Chern-Simons theory of the anisotropic quantum Heisenberg antiferromagnet on a square lattice

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(Received 21 January 1994)

We consider the anisotropic quantum Heisenberg antiferromagnet (with anisotropy λ) on a square lattice using a Chern-Simons (or Wigner-Jordan) approach. We show that the average field approximation (AFA) yields a phase diagram with two phases: a Néel state for $\lambda > \lambda_c$ and a flux phase for $\lambda < \lambda_c$ separated by a second-order transition at $\lambda_c < 1$. We show that this phase diagram does not describe the XY regime of the antiferromagnet. Fluctuations around the AFA induce relevant operators which yield the correct phase diagram. We find an equivalence between the antiferromagnet and a relativistic field theory of two self-interacting Dirac fermions coupled to a Chern-Simons gauge field. The field theory has a phase diagram with the correct number of Goldstone modes in each regime and a phase transition at a critical coupling $\lambda^* > \lambda_c$. We identify this transition with the isotropic Heisenberg point. It has a nonvanishing Néel order parameter, which drops to zero discontinuously for $\lambda < \lambda^*$.

I. INTRODUCTION

Since the discovery of high-T_c superconductors,[1] the two-dimensional quantum Heisenberg model has received considerable attention. This is largely due to well-established experimental facts which strongly suggest that these compounds can be described by a doped Heisenberg spin-1/2 quantum antiferromagnets.[2]

Dimensionality plays a crucial role in the properties of the quantum Heisenberg antiferromagnet. The $S = 1/2$ quantum antiferromagnetic chain can be solved exactly using the Bethe ansatz.[3] By using the Wigner-Jordan transformation,[4,5] this model can be mapped onto a system of spinless, interacting, fermions with a coupling constant equal to the anisotropy parameter. This model is particularly simple in the XY limit where the spin problem maps to free fermions. Although fairly reliable in general dimensions, in one space dimension the spin-wave theory is plagued by a number of notorious problems. This approximation is based on the Holstein-Primakov[6] transformation which maps $S = 1/2$ spins into hard core bosons. The spin-wave approximation[7] relaxes the hard core constraint and treats correctly the commutation relation between spins in different sites. For one-dimensional systems, spin-wave theory (or rather, the $1/S$ expansion) is infrared divergent order-by-order. This divergence is the manifestation of the fact that the continuous symmetry of global spin rotations cannot be broken in one space dimension. It also misses the essential fact that half-integer spin systems are critical while integer spin systems are always quantum disordered and have an energy gap.[8] These properties of the exact ground state of the system can be recovered in one-dimensional spin systems by using nonperturbative methods, such as the Wigner-Jordan transformation combined with bosonization.[9,10]

Much less is known for two-dimensional quantum antiferromagnets. First, there is no exact solution available in any limit of the spin-$\frac{1}{2}$ system. Spin-wave theory predicts a Néel ordered ground state for the isotropic antiferromagnet on a square lattice, although with a moment reduced to 50% of the classical value by quantum fluctuations.[11] The Hamiltonian for the anisotropic quantum Heisenberg antiferromagnet on a square lattice is

$$H = J \sum_{\langle \mathbf{x},\mathbf{x}' \rangle} \left\{ \lambda S_z(\mathbf{x}) S_z(\mathbf{x}') + \frac{1}{2} \left[S^+(\mathbf{x}) S^-(\mathbf{x}') + S^-(\mathbf{x}') S^+(\mathbf{x}') \right] \right\} , \tag{1.1}$$

where $\langle \mathbf{x},\mathbf{x}' \rangle$ denotes nearest-neighboring sites and λ is the anisotropy parameter ($\lambda = 1$ corresponds to the isotropic case).

The following facts are known to be true for this system. For $\lambda \gg 1$, the Ising term dominates and the ground state should be close to a *classical* antiferromagnet which has total $S_z = 0$. This state has an energy gap and an expansion in powers of $1/\lambda$ is rapidly con-

0163-1829/94/49(21)/15139(20)/$06.00

vergent. In the opposite XY limit, where $\lambda \to 0$, there is a theorem[12] due to Kennedy, Lieb, and Shastry and to Kubo and Kishi, which proves that there exists long range order with the spins lined up *on the XY plane* for $\lambda < \lambda_1$ (with $\lambda_1 \geq 0.13$). The same theorem proves that, in the Ising regime, the antiferromagnetic ground state extends *at least* down to an anisotropy parameter $\lambda \leq 1.78$. No theorem is known for the isotropic case $\lambda = 1$ and spin $S = \frac{1}{2}$. For $S \geq 1$ Dyson, Lieb, and Simon[13] proved a theorem which shows that there is Néel order *even* at the isotropic antiferromagnetic point. Finite size diagonalization,[14] quantum Monte Carlo[15] and variational estimates,[16] are more consistent with a Néel antiferromagnetic ground state for the isotropic antiferromagnet. For two-dimensional quantum antiferromagnets, the semiclassical $1/S$ expansion is free of the infrared divergencies found in one dimension. This approach predicts that the low energy limit of the isotropic antiferromagnet is a nonlinear σ model *without* a topological term.[17] This latter results have been confirmed by detailed renormalization group studies[18] which yield an excellent agreement with experiments on La_2CuO_4.

These results suggest that the anisotropic quantum antiferromagnet has a phase diagram with just two phases: (a) a Néel state with Ising anisotropy for $\lambda > 1$, and (b) an XY phase for $\lambda < 1$. For $\lambda > 1$ the Ising anisotropy should make all excitations massive (i.e., no Goldstone bosons for $\lambda > 1$). For $\lambda < 1$ the U(1) XY symmetry is spontaneously broken and there should be *one* Goldstone boson (spin wave). In this scenario, exactly *at* the isotropic point $\lambda = 1$, the SU(2)/U(1) global symmetry of the Heisenberg model is spontaneously broken and there should be *two* Goldstone bosons (spin waves), as predicted by the nonlinear σ model.[17] In some sense there is a phase transition at $\lambda = 1$ in that the Néel order parameter should jump *discontinuously* to zero as λ is *decreased* through $\lambda = 1$. Precisely *at* $\lambda = 1$ the XY and Néel orders are equivalent under an SU(2) rotation and, thus, there is still long range order. In contrast, the one-dimensional spin-$\frac{1}{2}$ chain has a *line of fixed points* for $\lambda \leq 1$ and Néel order in the massive phase $\lambda > 1$. In Sec. II we give an argument, based on the $1/S$ expansion, in support of this general scenario.

In this paper we investigate the anisotropic quantum Heisenberg antiferromagnet on a square lattice using a generalized Wigner-Jordan transformation constructed earlier by one of us.[19] In Ref. 19 it was shown that the Wigner-Jordan transformation in two dimensions is a special form of a statistics changing transformation which, quite generally, is achieved by coupling particles to (lattice) Chern-Simons gauge fields with a properly chosen coupling constant θ. The quantum Heisenberg antiferromagnet becomes equivalent to a system of spinless fermions which interact with each other (just as in the case of the spin chain) but which are also coupled to the Chern-Simons gauge field. Thus, even in the XY limit this system is interacting. Systems of this type have been considered recently in connection with the problem of anyon superfluidity[20–22] and the fractional quantum Hall effect (FQHE).[23,24] Unlike the case of the spin chain, the equivalent fermion problem is never free and a new type of approximation has to be found. The analog of mean-field theory in this context is known as the average field approximation (AFA) (defined below in Sec. III). We will also show that the physics of this problem is hidden at the mean-field level and it is only revealed by a careful consideration of the fluctuations.

