Invariants of the $1/r^2$ supersymmetric t-J models

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In this work, we have studied the invariants of motion of two SU(N) supersymmetric t-J models of $1/r^2$ hopping and exchange in one dimension. The first model is defined on a lattice of equal spaced sites, and the second on a nonequal spacing lattice. Using the "exchange operator formalism," we are able to construct all the invariants for the models, by mapping the systems onto mixtures of fermions and bosons. This identification shows that the supersymmetric t-J model on the chain with equal-spaced sites also belongs to Shastry-Sutherland's "super-lax-pair" family.

Since the independent works by Haldane and Shastry, there have been renewed interests in exactly solvable models of long-range interaction.[1–6] Of these systems, the one-dimensional (1D) supersymmetric t-J model of $1/r^2$ exchange and hopping has been studied intensively.[6–8,14] The system is identified as a free Luttinger liquid,[6–8] and the asymptotic correlation functions have been calculated through the finite-size-scaling technique.[7] The excitation spectrum of the system may be obtained with the help of the asymptotic Bethe ansatz.[7] In particular, for the SU(2) case, the asymptotic Bethe ansatz spectrum was explicitly shown to be exact, and the correct thermodynamics was given when the spinon rotation was properly taken into account.[8] In general, exact solvability implies the existence of an infinite number of constants of motion. For the long-range t-J models, the complete construction of invariants of motion has remained unknown. In this work, applying the "exchange operator formalism" to a mixture of fermions and bosons, we are able to systematically provide all the invariants for the SU(N) systems.

Let us first consider the $1/r^2$ supersymmetric t-J model on a one-dimensional lattice of equal spaced sites. The Hamiltonian for the one-dimensional t-J model is given by

$$H=P_G\left[-\sum_{1\le i\neq j\le L}\sum_{\sigma=1}^{N} t_{ij}(c_{i\sigma}^{\dagger}c_{j\sigma}) + \sum_{1\le i\neq j\le L} J_{ij}[P_{ij}-(1-n_i)(1-n_j)]\right]P_G\ , \qquad (1)$$

where we take the hopping matrix and the spin exchange interaction to be $t_{ij}/2=J_{ij}=1/d^2(i-j)$, and $d(n)=(L/\pi)\sin(n\pi/L)$ is the chord distance, with L the size of the lattice. The operator $c_{i\sigma}^{\dagger}$ is the fermionic operator to create an electron with spin component σ at site i, $c_{i\sigma}$ is the corresponding fermionic annihilation operator. Their anticommutation relations are given by $\{c_{i\sigma_i},c_{j\sigma_j}^{\dagger}\}_+=\delta_{ij}\delta_{\sigma_i\sigma_j}$, $\{c_{i\sigma_i},c_{j\sigma_j}\}_+=0$, $\{c_{i\sigma_i}^{\dagger},c_{j\sigma_j}^{\dagger}\}_+=0$. We assume that the spin component σ takes values from 1 to N. The Gutzwiller projection operator P_G projects out all the double or multiple occupancies, $P_G=\sum_{i=1}^{L}P_G(i)$, and $P_G(i)=\delta_{0,n_i}+\delta_{1,n_i}$, with $n_i=\sum_{\sigma=1}^{N}c_{i\sigma}^{\dagger}c_{i\sigma}$. The operator

$$P_{ij}=\sum_{\sigma=1}^{N}\sum_{\sigma'=1}^{N}c_{i\sigma}^{\dagger}c_{i\sigma'}c_{j\sigma'}^{\dagger}c_{j\sigma}$$

exchanges the spins of the electrons at sites i and j, if both sites are occupied. n_i and n_j are the electron number operators at sites i and j.

