Topological Landau-Ginzburg theory for vortices in superfluid ^{4}He

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(Received 19 January 1994)

We propose a modified Landau-Ginzburg theory for arbitrarily shaped vortex strings in superfluid ^{4}He. The theory contains a topological term and directly describes vortex dynamics. We introduce gauge fields in order to remove singularities from the Landau-Ginzburg order parameter of the superfluid, so that two kinds of gauge symmetries appear, making the continuity equation and conservation of the total vorticity manifest. The topological term gives rise to the Berry phase term in the vortex mechanical actions.

Since the existence of quantized vortices was predicted by Onsager and Feynman, vortices have been observed in superfluid ^{4}He and ^{3}He, and in superconductor systems. At low temperature in superfluid helium the quantized vortex obeys the classical hydrodynamical law that the vortex moves with the local velocity of the fluid, while the vorticity quantization comes from the fact that the superfluid is a quantum state described by a wave function. The vortex dynamics is governed by classical hydrodynamics and the quantum aspects of the system is governed by a nonlinear Schrödinger equation which is equivalent to the Landau-Ginzburg theory for superfluid developed by Ginzburg, Pitaevskii, and Gross (GPG).[1] By inserting a suitable form for the phase of the field by hand on a case-by-case basis it has been shown that a Landau-Ginzburg theory produces vortex dynamics.[2,3] However there is no satisfactory theory which describes both vortex dynamics and quantum properties of the vortex.

In this paper we propose a topological Landau-Ginzburg theory for vortices in superfluid ^{4}He. The characteristic features of our formalism are as follows. (i) We introduce a gauge field, A_μ, in the GPG theory in such a way that A_μ carries the singularities in the phase of the Landau-Ginzburg order parameter: the phase therefore becomes single valued. In order not to change physical observables we introduce A_μ gauge covariantly, and we choose the condition that the dual field strength of A_μ coincides with the vorticity tensor. This condition is imposed by using a rank two antisymmetric tensor Lagrange multiplier, $B_{\mu\nu}$. There are two kinds of gauge symmetries, which lead to the continuity equation and to the conservation of the total vorticity. (ii) A topological term, "BF term" ($\varepsilon^{\mu\nu\rho\lambda}B_{\mu\nu}F_{\rho\lambda}$), where $F_{\mu\nu}$ is the field strength, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and a coupling term, $B_{\mu\nu}J^{\mu\nu}$, to the vorticity tensor $J^{\mu\nu}$ are required to reproduce the Berry phase term. Because the BF term does not couple to the (3+1)-dimensional metric, it is a so-called topological term. In general the BF term is used in evaluating linking numbers which are topological numbers counting how many times a string and a membrane are entangled in (3+1) dimensions,[4] while the Berry phase term in the vortex mechanical action is similar to the Hopf term which counts the instanton number in the $O(3)$ nonlinear sigma model.[5] The topological BF term is a generalization of the Chern-Simons term which plays an important role in the study of the fractional quantized Hall effect and anyon systems in (2+1) dimensions.[5] It is desirable to include such a topological term since a vortex is a topological excitation in the sense that it is not obtained by a continuous deformation from the ground state. (iii) The vorticity tensor, whose time components, J^{0i}, correspond to a vorticity vector, has a general form so that it can describe arbitrarily shaped vortex strings or rings. Regularization, if needed, involves regularizing only this tensor, so that the density ρ and the velocity, $\boldsymbol{v}$, never become singular. (iv) Since the vorticity tensor contains vortex coordinates explicitly, this action directly leads to the equation of motion of vortices as well as the field equations for the order parameter. This action also reproduces the correct vortex mechanical action which contains the Berry phase term.