Just as in systems of anyons or in the FQHE, the energy spectrum predicted by the AFA consists of free fermions with an effective band structure generated by the self-consistent flux. We find that, for the sector with $S_z = 0$ (which corresponds to half-filling of either the hard core bosons or the fermions) the average uniform effective flux is equal to one-half of the flux quantum. Thus, on average, we find a *flux phase* analogous to that of the gauge theory approach to quantum antiferromagnets of Baskaran and Anderson,[25] Affleck and Marston,[26] and Kotliar.[27] The AFA also predicts that, for $\lambda \geq \lambda_c$ (where $\lambda_c \approx 0.39$), a gap opens up in the energy spectrum. In this regime the fermion density *and* the flux acquire a modulation with wave vector (π, π). The Wigner-Jordan transformation maps the z component of the spin $S_z(\mathbf{x})$ at $\mathbf{x}$ to the fermion occupation number $n(\mathbf{x})$ by $S_z(\mathbf{x}) = \frac{1}{2} - \mathbf{n}(\mathbf{x})$. Thus, we identify this regime with Néel antiferromagnetic order. For $\lambda < \lambda_c$ the AFA spectrum of fermions is massless. Recently, Wang[28] has studied the Heisenberg model on a square lattice using the Wigner-Jordan transformation of Ref. 19 combined with an approximation similar to the average field approximation discussed in Sec. IV. The results of the average field approximation (AFA) that we present here disagree with Wang's, mainly because his form of the AFA is not fully self-consistent.

This mean-field spectrum is incompatible with the scenario proposed above, based on the semiclassical expansion. There, the spectrum of low lying states contains *only* integer spin fluctuations (i.e., spin flips). Some of these states may be massless (as for $\lambda \leq 1$) or all massive (such as for $\lambda > 1$) but they are all *bosonic*. In particular, and in contrast with one-dimensional systems, the $1/S$ expansion predicts the existence of long range order.

This problem is solved by a careful consideration of the role of the fluctuations. As expected, symmetry plays a crucial role here. The main problem is that, for small values of λ, the AFA fermion spectrum is gapless. For arbitrary values of the Chern-Simons coupling θ, two types of gaps, even or odd under time reversal (T) or parity (P) can be generated by fluctuations. The Lagrangian at the level of the AFA is even under both P and T. Thus, we should expect that fluctuations will generate all terms with low scaling dimension (i.e., operators which are either relevant or marginal) which are compatible with the symmetries of the full system. Notice that here we encounter the *opposite* of the situation usually found with spontaneously broken symmetries where the mean-field theory has *less* symmetry than the full system.

The symmetry analysis becomes more transparent in terms of an effective theory for the low energy degrees of freedom. This effective continuum theory for the two-dimensional quantum antiferromagnet on a square lattice turns out to be a theory of two species of relativistic Dirac fermions (moving at the Fermi velocity defined in Sec. V) coupled to a Chern-Simons gauge field and to the fluctuations of an effective Néel order parameter field. This effective field theory is derived in Sec. V. This theory is a generalization to 2D of the well-known equivalence between the antiferromagnetic spin *chain* and a field theory of interacting relativistic fermions in 1+1 dimensions, known as the Luttinger-Thirring model.

In this language, the fermion spectrum can be understood in terms of the possible energy gaps, or *masses*, of the fermions and of the symmetry properties of the operators connected with these masses. Our analysis shows that the Néel order parameter acts like a mass operator which does not break P or T and which we will denote as $M_{\text{Néel}}$. In the Néel phase the two species of fermions acquire a mass, but with relative *opposite* sign. However, for general values of the coupling constant θ, the Chern-Simons term breaks both T and P. Thus, quantum fluctuations of the Chern-Simons field will necessarily generate all terms which break the same symmetries. A fermion mass term with the *same sign* for both species also breaks both P and T and we find that it does get generated by quantum fluctuations. We will refer to this as the *induced* fermion mass, M_{ind}. The actual phase diagram follows from the competition of these two mechanisms.

The generation of a parity breaking mass term changes radically the long distance behavior of the system. It is well known[29] that fermions with masses which break time reversal and parity induce Chern-Simons terms in the action of the gauge fields at length scales long compared with the correlation length of the fermions, i.e., the inverse of the parity breaking fermion mass. Also, the sign of the *induced* Chern-Simons term is equal to the sign of the mass of the fermion. Hence, at length scales long compared with $1/M_{\text{ind}}$, we expect to see a finite renormalization of the Chern-Simons coupling constant from its bare value θ to some effective value. This effective value depends on the pattern of symmetry breaking, i.e., on the relative signs of the induced masses which, in turn, are determined by the bare coupling θ itself and by the nature of the ground state. We find that the sign of M_{ind} is such that the induced Chern-Simons coupling tends to *reduce* the bare Chern-Simons coupling. This renormalization can be viewed as a tendency to screen the bare statistics. This is a manifestation of a more general property which we may think of as a "Lenz law of statistics".[30]

Thus, we arrive to the following scenario. At small λ, the fermions acquire a parity breaking mass M_{ind} through the fluctuations of the gauge field. In turn, at distances long compared with $1/M_{\text{ind}}$, a Chern-Simons term is induced with a coupling constant which tends to cancel the bare statistics. We will find in Sec. VI that for $\theta = 1/2\pi$ the cancellation is complete and the gauge fields are actually gapless at long distances. This scenario is reminiscent of anyon superfluidity. We will identify the gapless transverse gauge fluctuation (i.e., Laughlin's mode[20]) with the gapless *transverse* spin wave of the XY regime. Clearly two fermion masses with different symmetry necessarily compete with each other. We expect that when they become of comparable magnitude $|M_{\text{ind}}| \approx |M_{\text{Néel}}|$ a phase transition should occur. For this range of λ, one of the two species of fermions has a mass that is very small and, at least nominally, is going to vanish at some critical value of the anisotropy λ^*. We will identify this phase transition with the SU(2)-symmetric Heisenberg point.[31] This phase transition occurs at a value of the anisotropy $\lambda^* > \lambda_c$. This phase transition preempts the naïve second-order transition predicted by the AFA from taking place. This mechanism seems to bear a close analogy with a fluctuation-induced first order transition.[32,33]

To summarize, the Chern-Simons (or Wigner-Jordan) approach to the spin $S = \frac{1}{2}$ anisotropic quantum Heisenberg antiferromagnet yields a phase diagram which is in qualitative agreement with the predictions of the semiclassical $1/S$ expansion. The physically correct (PC) phase diagram is not found at the level of the average field approximation and, except in the regime of strong Ising anisotropy, it is due almost entirely to fluctuation effects. In this paper we show how this physical picture is realized in the context of the Chern-Simons theory. This is a (necessary) first step before these methods could be applied to more subtle problem such as frustrated systems.

This paper is organized as follows. In Sec. II we develop the semiclassical $1/S$ theory of the anisotropic antiferromagnet. In Sec. III we review the Wigner-Jordan construction and its connection with the Chern-Simons theory on a square lattice. In Sec. IV we present the results in the average field approximation and discuss the role of Gaussian fluctuations. In Sec. V the effective field theory is derived. The dynamical, nonperturbative, effects of fluctuations are discussed in Sec. VI where we give a justification of the phase diagram discussed in this (long) introduction. Section VII is devoted to the conclusions.