Now, on the lattice, we may introduce two new fields, the f and b fields. For the new fields, we have $\{f_{i\sigma},f_{j\sigma'}\}_+=0$, $\{f_{i\sigma},f_{j\sigma'}^{\dagger}\}_+=\delta_{ij}\delta_{\sigma\sigma'}$, $[b_i,b_j]=0$, $[b_i,b_j^{\dagger}]=\delta_{ij}$. The b fields always commute with the f fields. The size of the Hilbert space at each site is ∞ in this case. However, let us project out the zero occupancy and all the double or multiple occupancies, and work in the subspace where there is exactly one particle at each site. This new subspace can be shown to be equivalent to the subspace defined by the c field with no double or multiple occupancies. In particular, we may represent the fermionic electron operators $c_{i\sigma}^{\dagger}$ and $c_{i\sigma}$ in the following way:

$$P_G(i)c_{i\sigma}^{\dagger}P_G(i)=\delta_{1,n_b^i+n_f^i}f_{i\sigma}^{\dagger}b_i\delta_{1,n_b^i+n_f^i}\ ,$$
$$P_G(i)c_{i\sigma}P_G(i)=\delta_{1,n_b^i+n_f^i}b_i^{\dagger}f_{i\sigma}\delta_{1,n_b^i+n_f^i}\ , \qquad (2)$$

where $n_b^i+n_f^i=b_i^{\dagger}b_i+\sum_{\sigma=1}^{N}f_{i\sigma}^{\dagger}f_{i\sigma}$. In terms of the f and b fields, a state vector can be written as

$$|\phi\rangle=\sum_{\sigma_1,\sigma_2,\ldots,\sigma_{N_e}}\sum_{\{x\},\{y\}}\phi(x_1\sigma_1,x_2\sigma_2,\ldots,x_{N_e}\sigma_{N_e}|y_1,y_2,\ldots,y_Q)f_{x_1\sigma_1}^{\dagger}f_{x_2\sigma_2}^{\dagger}\cdots f_{x_{N_e}\sigma_{N_e}}^{\dagger}b_{y_1}^{\dagger}b_{y_2}^{\dagger}\cdots b_{y_Q}^{\dagger}|0\rangle\ , \qquad (3)$$

where N_e is the number of f fermions on the lattice, Q is the number of b bosons, and we require that $x_i\neq x_j\neq y_k\neq y_l$, and that the f fermion positions $\{x\}$ and the b boson positions $\{y\}$ span the whole chain. Obviously, N_e is also the number of electrons, and Q the number of holes on the lattice. The amplitude ϕ is antisymmetric when exchanging $(x_i\sigma_i)$ and $(x_j\sigma_j)$, and symmetric in the boson coordinates $\{y\}=(y_1,y_2,\ldots,y_Q)$. Using the mapping Eq. (2) in a straightforward way, the Hamiltonian of the supersymmetric t-J model can be written in terms of the fermionic f field and the bosonic b field.

With the above mapping, we can write the eigenenergy equation of the supersymmetric t-J model in the first quantized form. Define the "exchange operator" M_{ij} as

$$M_{ij}F(q_1,q_2,\ldots,q_i,\ldots,q_j,\ldots,q_L) = F(q_1,q_2,\ldots,q_j,\ldots,q_i,\ldots,q_L) ,$$

where the function F is an arbitrary function of some position variables $(q_1,q_2,\ldots,q_L)$, i.e., the operator M_{ij} exchanges the positions q_i,q_j of the particles i and j. In terms of such exchange operators, the eigenenergy equation of the t-J model takes the form as follows:[17]

$$-\left[\sum_{1\le i\neq j\le L} d^{-2}(q_i-q_j)M_{ij}\right]\phi(\{q\};\{\sigma\}) = E\phi(\{q\};\{\sigma\}) , \quad (4)$$

where

$$\{q\}=(q_1,q_2,\ldots,q_L)=(x_1,x_2,\ldots,x_{N_e},y_1,y_2,\ldots,y_Q)$$

and

$$\phi(\{q\};\{\sigma\})=\phi(q_1\sigma_1,q_2\sigma_2,\ldots,q_{N_e}\sigma_{N_e}|q_{N_e+1}q_{N_e+2}\cdots q_L)=\phi(x_1\sigma_1,x_2\sigma_2,\ldots,x_{N_e}\sigma_{N_e}|y_1,y_2,\ldots,y_Q)$$