In connection with point (ii), the rank two antisymmetric tensor field $B_{\mu\nu}$ (Kalb-Ramond field) has been used to describe the vortex dynamics in superfluid.[6] However the theory used was the Kalb-Ramond theory which completely differs from our approach. In the Kalb-Ramond theory,[7] the action contains the square of the field strength for $B_{\mu\nu}$, $H_{\mu\nu\rho}H^{\mu\nu\rho}$, so that $B_{\mu\nu}$ becomes dynamical and propagates in space time. The essential difference is that the form of B_{ij} is set by hand to be $B_{ij}=\epsilon_{ijk}x^k$ in the previous approach, while the condition $B_{ij}=\epsilon_{ijk}x^k$ appears naturally in our approach as we show later. Therefore our formalism is perhaps more natural when describing the vortex dynamics in a superfluid.

One of the motivations of this work is how to treat the nucleation of quantized vortices. Among the most interesting problems in vortex physics are the nucleation and annihilation of quantized vortices and the mechanism of breakdown of the superfluidity and the superconductivity through the vortex nucleation.[8] Nucleation involves the creation of a vortex from nothing, and annihilation destroys an existing vortex ring. Although recently interesting classical numerical simulations of vortex nucleation have been reported,[9] quantum field theory is required for a complete treatment of such processes. However, for a theory of these processes and for other problems involving vortices, we need to develop a suitable parametrization in which the slowly changing coordinates, such as the path of a vortex ring, separate easily from the high-frequency modes. Then by taking into account suitable instantonlike solutions of the low-frequency sector we can calculate the nucleation rates or other quantities of physical interest. We hope that our new theory will be useful for this purpose.

We begin by proposing a gauged Landau-Ginzburg theory with a topological term

$$S = \int d^4x \left[\hbar\psi^* \left(i\partial_0 + A_0\right)\psi - \frac{\hbar^2}{2m}\left|\left(i\partial_i + A_i\right)\psi\right|^2 - V(\psi) + \frac{\hbar}{2m}\varepsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma} + B_{\mu\nu}J^{\mu\nu}\right], \quad (1)$$

$$J^{\mu\nu}(x) = \sum_{a=1}^{N}\gamma_a \int d\tau d\sigma \frac{\partial X_a^{[\mu}}{\partial\tau}\frac{\partial X_a^{\nu]}}{\partial\sigma}\delta^{(4)}(x - X_a(\tau,\sigma)), \quad (2)$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the complete antisymmetric tensor and $V(\psi)$ is a potential which is a gauge-invariant function of ψ. In general, one assumes that $V(\psi) = \lambda(|\psi|^2 - \rho_0)^2$ so that the number density $|\psi|^2$ does not deviate too much from ρ_0 which is a constant. $J^{\mu\nu}$ is the vorticity tensor representing vortex strings or rings on which singularities appear while X_a^μ, γ_a, and N are the vortex position, the vorticity of the ath vortex and the total number of vortices, respectively, and $A^{[\mu}B^{\nu]} \equiv A^\mu B^\nu - A^\nu B^\mu$. The condition that the wave function be single valued results in quantization of γ_a; $\gamma_a = (h/m)n_a$ where n_a is some integer. Because of the form of the vorticity tensor, the variation of the action with respect to the vortex coordinates X_a^μ gives the equation of motion of the ath vortex directly. Since singularities appear only through the vorticity tensor, it is possible to regularize, if necessary, by modifying the form of the vorticity tensor, for example, the δ function can be replaced by some other distribution.

An important property of our action is that there are two gauge symmetries. The first one is a usual gauge symmetry; $A_\mu \to A_\mu + \partial_\mu\Lambda$ and $\psi \to e^{i\Lambda}\psi$ with a regular function Λ, which leads to the continuity equation $\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{v}) = 0$. The second is given by $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$, where Λ_μ is also a regular function. The corresponding conserved current is $J^{\mu\nu}$, so that the total vorticity $\int d^3x J^{0i}$ is conserved.