II. THE ANISOTROPIC HEISENBERG QUANTUM ANTIFERROMAGNET FOR LARGE S

In this section we discuss the semiclassical $1/S$ theory of the anisotropic antiferromagnet in a square lattice. The easiest way to get a path integral quantization for a spin system is to use coherent states. In this section we will follow the methods described in Ref. 34.

The set of coherent states $\{|\mathbf{n}\rangle\}$, labeled by the unit vector $\mathbf{n}$, is generated by a rotation of the highest weight vector $(|S,S\rangle)$ of an irreducible representation of the group SU(2) of spin s of the form

$$|\mathbf{n}\rangle = e^{i\theta(\mathbf{n}_0\times\mathbf{n})\cdot\mathbf{S}}|S,S\rangle, \tag{2.1}$$

where $\mathbf{n}_0$ is a unit vector along the quantization axis, θ is the colatitude $(\mathbf{n}\cdot\mathbf{n}_0 = \cos\theta)$ and S_i $(i = 1,2,3)$ are the three generators of SU(2) in the spin-s representation. The state $|\mathbf{n}\rangle$ can be expanded in a complete basis of the spin-s irreducible representation $\{|S,M\rangle\}$ where M labels the eigenvalues of S_3. The coefficients of the expansion are the representation matrices $D^{(S)}(\mathbf{n})_{M,S}$

$$|\mathbf{n}\rangle = \sum_{M=-S}^{S} D^{(S)}(\mathbf{n})_{M,S}|S,M\rangle. \quad (2.2)$$

The matrices $D^{(S)}$ do not form a group but satisfy

$$D^{(S)}(\mathbf{n}_1)\, D^{(S)}(\mathbf{n}_2) = D^{(S)}(\mathbf{n}_3)\, e^{i\Phi(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)S_3}, \quad (2.3)$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are three arbitrary unit vectors and $\Phi(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)$ is the area of the spherical triangle with vertices at $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$. Other useful properties of the spin coherent states are: (a) the inner product $\langle\mathbf{n}_1|\mathbf{n}_2\rangle$,

$$\langle\mathbf{n}_1|\mathbf{n}_2\rangle = e^{i\Phi(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_0)s}\left(\frac{1+\mathbf{n}_1\cdot\mathbf{n}_2}{2}\right)^s, \quad (2.4)$$

(b) the diagonal matrix elements of the generators $\mathbf{S}$,

$$\langle\mathbf{n}|\mathbf{S}|\mathbf{n}\rangle = s\mathbf{n}, \quad (2.5)$$

and (c) the resolution of the identity operator

$$\hat{I} = \int d\mu(\mathbf{n})\, |\mathbf{n}\rangle\langle\mathbf{n}|, \quad (2.6)$$

where we have used the integration measure

$$d\mu(\mathbf{n}) = \left(\frac{2s+1}{4\pi}\right) d^3\mathbf{n}\; \delta(\mathbf{n}^2-1). \quad (2.7)$$

Using these properties we can write an expression for the path-integral in this coherent state representation. The zero temperature partition function reads

$$\mathcal{Z} = \int \mathcal{D}\mathbf{n}\; e^{i\mathcal{S}_M[\mathbf{n}]}\quad , \quad (2.8)$$

where the action for the many-spin system in real time is given by

$$\mathcal{S}_M[\mathbf{n}] = s\sum_{\mathbf{r}} \mathcal{S}_{WZ}[\mathbf{n}(\mathbf{r})] - \int_0^T dx_0 J s^2 \sum_{\langle\mathbf{r},\mathbf{r}'\rangle} \{\mathbf{n}_\perp(\mathbf{r},x_0)\cdot\mathbf{n}_\perp(\mathbf{r}\,',x_0) + \lambda\mathbf{n}_3(\mathbf{r},x_0)\cdot\mathbf{n}_3(\mathbf{r}\,',x_0)\}\,. \quad (2.9)$$

To write this expression we have used the Hamiltonian for the anisotropic quantum Heisenberg antiferromagnet on a square lattice given by Eq. (1.1) (with $\langle\mathbf{r},\mathbf{r}\,'\rangle$ denoting nearest-neighboring sites), and we have assumed periodic boundary conditions. In Eq. (2.9) $\mathbf{n}_\perp$ is the projection of the vector $\mathbf{n}$ onto the 12 (or xy) plane.

The first term in Eq. (2.9) is just the sum of the Wess-Zumino terms (or *Berry phases*) for the individual spins. The contribution of each term to the action is

$$\mathcal{S}_{WZ}[\mathbf{n}] = \int_0^1 d\tau \int_0^\beta dt\; \mathbf{n}(t,\tau)\cdot[\partial_t\mathbf{n}(t,\tau)\times\partial_\tau\mathbf{n}(t,\tau)], \quad (2.10)$$

where $\beta = iT$. The effect of this term is to quantize the spin.

The effective action $\mathcal{S}_M[\mathbf{n}]$ scales like s, the spin representation. Therefore, in the large spin limit, the path integral equation (2.8) should be dominated by the stationary points of the action. This is the semiclassical limit. Corrections to the large-s limit can be arranged in an expansion in powers of $\frac{1}{s}$. Since we expect to be close to a Néel state, we will stagger the configuration

$$\mathbf{n}(\mathbf{r}) \to (-1)^{x_1+x_2}\mathbf{n}(\mathbf{r}). \quad (2.11)$$

The Wess-Zumino terms are odd under the replacement of Eq. (2.11) and thus get staggered. Up to an additive constant the action reads

$$\begin{aligned}\mathcal{S}_M[\mathbf{n}] = {} & s\sum_{\mathbf{r}}(-1)^{x_1+x_2}\mathcal{S}_{WZ}[\mathbf{n}(\mathbf{r})] \\ & -\frac{Js^2}{2}\sum_{\mathbf{r}}\sum_{j=1,2}\int_0^T dx_0 \left\{[\mathbf{n}_\perp(\mathbf{r},x_0)-\mathbf{n}_\perp(\mathbf{r}+\hat{e}_j,x_0)]^2 + [\mathbf{n}_\perp(\mathbf{r},x_0)-\mathbf{n}_\perp(\mathbf{r}-\hat{e}_j,x_0)]^2\right\} \\ & +\lambda\left\{[\mathbf{n}_3(\mathbf{r},x_0)-\mathbf{n}_3(\mathbf{r}+\hat{e}_j,x_0)]^2 + [\mathbf{n}_3(\mathbf{r},x_0)-\mathbf{n}_3(\mathbf{r}-\hat{e}_j,x_0)]^2\right\} \\ & +(1-\lambda)\left\{2[\mathbf{n}_3(\mathbf{r},x_0)]^2 + [\mathbf{n}_3(\mathbf{r}+\hat{e}_j,x_0)]^2 + [\mathbf{n}_3(\mathbf{r}-\hat{e}_j,x_0)]^2\right\}.\end{aligned} \quad (2.12)$$

We split the staggered spin field $\mathbf{n}$ in the following way:

$$\mathbf{n}(\mathbf{r}) = \mathbf{m}(\mathbf{r}) + (-1)^{x_1+x_2}a_0\boldsymbol{l}(\mathbf{r}), \quad (2.13)$$

where $\mathbf{m}(\mathbf{r})$ is a slowly varying piece, the order parameter field, and $\boldsymbol{l}(\mathbf{r})$ is a small rapidly varying part which roughly represents the average spin. The constraint $\mathbf{n}^2 = 1$ and the requirement that the order parameter field $\mathbf{m}$ should obey the same constraint, $\mathbf{m}^2 = 1$, demand that $\mathbf{m}\cdot\boldsymbol{l} = 0$. Using this property we can write the Lagrangian density for this theory in the long wavelength limit as

$$\mathcal{L}_M(\mathbf{m},\boldsymbol{l}) = \frac{s}{a_0}\boldsymbol{l}\cdot(\mathbf{m}\times\partial_0\mathbf{m}) - Js^2\left\{\sum_{j=1,2}[(\partial_j\mathbf{m}_\perp)^2 + \lambda(\partial_j\mathbf{m}_3)^2] + 8[(\boldsymbol{l}_\perp)^2 + \lambda(\boldsymbol{l}_3)^2] + 4(1-\lambda)\left[\frac{(\mathbf{m}_3)^2}{a_0^2} + (\boldsymbol{l}_3)^2\right]\right\}, \quad (2.14)$$

where a_0 is the lattice spacing.