is the amplitude of the state vector of Eq. (3). $\{\sigma\}=(\sigma_1,\sigma_2,\ldots,\sigma_{N_e})$ are the spin variables of the f fermions. The operation M_{ij} is defined in the conventional way:

$$M_{ij}\phi(\{q\});\{\sigma\})=\phi(\{q'\};\{\sigma\}) ,$$

with

$$\{q\}=(q_1,q_2,\ldots,q_i,\ldots,q_j,\ldots,q_L)$$

and

$$\{q'\}=(q_1,q_2,\ldots,q_j,\ldots,q_i,\ldots,q_L) .$$

Here, the sum in the Eq. (4) is over all pairs of particles. Thus, we see that using the f and b fields, we can write the original t-J model as an eigenvalue problem for a mixture of the f fermions and the spinless b bosons in terms of the "exchange operators."

Recently, Fowler and Minahan have considered a gas of identical bosons on a one-dimensional chain.[9] Using the so-called "exchange operator formalism,"[10] they have been able to construct explicitly all the invariants of motion for the SU(N) spin chain of Haldane and Shastry. Let us briefly review their results. Say M_{ij} is the exchange operator that interchanges q_i and q_j, the positions of the particle i and particle j, when operating M_{ij} on a wave function $F(q_1,q_2,\ldots,q_L)$. In terms of this operator, they have been able to construct an infinite set of quantities I_n that commute among themselves:

$$[I_n,I_m]=0 , \quad (5)$$

where $I_n=\sum_{i=1}^{L}\pi_i^n$, with $\pi_i=\sum_{j(\neq i)}(z_j/z_{ij})M_{ij}$, $z_i=e^{2\pi i q_i/L}$, $z_{ij}=z_i-z_j$, and $n,m=0,1,2,\ldots,\infty$. It was found that all these quantities commute with the Hamiltonian $H=\sum_{1\le i\neq j\le L}|z_i-z_j|^{-2}M_{ij}$ as long as the particles occupy the whole chain. For a system of identical bosons on the chain, the wave function is totally symmetric when we simultaneously interchange spins and positions of two particles. The effect of the exchange operator M_{ij} is just equivalent to the effect of the spin exchange operator alone. Using this method, they have successfully constructed all the invariants for the SU(N) Haldane-Shastry model.

We would like to stress that, in the language of the exchange operators M_{ij}, the commutation results proved by Fowler and Minahan hold as operator identities. The central issue is that the form of the wave functions of many-particle systems, as well as the statistics of the particles or the types of particles, do *not* matter in order for the commutators to hold, as long as the particles occupy the whole chain. We may then apply the "exchange operator formalism" to the wave functions of mixtures of fermions and bosons. Therefore, from the eigenequation, Eq. (4), we conclude that in the first quantization all the invariants of the t-J model are the same I_n's as constructed by Fowler and Minahan, which can be written in terms of the exchange operators M_{ij}'s.

With the permutation properties of the amplitude ϕ for the mixture of bosons and fermions, it is straightforward to write all the invariants of motion of the t-J model in the second quantization form using the I_n's. For instance, the exchange operation between the f fermion positions is equivalent to the spin exchange operation (minus sign involved), the exchange operation between b boson positions is equivalent to the hole-hole interaction term, and exchange operation between f fermion and b boson positions is equivalent to the electron hopping. Such a procedure to reduce an I_n to a second quantized form is quite simple, and we do not write all the details. Thus, we provide a systematic way to construct all the invariants of motion for the $1/r^2$ supersymmetric t-J model, either in first quantized or in second quantized forms.

