j}\times\nabla\varphi) = \oint d\mathbf{S}\cdot\varphi(\nabla\times\mathbf{j})$ (see Ref. 10). Combining the result with $\epsilon^{(a)}$ of (29), one obtains $\oint d\mathbf{S}\cdot\varphi(\mathbf{h} + 4\pi\lambda_L^2\mathrm{curl}\mathbf{j}/c)$. The expression in the parentheses is the LHS of the London equation (1), i.e., it is $\phi_0\hat{\mathbf{v}}\delta^{(2)}(\mathbf{r} - \mathbf{r}_v)$ where $\hat{\mathbf{v}}$ is the direction of the vortex crossing the surface at the point $\mathbf{r}_v$, and $\delta^{(2)}(\mathbf{r}-\mathbf{r}_v)$ is the two-dimensional δ function. We then obtain:

$$8\pi\epsilon = \phi_0[\varphi(\mathbf{r}_{\mathrm{ent}}) - \varphi(\mathbf{r}_{\mathrm{ex}})] - \frac{4\pi\lambda_L^2}{c}\oint_{\mathrm{core}} d\mathbf{S}\cdot(\mathbf{h}\times\mathbf{j})\,, \qquad (30)$$

with $\mathbf{r}_{\mathrm{ent}}$ and $\mathbf{r}_{\mathrm{ex}}$ being the positions of the vortex entry and exit at the sample surface (the vortex is assumed to cross the sample surface at right angles; otherwise, one should multiply the potentials by cosines of corresponding angles).

For thin films, the integral over the core surface ($\propto d$) can be neglected in Eq. (30). The potential $\varphi(\mathbf{r}, z = +0)$ at the upper face of the film, is simply related to the function $G(\mathbf{r})$ introduced above; Eqs. (4) and (5) yield

$$\varphi(\mathbf{r},+0) = -2\pi G(\mathbf{r})/c \quad (31)$$

(a possible additive constant is set zero since both φ and G are gauged to zero at ∞). Therefore, the energy in question is

$$\epsilon = -\frac{\phi_0}{4\pi}\varphi(\mathbf{a},+0) = \frac{\phi_0}{2c}G(\mathbf{r}\to\mathbf{a})\,. \quad (32)$$

As is seen from Eq. (14), $G(\mathbf{r}\to\mathbf{a})$ is logarithmically divergent, and one has to introduce the standard cutoff at $\rho \approx \xi$:

$$\epsilon = \frac{\phi_0^2}{16\pi^2\lambda_p}\left[\ln\frac{8\lambda_p}{e^\gamma\xi} - \frac{\pi}{2}\Phi_0\left(\frac{a}{2\lambda_p}\right)\right], \quad (33)$$

where Φ_0 is defined following Eq. (12). As expected, the London description fails when the distance a from the edge is on the order ξ. The energy (33) describes the Coulomb attraction of the vortex and the edge at $a \gg 2\lambda_p$: $-\partial_a\epsilon \simeq -\phi_0^2/4\pi a^2$; compare this with a vortex in the bulk parallel to the sample surface where the interaction $\propto \exp(-a/\lambda_L)$. At short distances, $a \ll 2\lambda_p$, the force is $-\phi_0^2/16\pi^2\lambda_p a$ (the a dependence is the same as in the bulk case, but the coefficient is much smaller).

The formal method employed above is designed for the semi-infinite film; it cannot be applied to other film shapes and in particular to thin-film strips, the case of practical interest. The problem seems difficult for strips of an arbitrary width W; however, for narrow strips, $W \ll \lambda_p$, it becomes manageable again because the field h_z in Eq. (2) can be disregarded with respect to the term with current derivatives.[7] One can see that at short distances from the vortex core, $r \ll \lambda_p$, the field $h_z \sim \phi_0/r\lambda_p$, the sheet current $g \sim c\phi_0/r\lambda_p$, and the ratio of the second term in Eq. (2) to the first one is of the order $\lambda_p/r \gg 1$. Then, Eq. (7) for G reads

$$\nabla^2 G = -(c\phi_0/4\pi\lambda_p)\delta(\mathbf{r}-\mathbf{a}) \quad (34)$$

with the boundary condition $G = 0$ at the strip edges $x = 0$ and $x = W$. Thus, the problem is equivalent to that of the electrostatic potential of a linear charge $q = c\phi_0/16\pi^2\lambda_p$ situated at $\mathbf{r} = \mathbf{a}$ between grounded metal plates at $x = 0$ and $x = W$ (see, e.g., Refs. 14 or 15):

$$\tanh\frac{G}{2q} = \frac{\sin(\pi a/W)\sin(\pi x/W)}{\cosh(\pi y/W) - \cos(\pi a/W)\cos(\pi x/W)}\,. \quad (35)$$

A rich library of the 2D electrostatics is instrumental in solving for vortex currents for a variety of film shapes with linear dimensions less than λ_p; see, e.g., Ref. 15 for the solution of Eq. (34) for a rectangular film.