In the long wavelength limit, the Wess-Zumino action can be written as a sum of a topological term and the first term in Eq. (2.14). It has been shown (Ref. 17) that if we expect to have Néel order, the topological term does not contribute to the action in two space dimensions. Notice that, in the one-dimensional case, this same procedure leads to a σ model *with* a topological term.

After integrating out the fast modes, i.e., the components $\boldsymbol{l}_\perp$ and l_3 of $\boldsymbol{l}$, in the partition function, the resulting Lagrangian is

$$\mathcal{L}_M(\mathbf{m}) = \frac{1}{2g}\left(\frac{1}{v_s}(\partial_0\mathbf{m}_\perp)^2 - v_s\sum_{j=1,2}(\partial_j\mathbf{m}_\perp)^2\right) + \frac{1}{2g}\left(\frac{1+\lambda}{2v_s}(\partial_0\mathbf{m}_3)^2 - \lambda v_s\sum_{j=1,2}(\partial_j\mathbf{m}_3)^2\right)$$
$$-\frac{8(1-\lambda)}{a_0^2}\frac{v_s}{2g}(\mathbf{m}_3)^2 + \frac{(1-\lambda)}{2}\frac{1}{2gv_s}(\mathbf{m}_3)^2[(\partial_0\mathbf{m}_\perp)^2 - (\partial_0\mathbf{m}_3)^2], \tag{2.15}$$

where the coupling constant g and the spin-wave velocity v_s are given by

$$g = \frac{2}{s}a_0(1+\lambda)^{\frac{1}{2}}, \tag{2.16}$$

$$v_s = 4Jsa_0(1+\lambda)^{\frac{1}{2}}. \tag{2.17}$$

We can Wick rotate back to imaginary time (i.e., $x_3 = ix_0$), and write the Euclidean Lagrangian density $\mathcal{L}_E$ as

$$\mathcal{L}_E(\mathbf{m}) = \frac{1}{2g}\left(\frac{1}{v_s}(\partial_3\mathbf{m}_\perp)^2 + v_s\sum_{j=1,2}(\partial_j\mathbf{m}_\perp)^2\right) + \frac{1}{2g}\left(\frac{1+\lambda}{2v_s}(\partial_3\mathbf{m}_3)^2 + \lambda v_s\sum_{j=1,2}(\partial_j\mathbf{m}_3)^2\right)$$
$$+\frac{8(1-\lambda)}{a_0^2}\frac{v_s}{2g}(\mathbf{m}_3)^2 + \frac{(1-\lambda)}{2}\frac{1}{2gv_s}(\mathbf{m}_3)^2[(\partial_3\mathbf{m}_\perp)^2 - (\partial_3\mathbf{m}_3)^2]. \tag{2.18}$$

By direct inspection of Eq. (2.18) one can see that the third term of this action is relevant in the long wavelength limit. The physics of this term is the following. For $\lambda < 1$, the system will lower its energy by making $m_3 \to 0$, i.e., the system is in the XY limit. On the other hand, if $\lambda > 1$ the energy will be maximized when m_3 acquires its maximum possible value, i.e., $m_3 = 1$ and the system is in the Néel state with Ising anisotropy.

The phase diagram for the anisotropic quantum antiferromagnet suggested by these results has only two phases. For $\lambda > 1$ the system is in a Néel state with Ising anisotropy, and all the excitations are massive. For $\lambda < 1$ the systems are in an XY phase, the U(1) XY symmetry is spontaneously broken and there should be one Goldstone boson. Exactly at the isotropic point ($\lambda = 1$) there should be two Goldstone bosons as predicted by the nonlinear σ model.[17]

III. CHERN-SIMONS ON A LATTICE

We begin by reviewing the path integral picture of a spin system on a two-dimensional lattice in terms of fermions coupled to Chern-Simons gauge fields, introduced in Ref. 19.

The Wigner-Jordan transformation is based on the identification of a system of *hard core bosons* (i.e., spin flips) with an equivalent system of *fermions* each of them rigidly attached with solenoids that carry one-half of the flux quantum. Mathematically, the equivalent system is a theory of fermions coupled to a Chern-Simons gauge field on the square lattice. Chern-Simons theories[35] have been used with great success in the fractional quantum Hall effect[23,24] and in anyon superfluidity.[20–22] The presence of the lattice introduces a number of subtleties not found in continuum systems. The role of the Chern-Simons gauge fields is to enforce the constraint that attaches particles to fluxes locally, *and* a set of commutation relations among the gauge fields compatible with these constraints.[36] These two features are key ingredients for the Wigner-Jordan transformation to work.

However, unlike the one-dimensional spin chain, in two dimensions the equivalent system of fermions is coupled to a gauge field which can have *local* flux. Hence the fermions are always interacting, even in the XY limit, and approximations become necessary. Written in the fermion language, the system can then be described in terms of a theory of fermions which interact with each other and with a Chern-Simons gauge field.

In what follows we will use a path integral description. The zero temperature partition function for this problem has the form

$$\mathcal{Z} = \int \mathcal{D}\psi^*\mathcal{D}\psi\mathcal{D}\mathcal{A}_\mu\, e^{iS}, \tag{3.1}$$

where $\psi(\mathbf{x},t)$ is a Fermi field (i.e., Grassmann variables in the path integral) defined on the sites $\{\mathbf{x}\}$ of the square lattice and $\mathcal{A}_\mu$ are the statistical or Chern-Simons gauge fields. The space components $\mathcal{A}_j(\mathbf{x},t)$ are defined on the links of the lattice while the time component $\mathcal{A}_0(\mathbf{x},t)$ is defined on the sites. The role of the gauge field is to change the statistics. The action S is given by

$$S = \int dt \Big\{ \sum_{\mathbf{x}} \psi^*(\mathbf{x},t)[iD_0 + \mu]\psi(\mathbf{x},t) - \frac{J}{2} \sum_{j=1,2} ([\psi^*(\mathbf{x}+e_j,t)e^{-i\mathcal{A}_j(\mathbf{x},t)}\psi(\mathbf{x},t) + \text{c.c.}] + 2\lambda[|\psi(\mathbf{x},t)|^2 - \tfrac{1}{2}][|\psi(\mathbf{x}+e_j,t)|^2 - \tfrac{1}{2}]) \Big\} + \theta S_{\text{CS}}(\mathcal{A}), \tag{3.2}$$

where $D_0 = \partial_0 + i\mathcal{A}_0$ is the covariant time derivative, and the spatial covariant derivative is in this case the gauge covariant lattice difference implied by the hopping term. The action S of Eq. (3.2) describes self-interacting fermions which are coupled to a fluctuating Chern-Simons gauge field. The self-interaction is represented by the quartic term in fermions in the action and it corresponds to the S_zS_z Ising interaction of the Heisenberg model. We will call this term S_{int}.