Recently, Shastry and Sutherland have studied the interesting relation between supersymmetry and integrability, through the so-called "super-lax-pair" approach.[4,5] For this equal-spacing chain, using the mapping Eq. (2), we have been able to write the t-J model Hamiltonian, Eq. (1), in terms of the exchange operators as Eq. (4). This identification shows that the "super-lax-pair" results obtained by Shastry and Sutherland may apply to this t-J model.[4,5]

Besides the above integrable t-J model on equally spaced sites, let us consider another supersymmetric t-J model of $1/r^2$ hopping and exchange on a chain with sites not equally spaced. The positions of the sites $x_1,x_2,\ldots,x_L$ are determined by the equation

$$x_i=\sum_{1\le j(\neq i)\le L} 2/(x_i-x_j)^3 . \quad (6)$$

This equation has appeared in a paper discussing a long-range spin chain of Haldane-Shastry type.[10] Doping this spin chain, we are led to the following supersymmetric t-J model:

$$H=P_G\left[-\sum_{1\le i\neq j\le L}\sum_{\sigma=1}^{N}t_{ij}(c_{i\sigma}^{\dagger}c_{j\sigma})+\sum_{1\le i\neq j\le L}J_{ij}[P_{ij}-(1-n_i)(1-n_j)]\right]P_G\ , \qquad (7)$$

where the hopping matrix and the antiferromagnetic exchange interaction are given by $t_{ij}/2=J_{ij}=1/(x_i-x_j)^2$, and each site is occupied at most by one electron.

In the half-filled case $N_e=L$, this system reduces to the spin chain that has been studied before, which is completely solvable and a similar exchange operator formalism has been developed.[10,11] Let us just write down the results obtained by Polychronakos: $[I_n,I_m]=0$, $[I_n,H]=0$, where $I_n=\sum_{i=1}^{L}h_i^n$, $h_i=a_i^{\dagger}a_i$, and $a_i^{\dagger}=\pi_i^{\dagger}+iq_i$, $a_i=\pi_i-iq_i$, with $\pi_i=\sum_{j(\neq i)}i(q_i-q_j)^{-1}M_{ij}$, $H=\sum_{i\neq j}(q_i-q_j)^{-2}M_{ij}$, and $n,m=0,1,2,\ldots,\infty$. Here, all the particles are put on the chain where the sites are positioned as determined by Eq. (6). We may relate the operation of exchanging particle positions to the operation of exchanging particle spins, by assuming that we have identical bosons again, for which the wave functions are totally symmetric when we exchange the spins and positions of two particles simultaneously.[10] With this assumption, from I_n's, we thus can derive all the invariants of motion written in terms of the spin exchange operators alone.

Again, all commutation results written in terms of the exchange operators M_{ij} obtained by Polychronakos hold as operator identities, as long as the particles occupy the whole chain of the sites positioned in the special way. The forms of the wave functions do *not* matter at all. Thus, the commutation results can be applied to wave functions of particles of arbitrary statistics or wave functions of mixtures of particles of different statistics on the chain. Mapping our supersymmetric t-J model in terms of the b and f fields, we can also write the eigenenergy equation in first quantized form. In terms of the exchange operators between the positions of the bosons and fermions, the Hamiltonian takes the form

$$H=-\sum_{1\le i\neq j\le L}(q_i-q_j)^{-2}M_{ij}\ . \qquad (8)$$

Applying the formalism to this t-J model, in a similar way we obtain all the invariants, either in first quantized or in the second quantized form, which commute among themselves and with the Hamiltonian. Thus, this supersymmetric t-J model is also completely integrable.

In conclusion, we have studied the invariants of motion of two SU(N) supersymmetric t-J models of long-range hopping and exchange. The first system is on the chain of equal-spaced sites, and the other on a chain of nonequal-spaced sites. Mapping the corresponding t-J model Hamiltonians to those written in terms of mixed fermionic and bosonic fields, then applying the "exchange operator formalism," we were able to construct systematically all the invariants of the original Hamiltonians.

Finally, we wish to point out that, the second t-J model has also a metal-insulator phase transition at half filling. Away from half filling, we expect to have decoupled spin and charge excitations near the ground state. The system would be a Luttinger liquid. The study of the physical properties of the t-J model, such as its full excitation spectrum, is reported in our forthcoming paper. We obtain Jastrow product ground-state and excited-state wave functions, as in the case for the model on the chain with equally spaced sites. It would also be very interesting to find out possible Shastry-Sutherland type "super-lax-pair" for this supersymmetric t-J model on a nonequal-spacing chain. We will return to the issue of constructing invariants of the nonsupersymmetric t-J models with $1/r^2$ hopping and exchange in future.

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