If we consider three-dimensional vortex rings, the trajectory of a vortex ring, which is a sheet, may be parametrized by τ and σ. Since the trajectory of a vortex ring is a sheet, the vorticity tensor which describes vortex trajectories becomes a rank two tensor. On the other hand, for the trajectory of a point vortex it becomes a vector. In order to describe the nonrelativistic situation we can identify τ and X_a^0 with the time axis, that is, $\tau = X_a^0 = t$. We may rewrite $J^{\mu\nu}$ in terms of two vectors $\boldsymbol{J} = (J^{0i})$ and $\boldsymbol{j} = (\frac{1}{2}\epsilon^{ijk}J^{jk})$, and $\boldsymbol{J}$ just coincides with a usual vorticity vector

$$\boldsymbol{J}(\boldsymbol{x}) = \sum_{a=1}^{N}\gamma_a \oint_{\Gamma_a} d\boldsymbol{X}_a \delta^{(3)}(\boldsymbol{x} - \boldsymbol{X}_a(t,\sigma)), \quad (3)$$

$$\boldsymbol{j}(\boldsymbol{x}) = \sum_{a=1}^{N}\gamma_a \oint_{\Gamma_a} \dot{\boldsymbol{X}}_a \times d\boldsymbol{X}_a \delta^{(3)}(\boldsymbol{x} - \boldsymbol{X}_a(t,\sigma)), \quad (4)$$

where Γ_a is a ring configuration of the ath vortex ring. The conservation law of $J^{\mu\nu}$ written in terms of these vectors becomes $\nabla\cdot\boldsymbol{J} = 0$ and $\partial\boldsymbol{J}/\partial t = \nabla\times\boldsymbol{j}$.

An important feature of our theory is that the equation of motion of vortices can be obtained from our action directly. The variation of the action with respect to the vortex coordinate X_a^i gives

$$H_{0ij}\frac{\partial X_a^j}{\partial\sigma} - H_{ijk}\frac{\partial X_a^j}{\partial t}\frac{\partial X_a^k}{\partial\sigma} = 0, \quad (5)$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, and H_{0ij} and H_{ijk} are determined by the equations of motion of A_0 and A_i to be

$$H_{ijk} = -\frac{m\rho}{2}\epsilon_{ijk}, \quad (6)$$

$$H_{0ij} = \frac{m\rho}{2}\epsilon_{ijk}v^k, \quad (7)$$

where $\boldsymbol{v} = (\hbar/m)(\nabla\theta + \boldsymbol{A})$. Here, θ is a regular phase of ψ. Substituting Eqs. (6) and (7) into (5), we obtain the equation of motion of the vortex as

$$\frac{\partial\boldsymbol{X}_a}{\partial t} = \boldsymbol{v}(\boldsymbol{X}_a) + \alpha_a\frac{\partial\boldsymbol{X}_a}{\partial\sigma}, \quad (8)$$

where α_a are arbitrary coefficients reflecting the reparametrization freedom of σ: there is a parametrization of σ such that the last term in (8) vanishes. The vortex equation of motion is the same as the equation obtained in classical hydrodynamics, and specifies that vortices move with the local fluid velocity. Note that we have also shown that the equation $\partial\boldsymbol{X}_a/\partial t = \boldsymbol{v}(\boldsymbol{X}_a)$ holds not only in an incompressible superfluid but also in a compressible superfluid, since we did not assume any condition such as $\rho = \rho_0$ in the above derivation.

As well as the equation of motion for vortices, we have the following field equations, obtained by varying the density $\rho(x)$ which is $|\psi|^2$ and the phase of ψ, $\theta(x)$:

$$\hbar\dot{\theta} - \hbar A_0 + \frac{m}{2}\boldsymbol{v}^2 - \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\Delta\sqrt{\rho} + \frac{\partial V}{\partial\rho} = 0, \quad (9)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0. \quad (10)$$

The first equation is similar to the Bernoulli theorem and the second is the continuity equation. The field equation for ρ gives the dependence of ρ on the position of the vortex ring $\boldsymbol{X}_a$. In general, $\rho(\boldsymbol{x};\{\boldsymbol{X}_{a'}\}) = \rho_s(\boldsymbol{x})+\delta\rho(\boldsymbol{x},\boldsymbol{X}_a)$ where $\rho_s(\boldsymbol{x})$ is the density in the absence of the ath vortex ring and $\delta\rho(\boldsymbol{x},\boldsymbol{X}_a)$ is the modification due to the presence of the ath vortex ring. Indeed $\delta\rho(\boldsymbol{x},\boldsymbol{X}_a)$ contains the contribution from the core of the ath vortex ring. Variations with respect to B_{i0} and B_{ij} lead to

$$\nabla \times \boldsymbol{A} = \frac{m}{\hbar}\boldsymbol{J}, \tag{11}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} + \nabla A^0 = \frac{m}{\hbar}\boldsymbol{j}. \tag{12}$$

The first equation is a constraint which restricts the space component of the gauge field and the second equation determines the time component of the gauge field.