The lattice Chern-Simons $S_{\text{CS}}(\mathcal{A})$ action was defined in Refs. 19 and 36. Its explicit form will be given below. The coupling constant θ is chosen to be $\theta = \frac{1}{2\pi}$ so that the statistics corresponds to bosons (with hard cores). For general Chern-Simons coupling θ this system is equivalent to a system of interacting anyons with statistical angle $\delta \equiv \frac{1}{2\theta}$, on a square lattice.[19] Lattice anyons have been studied numerically by Canright *et al.*[37] and analytically by Fradkin.[22] We will see in Sec. IV that the problem at hand is an example of the degenerate solution found in Ref. 22.

In the representation of the Heisenberg model in terms of fermions coupled to gauge fields, with the action of Eq. (3.2), the natural mean-field approximation consists of detaching the fermions from their local fluxes and to replace this dynamical flux by a static average background. Unlike spin-wave theory, in this mean-field approximation the hard core constraint is taken into account exactly. The phase factors present in the hopping amplitudes of the equivalent system of fermions, whose role is to enforce the original bosonic commutation relations, are treated approximately. The *average field approximation* (AFA), as this mean-field theory has come to be known, was first introduced by Laughlin in the context of his study of the anyon gas[20] and subsequently used quite extensively in the context of the FQHE by two of us.[24] A peculiar feature of this mean-field theory is that it breaks a number of space-time symmetries in a very explicit manner. For example, in the anyon gas the ground state obtained at the AFA level breaks Galilean invariance while the actual ground state does not. The fluctuations around the AFA restore Galilean invariance. Likewise, in the context of the FQHE Galilean (or rather, *magnetic*) invariance is broken at the level of the AFA but it is also restored by Gaussian fluctuations.[38]

As stated above, the role of the Chern-Simons gauge fields is to change the statistics from (hard core) bosons to fermions. Here we will follow the approach of Refs. 19 36. The effect of the Chern-Simons action is twofold: (a) a constraint on the allowed states which are required to satisfy a relation between the local particle density and the local statistical flux and (b) a set of commutation relations for the gauge fields.[19] With the sign conventions of the action of Eq. (3.2), the constraint reads

$$\rho(\mathbf{x},t) = \theta\mathcal{B}(\mathbf{x},t) \tag{3.3}$$

with $\rho(\mathbf{x},t) \equiv \psi^*(\mathbf{x},t)\psi(\mathbf{x},t)$. This is a constraint on the allowed states in the Hilbert space and it plays the same role as Gauss' Law in Maxwell's electrodynamics. In Eq. (3.3) the particles live on the sites of the square lattice whereas the flux $\mathcal{B}(\mathbf{x},t)$ is defined on the sites of the dual of the square lattice, i.e., the center of the plaquette "north-east" of the lattice site $\mathbf{x}$. The Chern-Simons gauge fields are defined to be on the links of the square lattice.

The lattice form of the Chern-Simons action $S_{\text{CS}}(\mathcal{A})$ can be written as the sum of two terms

$$S_{\text{CS}}(\mathcal{A}) = S_{\text{CS}}^{(1)} + S_{\text{CS}}^{(2)}, \tag{3.4}$$

where $S_{\text{CS}}^{(1)}$ and $S_{\text{CS}}^{(2)}$ are responsible for enforcing the constraint and for the determination of the commutation relations, respectively. Here we will use the form given by Eliezer and Semenoff[36]

$$S_{\text{CS}}^{(1)} = \int dt \sum_{\mathbf{x}} A_0(\mathbf{x},t)\,\epsilon^{ij} d_i A_j(\mathbf{x},t), \tag{3.5}$$

$$S_{\text{CS}}^{(2)} = -\frac{1}{2}\int dt \sum_{\mathbf{x}} A_i(\mathbf{x},t)\,K^{ij}\frac{\partial}{\partial t}A_j(\mathbf{x},t). \tag{3.6}$$

Here, we have used the forward difference operator d_i which acts on functions $f(\mathbf{x})$ defined on the sites as $d_i f(\mathbf{x}) \equiv f(\mathbf{x}+\hat{e}_i) - f(\mathbf{x})$, where $\hat{e}_i$ is a unit vector pointing towards the direction $i = 1,2$ of the square lattice. Similarly the backward difference operator $\hat{d}_i$ acts like $\hat{d}_i \equiv f(\mathbf{x}) - f(\mathbf{x}-\hat{e}_i)$. The kernel K^{ij} is found to be given by the matrix[36]

$$K^{ij} = -\frac{1}{2}\begin{pmatrix} d_2 + \hat{d}_2 & -2 - 2d_1 + 2\hat{d}_2 + \hat{d}_2 d_1 \\ 2 + 2d_2 - 2\hat{d}_1 - \hat{d}_1 d_2 & -d_1 - \hat{d}_1 \end{pmatrix}. \tag{3.7}$$

The quartic term in the fermion part of the action represents the S_zS_z interaction. The constraint of Eq. (3.3) restricts the space of configurations to those in which the fermion occupation number at a site is equal to the flux at the plaquette north-east of the site divided by θ. Hence, it is legitimate to replace in the action the fermion density by $\mathcal{B}/\theta$. Therefore, the interaction term in the action, S_{int}, becomes only a function of the configuration of the gauge fields

$$S_{\text{int}} = -\tfrac{1}{2}\sum_{\mathbf{x},\mathbf{x}'}[\theta\mathcal{B}(\mathbf{x},t) - \tfrac{1}{2}]V(\mathbf{x}-\mathbf{x}')[\theta\mathcal{B}(\mathbf{x}',t) - \tfrac{1}{2}] \tag{3.8}$$

with a pair potential $V(\mathbf{x}-\mathbf{x}')$ given by

$$V(\mathbf{x}-\mathbf{x}') = \begin{cases} J\lambda & \text{if } \mathbf{x}' = \mathbf{x} \pm \mathbf{e}_j \quad (j=1,2), \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Notice that the interaction term is *bilinear* in the gauge fields instead of a quartic functional of the Fermi fields. This result was obtained before, in the context of the FQHE, in Ref. 38. Alternatively, one could use a Hubbard-Stratonovich transformation and arrive at the same result.[38]

By putting all the terms together we arrive to the final form of the action

$$S = S_F(\psi, \psi^*, \mathcal{A}_\mu) + S_{\text{int}}(\mathcal{A}_\mu) + \theta S_{\text{CS}}(\mathcal{A}_\mu), \tag{3.10}$$

where $S_F(\psi,\psi^*,\mathcal{A}_\mu)$ is the action for the fermions coupled to the gauge field

$$S_F = \int dt \sum_{\mathbf{x}} \Bigg\{ \psi^*(\mathbf{x},t)[iD_0+\mu]\psi(\mathbf{x},t) - \frac{J}{2}\sum_{j=1,2}\left[\psi^*(\mathbf{x}+e_j,t)e^{-i\mathcal{A}_j(\mathbf{x},t),t)}\psi(\mathbf{x},t) + \text{c.c.}\right]\Bigg\}. \tag{3.11}$$

$S_{\text{int}}(\mathcal{A}_\mu)$ and S_{CS} are defined in Eqs. (3.8) and (3.4), respectively. From now on we will use the action of Eq. (3.10).