The FT of the solution (35) can be obtained directly from Eq. (34): Extend the domain where G is defined to the whole plane and introduce two systems of images to satisfy the boundary conditions at the edges, so that the source term reads $-4\pi q\delta(y)[\sum_m \delta(x-x_m) - \sum_n \delta(x-x_n)]$ with $x_m = a + 2mW$ and $x_n = -a + 2nW$ (m, n are integers). The function $G(x)$ is then periodic with a unit cell $(-W, W)$ with the FT

$$G(\mathbf{k}) = \frac{c\phi_0 \sin k_x a}{2i\lambda_p k^2 W}\sum_{m=-\infty}^{\infty}\delta\left(k_x - \frac{\pi}{W}m\right). \quad (36)$$

To evaluate the flux ϕ_z through the strip, one starts with the Biot-Savart equation (3) and expresses h_z in terms of G:

$$h_z = \frac{1}{c}\int d^2\mathbf{r}'\,\nabla' G(\mathbf{r}')\cdot\frac{\mathbf{R}}{R^3} = -\frac{1}{c}\int d^2\mathbf{r}'\,\frac{G(\mathbf{r}')}{R^3}\,, \quad (37)$$

where the integral is over the strip $0 < x' < W$ (see Ref. 16). Then, one obtains for $\phi_z = \int h_z d^2\mathbf{r}$

$$\frac{c\phi_z}{2W} = \int d^2\mathbf{r}'\frac{G(\mathbf{r}')}{x'(W-x')} = \int_0^W \frac{dx}{x(W-x)}J(x)\,, \quad (38)$$

where $J(x) = \int_{-\infty}^{\infty} G(x,y)dy$ can be evaluated by integrating Eq. (34) over y:

$$J''(x) + [\partial_y G]_{-\infty}^{\infty} = -4\pi q\delta(x-a)\,. \quad (39)$$

The term $\partial_y G = g_x$ vanishes at $y = \pm\infty$, and therefore J is a linear function of x. Since $J(0) = J(W) = 0$, the function $J(x)$ (which is continuous[17]) must have a break at $x = a$:

$$J(x) = J_0\cdot\begin{Bmatrix} x, & 0 < x < a, \\ a\frac{W-x}{W-a}, & a < x < W \end{Bmatrix}. \quad (40)$$

Coefficient J_0 can be obtained making use of Gauss' theorem for the "potential" G: take a surface containing the linear "charge" q as made of two yz planes at $x = x_1 < a$ and at $x = x_2 > a$:

$$\int_{-\infty}^{\infty} dy\,\partial_x G(x_1,y) - \int_{-\infty}^{\infty} dy\,\partial_x G(x_2,y) = 4\pi q, \quad (41)$$

or $J'(x_1) - J'(x_2) = 4\pi q$. This yields

$$J_0 = \frac{c\phi_0}{4\pi\lambda_p}\,\frac{W-a}{W}. \quad (42)$$

Substituting $J(x)$ thus obtained in Eq. (38), one has

$$\phi_z(a) = \frac{\phi_0}{2\pi\lambda_p}\left(a\ln\frac{W-a}{a} + W\ln\frac{W}{W-a}\right). \quad (43)$$

The flux $\phi_z(a)$ turns zero at the edges (as $-a\ln a$ at $a \to 0$) and reaches maximum of $\phi_0 W\ln 2/2\pi\lambda_p$ in the strip middle. Thus, the flux carried by a vortex in a narrow ($W \ll \lambda_p$) thin-film bridge scales with the ratio W/λ_p, depends on the vortex position, and is much smaller than the flux quantum.

Similar to ϕ_z, one can estimate the flux which goes around the film edges from the half-space above the strip to the lower half-space. With the help of Eq. (37), one obtains for the flux $\phi_z^L = \int_{x<0} h_z d^2\mathbf{r}$ crossing the plane $z = 0$ left of the edge $x = 0$:

$$\phi_z^L = -\frac{2}{c}\int d^2\mathbf{r}'\,\frac{G(\mathbf{r}')}{x'} = -\frac{\phi_0 a}{2\pi\lambda_p}\ln\frac{W}{a}\,. \quad (44)$$

The flux ϕ_z^L drops fast when the vortex approaches the left edge ($\phi_z^L \propto a\ln a$), whereas the decrease is slow for the vortex moving toward the opposite edge: $\phi_z^L \propto (W - a)$. The flux $\phi_z^R = \int_{x>W} h_z d^2\mathbf{r}$ at the right of $x = W$ is evaluated in a similar manner to show that $\phi_z^L+\phi_z^R+\phi_z=0$. Moreover, one can show that the total flux crossing any plane $z=$const vanishes, unlike the case of a vortex in an infinite film where it is ϕ_0.

For the vortex magnetic moment in the narrow strip one obtains utilizing again the representation of $\mathbf{g}$ in terms of G

$$\mu_z(a)=\frac{\phi_0}{8\pi\lambda_p}(W-a)a\,. \tag{45}$$

Note that unlike the bulk situation, μ is size and position dependent (as for a vortex in a semi-infinite film).