Now we will show that our new action reproduces the vortex mechanical action. Since this theory is a macroscopic theory, there is a cutoff of the theory which must be bigger than the atomic scale $\sim$ Å of the underlying microscopic dynamics. Since the coherence length of the vortex core in the superfluid, ξ, is of the order of 1 Å which is comparable with the cutoff of this theory, we neglect contributions from vortex cores which are $O(\xi^3/L^3)$ with a container size L. We assume that time and spatial derivatives of $\boldsymbol{X}$ are small. We also assume incompressibility, $\rho(x) = \rho_0 =$ const, and that the phase of ψ, θ, be zero for simplicity, so that the velocity $\boldsymbol{v}$ almost coincides with $(\hbar/m)\boldsymbol{A}$ except at the vortex cores. According to the constraint (6) which is $\nabla\cdot\boldsymbol{b}(\boldsymbol{x}) = -(m/2)\rho(\boldsymbol{x})$, the integrand of $\int d^4x \boldsymbol{b}\cdot\partial\boldsymbol{A}/\partial t$, which is a part of the BF term, is a total time derivative except near the vortex cores, so that this contribution is also neglected. Using the conditions (6) and (11) with the above assumptions, then the original action (1) reduces to

$$S = \int d^4x \left[2\boldsymbol{b}\cdot\boldsymbol{j} - \frac{\hbar^2\rho_0}{2m}\boldsymbol{A}^2\right], \tag{13}$$

where $\boldsymbol{b} = (\frac{1}{2}\epsilon^{ijk}B^{jk})$ and $2\boldsymbol{b}\cdot\boldsymbol{j} = B_{ij}J^{ij}$. Imposing gauge fixing conditions $\nabla\cdot\boldsymbol{A} = 0$ and $\nabla\times\boldsymbol{b} = 0$ and solving (11) and (6), we obtain

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{m}{2h}\sum_{a=1}^{N}\gamma_a \oint d\boldsymbol{X}_a \times \frac{\boldsymbol{x}-\boldsymbol{X}_a}{|\boldsymbol{x}-\boldsymbol{X}_a|^3}, \tag{14}$$

$$\boldsymbol{b}(\boldsymbol{x}) = -\frac{\rho_0 m}{6}\boldsymbol{x}. \tag{15}$$

Here we stress that these solutions are fully determined by the theory.

We substitute $\boldsymbol{A}$ and $\boldsymbol{b}$ into the action (13), which becomes

$$S = \rho_0 m \int dt \left[\frac{1}{3}\sum_{a=1}^{N}\gamma_a \oint d\boldsymbol{X}_a\cdot(\dot{\boldsymbol{X}}_a\times\boldsymbol{X}_a) - \frac{1}{8\pi}\sum_{a,b}\gamma_a\gamma_b \oint\oint \frac{d\boldsymbol{X}_a\cdot d\boldsymbol{X}_b}{|\boldsymbol{X}_a-\boldsymbol{X}_b|}\right]. \tag{16}$$

This is of the same form as the action of a vortex ring in an incompressible perfect fluid.[10] If we consider an adiabatic process in which the final configuration coincides with the initial configuration, the first term in (16) can be interpreted as the Berry phase,[11] which is $-i\hbar\oint dt\,\langle\Psi(X_a(t))|d\Psi(X_a(t))/dt\rangle$ where Ψ is a microscopic Feynman-type wave function. The second term in (16), which was a kinetic term of the local fluid corresponding to $-(\hbar^2\rho_0/2m)\int d^4x\boldsymbol{A}^2$ in (13), represents the interaction between vortex rings. Thus our new theory naturally includes the vortex dynamics. Note that by using field equations there is an interesting relation $\int d^4x B_{ij}J^{ij} = \hbar\int d^4x\rho A_0$: it is clear that the left-hand side gives the phase change when the vortex ring is moved around a closed trajectory, that is the Berry phase, but it is not obvious that the right-hand side also does.