IV. MEAN-FIELD THEORY AND SEMICLASSICAL EXPANSION

We will now proceed to derive a mean-field theory in the usual fashion. The fermionic part in the action (3.10), being bilinear, can be integrated out yielding a fermion determinant. The resulting effective action S_{eff} is given by

$$S_{\text{eff}} = -i\text{tr}\ \ln[iD_0 + \mu - h(\mathcal{A})] + \theta S_{\text{CS}}(\mathcal{A}_\mu) - \frac{1}{2}\int dt \sum_{\mathbf{x},\mathbf{x}'}[\theta\mathcal{B}(\mathbf{x},t) - \tfrac{1}{2}]V(\mathbf{x}-\mathbf{x}') \times[\theta\mathcal{B}(\mathbf{x}',t) - \tfrac{1}{2}], \tag{4.1}$$

where $h(\mathcal{A})$ is the kinetic part that we can write in operator form as

$$h(\mathcal{A}) = \frac{J}{2}\sum_{\mathbf{x}}\sum_{j=1,2} |\mathbf{x},t\rangle e^{i\mathcal{A}_j(\mathbf{x},t)}\langle \mathbf{x}+\mathbf{e}_j,t| + \text{H.c.} \tag{4.2}$$

The semiclassical approximation is obtained by expanding around stationary configurations of fields that minimize the effective action. It is worthwhile to note that this effective action does not contain any small parameter to control this expansion. This is a problem that was also found in the context of the anyon superfluid as well as in the fermion Chern-Simons approach to the FQHE of Refs. 24 and 38. There it was found that if the AFA had a spectrum which is fully gapped, the fluctuations restore the correct spectrum at long wavelengths. We will see that, for the problem at hand, the AFA yields a gapless spectrum at least for a range of values of the anisotropy parameter λ. Thus, the validity of the AFA, even qualitatively, is questionable for that range. Indeed we will find it necessary to go well beyond the AFA in order to obtain asymptotically exact results.

The average field approximation is realized by the solutions of the saddle-point equations

$$\left.\frac{\delta S_{\text{eff}}}{\delta \mathcal{A}_\mu(\mathbf{x},t)}\right|_{\bar{\mathcal{A}}} = 0. \tag{4.3}$$

As usual, the variations of the fermionic part S_F of the action S_{eff} [i.e., the first term in Eq. (4.1)] with respect to the components of the gauge field $\mathcal{A}_\mu$ gives AFA expressions for the charge density $n(\mathbf{x},t)$,

$$\langle n(\mathbf{x},t)\rangle = \langle \psi^*(\mathbf{x},t)\psi(\mathbf{x},t)\rangle = -\frac{\delta S_F}{\delta\mathcal{A}_0(\mathbf{x},t)}, \tag{4.4}$$

and current density $j_k(\mathbf{x},t)$,

$$\langle j_k(\mathbf{x},t)\rangle = \left\langle \frac{iJ}{2}\left[\psi^*(\mathbf{x},t)e^{i\mathcal{A}_k(\mathbf{x},t)}\psi(\mathbf{x}+\mathbf{e}_k,t) - \psi^*(\mathbf{x}+\mathbf{e}_k,t)e^{-i\mathcal{A}_k(\mathbf{x},t)}\psi(\mathbf{x},t)\right]\right\rangle = -\frac{\delta S_F}{\delta\mathcal{A}_k(\mathbf{x},t)}. \tag{4.5}$$

Within the AFA, we find that the average density and currents are given by

$$\langle n(\mathbf{x},t)\rangle_{\text{AFA}} = -\frac{\delta S_F}{\delta\mathcal{A}_0(\mathbf{x},t)} = -iS(\mathbf{x},t;\mathbf{x},t), \tag{4.6}$$

$$\langle j_k(\mathbf{x},t)\rangle_{\text{AFA}} = -\frac{\delta S_F}{\delta\mathcal{A}_j(\mathbf{x},t)} = \frac{J}{2}\left(S(\mathbf{x}+\mathbf{e}_j,t;\mathbf{x},t)e^{i\bar{\mathcal{A}}_j(\mathbf{x},t)} - S(\mathbf{x},t;\mathbf{x}\mathbf{e}_j,t)e^{-i\bar{\mathcal{A}}_j(\mathbf{x},t)}\right), \tag{4.7}$$

where $\bar{D}_0 = \partial_0 + i\bar{\mathcal{A}}_0$ and $\bar{\mathcal{A}}_\mu$ ($\mu = 0,1,2$) is the expectation values of the components of the gauge fields within the AFA. The function $S(\mathbf{x},t;\mathbf{x}',t')$, which appears in (4.6) and (4.7), is the Green function for the fermions moving in the background field $\bar{\mathcal{A}}_\mu$ which is the solution of the lattice differential equation

$$\left(i\bar{D}_0 + \mu - h(\bar{\mathcal{A}})\right) S(\mathbf{x},t;\mathbf{x}',t') = \delta_{\mathbf{x},\mathbf{x}'}\delta(t-t'). \tag{4.8}$$

Below we give the explicit form of this Green function.

By varying S_{eff} with respect to $\mathcal{A}_0$ we recover the constraint equation, now as a condition for the value of the local density of the stationary configurations

$$\langle n(\mathbf{x})\rangle = \theta\langle\mathcal{B}(\mathbf{x})\rangle. \tag{4.9}$$

Likewise, by differentiating with respect to the spacial components $\mathcal{A}_k$, we find an equation for the fermion current

$$\langle j_k(\mathbf{x})\rangle = \theta\epsilon^{kl} d_l \langle \mathcal{A}_0(\mathbf{x})\rangle + \frac{\theta}{2}\left(K^{lk} - K^{kl}\right)\partial_0 \langle \mathcal{A}_l(\mathbf{x})\rangle - \theta^2 \epsilon_{kl} \sum_{\mathbf{x}'} V(\mathbf{x}-\mathbf{x}')\hat{d}_l \langle \mathcal{B}(\mathbf{x}')\rangle. \tag{4.10}$$

In terms of the pair potential of Eq. (3.9), we can write the expectation value of the current in the form

$$\langle j_k(\mathbf{x})\rangle = \theta\epsilon^{kl} d_l \langle \mathcal{A}_0(\mathbf{x})\rangle + \frac{\theta}{2}\left(K^{lk} - K^{kl}\right)\partial_0 \langle \mathcal{A}_l(\mathbf{x})\rangle - \theta^2 J\lambda \epsilon_{kl}\hat{d}_l \sum_{j=1,2} [\langle \mathcal{B}(\mathbf{x}+\mathbf{e}_j)\rangle + \langle \mathcal{B}(\mathbf{x}-\mathbf{e}_j)\rangle]. \tag{4.11}$$

In Eqs. (4.10) and (4.11) K^{lk} is the operator matrix defined in Eq. (3.7). Using this definition explicitly, the terms in Eq. (4.11) which depend on K^{lk} are given by

$$\theta\left(K^{lk} - K^{kl}\right)\partial_0 \langle \mathcal{A}_l(\mathbf{x})\rangle = -\frac{\theta}{2}\epsilon^{kl}[4 + 2(d_2 - \hat{d}_2) + 2(d_1 - \hat{d}_1) + \hat{d}_2 d_1 + \hat{d}_1 d_2]\partial_0 \langle \mathcal{A}_l(\mathbf{x})\rangle. \tag{4.12}$$

In the continuum limit, the first two terms in Eq. (4.11) are equal to the conventional Chern-Simons current, i.e., $\theta\epsilon^{kl}\langle \mathcal{E}_l(\mathbf{x})\rangle$.