It is worth noting that though a vortex in a strip carries less than ϕ_0 of a flux, the interaction of such a vortex with the transport current is still described in terms of the common Lorentz force. Consider as an example the interaction energy of two vortices at $\mathbf{a}_1=(a_1,0)$ and $\mathbf{a}_2=(a_2,0)$; according to Eq. (32)

$$\epsilon_{\rm int}=\frac{\phi_0}{2c}[G_1(a_2)+G_2(a_1)]=\frac{\phi_0}{c}G_1(a_2)\,. \tag{46}$$

Then the force of the first vortex upon the second is

$$-\frac{\partial\epsilon_{\rm int}}{\partial a_2}=-\frac{\phi_0}{c}\frac{\partial G_1}{\partial a_2}=-\frac{\phi_0}{c}g_{1y}(a_2)\,. \tag{47}$$

The energy of a vortex in a narrow bridge is obtained directly from Eqs. (32) and (35) by introducing the cutoff at $|\mathbf{r}-\mathbf{a}|=\xi$ to a divergent $G(\mathbf{r}\to\mathbf{a})$:[15]

$$\epsilon(a)=\frac{\phi_0^2}{16\pi^2\lambda_p}\ln\left(\frac{2W}{\pi\xi}\sin\frac{\pi a}{W}\right)\,. \tag{48}$$

As expected, this expression fails at distances of the order ξ from the edges.

In short, analytic expressions are obtained for the current distribution of a vortex near the edge of a semi-infinite thin film and in a narrow thin-film strip. It is shown that the magnetic flux ϕ_z carried by vortices is reduced relative to the flux quantum ϕ_0; the region adjacent to edges where this happens may become macroscopic in sufficiently thin films. In narrow thin-film strips, the vortex flux ϕ_z scales with the small ratio of the width W to the film penetration depth λ_p; in other words, $\phi_z\ll\phi_0$. Closed formulas for the position-dependent energy and the magnetic moment of the vortex are provided.

The author is grateful to N. Nakagawa, J. Clem, M. Ledvij, L. Bulaevskii, T. Pe, and J. McDonald for discussions which helped to improve various parts of the text. The work was supported by the Office of Basic Energy Sciences of DOE and in part by the NSF Grant No. DMR9307581.

---

[1] S. Tahara, S. M. Anlage, J. Halbritter, C.-B. Eom, D. K. Fork, T. H. Geballe, and M. R. Beasley, Phys. Rev. B **41**, 11 203 (1990).

[2] D. K. Christen, C. E. Klabunde, J. R. Thompson, H. R. Kerchner, S. T. Sekula, R. Feenstra, and J. D. Budai, in *Proceedings of the International Conference on Materials and Mechanisms of Superconductivity, High Temperature Supercondutors II*, edited by R. N. Shelton, W. A. Harrison, and N. E. Phillips (North-Holland, Amsterdam, 1989), Part 1, p. 653.

[3] W. R. White, A. Kapitulnik, and M. R. Beasley, Phys. Rev. Lett. **70**, 670 (1993).

[4] G. Stejic, A. Gurevich, E. Kadyrov, D. Christen, R. Joynt, and D. C. Larbalestier, Phys. Rev. B **49**, 1274 (1994).

[5] S. L. Miller, K. R. Biagi, J. R. Clem, and D. K. Finnemore, Phys. Rev. B **31**, 2684 (1985).

[6] J. Pearl, Appl. Phys. Lett. **5**, 65 (1964).

[7] K. K. Likharev, Izv. Vuzov Radiofiz. **14**, 909, 919 (1971) [Radiophys. Quantum Electron. **14**, 714, 722 (1971)].

[8] E. H. Brandt, Phys. Rev. B **46**, 8628 (1993).

[9] A. Larkin and Yu. Ovchinnikov, Zh. Eksp. Teor. Fiz. **61**, 1221 (1971) [Sov. Phys. JETP **34**, 651 (1972)].

[10] The part $\oint d\mathbf{S}\cdot(\nabla\times\varphi\mathbf{j})=0$: take a closed contour at the sample surface, consider the total sample surface as made of two pieces supported by this contour, and apply the Stokes theorem to the integration over each piece.

[11] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals Series & Products* (Academic, New York, 1980).

[12] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and A. Stegun, Natl. Bur. Stand. Appl. Math Ser. No. 55 (U.S. GPO, Washington, DC, 1965).

[13] V. V. Shmidt and I. M. Mkrtchyan, Usp. Fiz. Nauk, **112**, 459 (1975) [Sov. Phys. Usp. **17**, 170 (1974)].

[14] W. R. Smythe, *Static and Dynamic Electricity* (McGraw Hill, New York, 1950), Chap. IV, Sec. 21.

[15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. 2, Chap. 10.

[16] Integrating Eq. (2) (to find ϕ_z) and using G of Eq. (35) yield $\phi_z=0$ because h_z has been set zero in obtaining this G. The result (35) is an approximate solution of Eq. (2), and the procedure of evaluating ϕ_z in the text corresponds to the first perturbation correction.

[17] Writing explicitly $J=\int_{-\infty}^{\infty}G(x,y)dy$ with G of Eq. (35), one can see that $J(x)$ is continuous at $x=a$.