It is easy to get the vortex mechanical action in the compressible case. For example, the term corresponding to the Berry phase term $\int d^4x 2\boldsymbol{b}\cdot\boldsymbol{j} = \int d^4x B_{ij}J^{ij}$ is given by

$$\frac{m}{4\pi}\sum_{a=1}^{N}\gamma_a\int d^4x \oint d\boldsymbol{X}_a\cdot\left(\dot{\boldsymbol{X}}_a\times\frac{\boldsymbol{x}-\boldsymbol{X}_a}{|\boldsymbol{x}-\boldsymbol{X}_a|^3}\right)\rho(x). \tag{17}$$

The Berry phase for a compressible case can be calculated by using this formula.

We now proceed to discuss the more familiar two-dimensional case in which point vortices move in a thin superfluid film which is lying vertical to the z axis. The vortex string is directed to the z axis and has no z-coordinate dependence, so that σ is identified as $\sigma = X_a^3$. The vorticity tensor in two dimensions is then given as

$$\boldsymbol{J}(\boldsymbol{r}) = \sum_{a=1}^{N}\gamma_a\boldsymbol{e}_z\delta^{(2)}(\boldsymbol{r}-\boldsymbol{R}_a(t)), \tag{18}$$

$$\boldsymbol{j}(\boldsymbol{r}) = \sum_{a=1}^{N}\gamma_a\dot{\boldsymbol{R}}_a\times\boldsymbol{e}_z\delta^{(2)}(\boldsymbol{r}-\boldsymbol{R}_a(t)), \tag{19}$$

where $\boldsymbol{e}_z$ is a unit vector along the z axis, $\boldsymbol{r}$ is the two-dimensional coordinate, and $\boldsymbol{R}_a$ is the two-dimensional position of the ath point vortex. Solving the constraint conditions (6) and (11) analogous to the three-dimensional case with the same conditions in the three-dimensional case, we get

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{m}{h}\sum_{a=1}^{N}\gamma_a\boldsymbol{e}_z\times\frac{\boldsymbol{r}-\boldsymbol{R}_a(t)}{|\boldsymbol{r}-\boldsymbol{R}_a(t)|}, \tag{20}$$

$$\boldsymbol{b}(\boldsymbol{r}) = -\frac{m\rho_0}{4}\boldsymbol{r}. \tag{21}$$

As a result the action for point vortices in incompressible two-dimensional superfluid becomes

$$S_2 = \rho_0 m\int dt\left[\frac{1}{2}\sum_{a=1}^{N}\gamma_a\boldsymbol{e}_z\cdot(\dot{\boldsymbol{R}}_a\times\boldsymbol{R}_a) + \frac{1}{4\pi}\sum_{a\neq b}\gamma_a\gamma_b\ln|\boldsymbol{R}_a-\boldsymbol{R}_b|\right]. \tag{22}$$

Taking into account the equation of motion, we can check that the velocity of the vortex $\boldsymbol{R}_a$ coincides with the local velocity field at $\boldsymbol{R}_a$ (see, for example, Refs. 12 and 13).

It is interesting to formulate (2+1)-dimensional theory by dimensional reduction of the original (3+1)-dimensional theory: assuming z independence of the fields ψ, A_μ, and $B_{\mu\nu}$ and $A_3 = 0$, surviving terms $\varepsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma}$ and $B_{\mu\nu}J^{\mu\nu}$ in (3 + 1)-dimensional action (1) become $-2\epsilon^{\mu\nu\rho}B_\mu F_{\nu\rho}$ and $2B_\mu J^\mu$ in (2 + 1)-dimensional action, respectively, where $B_\mu \equiv B_{3\mu}$ and $J^\mu \equiv J^{3\mu}$, $\mu = 0, 1, 2$. Using this (2+1)-dimensional BF term, we can also obtain the vortex dynamics.