The saddle-point Equations (4.9) and (4.11) have many solutions. For a half-filled system (i.e., $S_z = 0$), the solution with lowest energy corresponds to a stationary state with a modulated charge density and with zero current. In the magnetic language this is a state with a nonzero Néel order parameter Δ, such that

$$\langle n(\mathbf{x})\rangle = \frac{1}{2} - \Delta e^{i\pi\cdot\mathbf{x}}. \tag{4.13}$$

Using the constraint of Eq. (4.9) we get

$$\langle B(\mathbf{x})\rangle = \frac{1}{\theta}\langle n(\mathbf{x})\rangle \equiv 2\pi\langle n(\mathbf{x})\rangle. \tag{4.14}$$

This state is time independent and it does not support any current, local or global. For a square lattice with $N\times N$ sites (with N even), periodic boundary conditions, and $\theta = 1/2\pi$, we can satisfy this static constraint with the following choice of gauge fields:

$$\mathcal{A}_0 = \tilde{A}_0 e^{i\pi\cdot\mathbf{x}}, \quad \mathcal{A}_j(\mathbf{x}) = \delta_{j,2}\left(\frac{\pi}{2} + \pi\cdot\mathbf{x} + \pi\Delta e^{i\pi\cdot\mathbf{x}}\right). \tag{4.15}$$

This solution corresponds to a problem in which a fermion moves in the presence of a modulated magnetic field (with an average of *half flux quantum per plaquette*) and a periodic potential $\mathcal{A}$, with the *same* modulation. By solving these equations we find a solution in which

$$\tilde{A}_0 = 4\lambda J\Delta. \tag{4.16}$$

For ground states with a modulation in the effective magnetic field, a periodic site potential $\langle \mathcal{A}_0\rangle$ appears which is commensurate with the variation of the field. The charge and the field vary in the same way, as required by the constraint. Thus, everything is self-consistent. Let us note in passing that for sectors with $S_z \neq 0$ the average flux is not equal to one-half of the quantum. At the level of the AFA this problem now becomes equivalent to a general Hofstadter problem.[39] This problem is known to a have a very complex spectrum which we will not explore here. One of us[22] discussed a similar problem in his treatment of the lattice anyon gas. In what follows we will only discuss the case $S_z = 0$.

The Green functions of the saddle-point problem are obtained by solving the lattice differential equation of Eq. (4.8). In the gauge that we have chosen, $\bar{A}_1 = 0$ and with the configuration of gauge fields of Eq. (4.15), the solution of Eq. (4.8) is the Green function of a one-particle system with the effective Hamiltonian $H_{\rm MF} = h(\bar{A}_j) + \sum_{\mathbf{x}} \bar{A}_0(\mathbf{x})$, which takes the form

$$H_{\rm MF} = \sum_{\mathbf{x}} \left\{ 4\lambda J\Delta e^{i\pi\cdot\mathbf{x}} \ |\mathbf{x}\rangle\langle\mathbf{x}| + \frac{J}{2} \ \left(|\mathbf{x}\rangle\langle\mathbf{x}+e_1| + a(\mathbf{x})|\mathbf{x}\rangle\langle\mathbf{x}+e_2| + \text{H.c.}\right)\right\} \tag{4.17}$$

with

$$a(\mathbf{x}) = e^{i\bar{A}_2(\mathbf{x})} = -\sin\pi\Delta + ie^{i\pi\cdot\mathbf{x}}\cos\pi\Delta \ . \tag{4.18}$$

Note that $a(\mathbf{x})$ is equal to $a = ie^{i\pi\Delta}$ when $\mathbf{x}$ belongs to the A sublattice and to $a^* = -ie^{-i\pi\Delta}$ when $\mathbf{x}$ belongs to the B sublattice. For $\Delta = 0$ we recover the flux-phase state of (uniform) half-flux quantum per plaquette.[26,27]

The Hamiltonian of Eq. (4.17) can be diagonalized in Fourier space. The charge density and the effective flux are periodic functions which take different values on the sublattices A and B. Thus, the fermion Green function is a matrix whose entries label the sublattice dependence of its arguments. It has the form

$$S_{\alpha\beta}(x,x') = \begin{bmatrix} S_{AA}(x,x') & S_{AB}(x,x') \\ S_{BA}(x,x') & S_{BB}(x,x') \end{bmatrix}, \tag{4.19}$$

where $x \equiv (\mathbf{x}, t)$ and $\alpha,\beta = A, B$. We find

$$S_{\alpha\beta}(x,x') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{|k_i|\le\frac{\pi}{2}} \frac{d\mathbf{k}}{\pi^2}\frac{e^{i\omega(t-t')-i(\mathbf{x}-\mathbf{x}')}}{\omega^2 - E^2(\mathbf{k}) + i\epsilon}\begin{pmatrix} \omega + 4J\lambda\Delta & J(\cos k_1 + a\cos k_2) \\ J(\cos k_1 + a^*\cos k_2) & \omega - 4J\lambda\Delta \end{pmatrix}. \tag{4.20}$$

Notice that the momentum integrals are done over the range $|k_i| \le \frac{\pi}{2}$ $(i = 1, 2)$. The fermion dispersion $E(\mathbf{k})$ is given by

$$E(\mathbf{k}) = +J\sqrt{\cos^2 k_1 + \cos^2 k_2 - 2\sin(\pi\Delta)\cos k_1 \cos k_2 + (4\lambda\Delta)^2}. \tag{4.21}$$

Recently, Wang[28] obtained similar results but without self-consistency between charge and flux modulations.

The dependence of $E(\mathbf{k})$ in $\pi\Delta$ as given by Eq. (4.21) is a consequence of the self-consistency. Once the Green function is known, the parameter Δ can be calculated by demanding that the saddle-point equations be satisfied. Thus, the average density on one sublattice, say A, must be given by

$$\begin{aligned}\langle n(\mathbf{x},t)\rangle_A &= -iS_{AA}(\mathbf{x},t;\mathbf{x},t)\\ &= -i\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{|k_i|\le\frac{\pi}{2}}\frac{d\mathbf{k}}{\pi^2}\frac{\omega+4J\lambda\Delta}{\omega^2-E^2(\mathbf{k})+i\epsilon}\\ &= \frac{1}{2} - 2J\lambda\Delta\int_{|k_i|\le\frac{\pi}{2}}\frac{d\mathbf{k}}{\pi^2}\frac{1}{E(\mathbf{k})}.\end{aligned} \tag{4.22}$$

From this expression we arrive at the *gap equation*

$$\Delta = 2J\lambda\Delta\int_{|k_i|\le\frac{\pi}{2}}\frac{d\mathbf{k}}{\pi^2}\frac{1}{E(\mathbf{k})}. \tag{4.23}$$

We now discuss the properties of the spectrum found in the AFA. For general values of λ, the spectrum of Eq. (4.21) has a gap $E_g = 4J\lambda\Delta$ at the four points in k space $(\pm\pi/2, \pm\pi/2)$. Equation (4.13) is a self-consistent equation for Δ. Qualitatively, its solution as a function of the anisotropy parameter λ is a monotonically increasing function which begins at a critical value of λ. From Eq. (4.23), it follows that there exists a critical value of the anisotropy parameter λ_c given by

$$\frac{1}{\lambda_c} = 2\int_{|k_i|\le\frac{\pi}{2}}\frac{d\mathbf{k}}{\pi^2}\frac{1}{\sqrt{\cos k_1^2+\cos k_2^2}} \simeq 2\times 1.285 = 2.570, \tag{4.24}$$

below which $\Delta = 0$, and the spectrum is gapless. Thus, the AFA predicts that the spectrum of fermions has a gap above a critical anisotropy $\lambda_c \simeq 0.39$. For small and positive values of $\lambda - \lambda_c$, the gap has the dependence

$$\Delta \simeq \text{const}(\lambda - \lambda_c) \tag{4.25}$$

and it vanishes for *all* $\lambda \le \lambda_c$.