Now let us compare our formalism with that of the GPG theory. The GPG action is given by

$$S_{\rm GPG} = \int d^4x \left[i\hbar\phi^*\partial_0\phi - \frac{\hbar^2}{2m}|\nabla\phi|^2 - \lambda(|\phi|^2 - \rho_0)^2 \right], \tag{23}$$

where ϕ is a complex Bose field which is allowed to have a singular phase. The phase $\theta_{\rm GPG}$ of ϕ includes both a regular part and a singular part which is multivalued:

$$\theta_{\rm GPG} = \theta_{\rm reg} + \theta_{\rm sing}. \tag{24}$$

The singular phase $\theta_{\rm sing}$ is the sum of the solid angles subtended by the vortex rings

$$\theta_{\rm sing} = \frac{1}{2}\sum_{a=1}^{N} n_a \int_{S_a} d\boldsymbol{S}'_a \cdot \nabla \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{25}$$

where S_a is any surface bounded by the ath vortex ring, $\partial S_a = \Gamma_a$.[2] An important property of $\theta_{\rm sing}$ is the noncommutativity of the differential on it

$$(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\theta_{\rm sing} = -\frac{m}{2\hbar}\varepsilon_{\mu\nu\rho\sigma}J^{\rho\sigma}. \tag{26}$$

If we take $-\partial_\mu\theta_{\rm sing}$ as the gauge field A_μ, Eq. (26) becomes the same as the constraint condition which is given by varying our action (1) with respect to $B_{\mu\nu}$. Furthermore, substituting $e^{i\theta_{\rm sing}}\phi_{\rm reg}$ into ϕ we obtain $\phi^*\partial_0\phi = \phi^*_{\rm reg}(\partial_0 - iA_0)\phi_{\rm reg}$ and $|\nabla\phi|^2 = |(\nabla - i\boldsymbol{A})\phi_{\rm reg}|^2$. Therefore, it turns out that if one solves the field equations of A_μ and $B_{\mu\nu}$ in our formalism and our field ψ is taken as $\phi_{\rm reg}$, then our action (1) becomes equivalent to the GPG action (23) whose phase is $\theta_{\rm GPG} = \theta_{\rm reg} + \theta_{\rm sing}$ in the case where $V(\psi) = \lambda(|\psi|^2 - \rho_0)^2$. Our formalism, however, has some advantages as previously mentioned. Since the dependence of the vortex coordinates appears only in $J^{\mu\nu}$, ours is useful to investigate the vortex dynamics. The topological BF term and the gauge invariance play important roles.

Towards the nucleation of the vortex, the next step is to find an instantonlike solution which makes action finite. As for future problems, the statistics of three-dimensional vortex rings are interesting because they are directly related to linking number. Our theory can be applied to the superconductor in which the contribution from vortex cores may be important since the coherent length is relatively large. Applications to the quantum Hall system are also interesting.

The topological BF term appears in various areas of theoretical physics, for example, it is induced by one-loop effects in a model with an anomalous $U(1)$ charge in superstring theory[14] and it is also used in a gravity theory which is called 2-form gravity.[15] Therefore it is interesting to consider some connections between such theories and the present theory on the vortex in the superfluid.

M.H. would like to thank the condensed matter group for their hospitality during her stay at University of Washington. She also expresses her thanks to T. Hatsuda for helpful discussions and encouragements and to H. Kuratsuji for informing her of a useful list of papers. S.Y. would like to acknowledge R. Ikeda, T. Muto, and M. Sato for useful discussions. We are grateful to M. G. Mitchard for reading the manuscript carefully. S.Y. was supported in part by Grant-in-Aid for Scientific Research from Ministry of Education, Science and Culture (Nos. 05230037 and 05740175). P.A. and D.T. were supported by NSF Grant No. DMR-9220733.

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