The vanishing of the gap for $\lambda \to \lambda_c$, and in particular the exponent of Eq. (4.25), results from the collapse of the Fermi surface to four Fermi points and from the linear dispersion near the Fermi points. Since the density of one-particle states vanishes in the middle of the band ($E = 0$) for a system with a linear, relativisticlike, energy-momentum dispersion in two space dimensions, all instabilities are pushed to finite values of the coupling constants and there is a critical coupling. In contrast, in conventional mean-field theories of interacting fermions at finite density on a lattice (but with zero gauge flux) there is no critical coupling and the spectrum is always gapped in the presence of nesting. In such cases, the gap dependence for small λ would be of the form $\Delta \sim e^{-\frac{1}{\lambda}}$. Thus, the existence of the critical value λ_c is a consequence of the inclusion of the gauge flux that removes the van Hove singularity in the density of states characteristic of the two-dimensional square lattice.

The exponent of Eq. (4.25) is valid only at the level of the AFA. In critical phenomena, it is usually the case that the exponents found at the level of "classical" approximations such as mean-field theory or in the large-N limit, get modified due to the effects of fluctuations. The AFA is a semiclassical approximation. Since the dimensionality of space-time of this system is $2+1$ we should expect nonclassical behavior. In fact, the exponent of Eq. (4.25) would be correct for a theory of N species of self-interacting relativistic fermions in the $N \to \infty$ limit. We will see in Sec. V that the system that we are studying here is indeed related to a theory of self-interacting relativistic fermions but *not* in the large-N limit. Thus, in principle, fluctuations are expected to correct this exponent. However, we will also show that this second-order phase transition is never reached and that it is preempted by a fluctuation-induced first-order transition at a value of lambda λ^* strictly larger than λ_c. We will also show that, at this fluctuation-induced first-order transition, the system has the expected physical properties of the isotropic Heisenberg antiferromagnet in a Néel state.

When applied to the isotropic case ($\lambda = 1$), the results of the AFA imply a Néel state with a value for the order parameter $\Delta = 0.442$ and the energy $E = -0.314J$ per bond. These results should be compared with the best numerical estimate of the energy, $-0.334J$ per bond.[16,40] Nevertheless, it is interesting that this approximation yields a Néel state instead of a flux phase, as one might have guessed beforehand.

But, is this the correct spectrum? According to the AFA, the spectrum consists of free fermions with an energy gap for $\lambda > \lambda_c$ but gapless otherwise. Clearly this spectrum has nothing to do with what we found semiclassically in Sec. II. Earlier work in anyons and in the FQHE suggests that fluctuations should play a crucial role. However, notice that since for $\lambda < \lambda_c$ the AFA spectrum is gapless, the fluctuations may yield much more important effects than what we have described with the AFA. In particular, this gapless case was found in Ref. 22 in the case of semions at half-filling and was found that the state was not determined by the AFA alone.

Hence, the next logical step is to look at the effects of fluctuations around the solutions of the AFA. While it is possible to carry out this calculation and to keep the full lattice effects at the same time, the expressions are very cumbersome and not amenable to an analytic treatment. Instead, we will resort to a different approach in which only the low energy degrees of freedom are kept. This is equivalent to replace the system by an effective continuum field theory. These methods are accurate provided that the gaps do not become too large, in which case the lattice effects may become dominant. This limitation will force us to work at values of λ close to λ_c. Nevertheless, our results will be qualitatively correct even away from this regime. This approach is pursued in Sec.V.

V. EFFECTIVE FIELD THEORY NEAR λ_c

In this section we will consider the behavior of the system near the critical anisotropy λ_c. The mean-field theory of Sec. IV yielded an energy gap for the *fermions* which vanishes linearly as $\lambda \to \lambda_c$ from above. The problem that we want to address here is the nature of this phase transition. We will find that the transition at λ_c does not take place as a result of radiative corrections, namely of fluctuations of the gauge field. Instead, a different transition with a *larger* critical value λ^* will actually occur and supersede the AFA-predicted transition at λ_c. By counting Goldstone modes we will be able to identify the transition at λ^* with the isotropic Heisenberg point.

We will now develop an effective theory for the low energy degrees of freedom and construct an effective field theory for the fluctuations about the AFA. To do so we will adopt a point of view which follows quite closely the treatment of the one-dimensional anisotropic spin-1/2 Heisenberg antiferromagnetic chain.[9] However, the physics that we find is very different. The Jordan-Wigner approach that we have used in this paper closely resembles the Jordan-Wigner transformation of one-dimensional systems. The main difference is that, in one space dimension, the only flux that can be defined is the one that is trapped by the *entire* chain and, hence, it is equivalent to a boundary condition. In two dimensions, in addition to global or topological flux, it is possible to generate local flux. Local fluxes cannot be reduced just to boundary conditions and a local, fluctuating, gauge degree of freedom appears necessarily in the effective theory.

We begin by first summarizing the standard procedure used in one-dimensional systems, first introduced by Luther.[9] We will next follow that construction for the two-dimensional case and use it to discuss the critical behavior near λ_c.

The fermionization of the one-dimensional spin chains is usually done in the following manner.[5] First, the algebra of spin-1/2 operators is realized in terms of canonical fermion operators *via* the use of the Jordan-Wigner transformation. The resulting system consists of a set of *spinless* fermions hopping between the nearest-neighboring sites of a one-dimensional chain of atoms. Two fermions cannot share the same site (Pauli principle) and only interact when on nearest-neighboring sites with a coupling constant equal to twice the strength of the S_zS_z coupling constant. The total fermion number is equal to the number of down spins (depending on the definitions). There are some subtleties concerning boundary conditions which are important but are not related to the issues that are being discussed here. Thus, the sector of the spin system with total $S_z = 0$ maps onto the half-filled portion of the Hilbert space of the Fermi system. Notice that in one dimension the fermion degrees of freedom exhaust the Hilbert space and no other degrees of freedom are needed to represent the states of the spin chain. The second step[9] consists of finding a quantum field theory in one space dimension which yields the exact long distance critical behavior of this system of interacting spinless fermions.[41] This is done by separating the slow from the fast components of the Fermi fields. In one dimension, the fermions can either move to the left (left movers) or to the right (right movers). The noninteracting Fermi system (equivalent to the spin-1/2 chain) has two Fermi points with momenta $p = \pm p_F = \pm\frac{\pi}{2}$. The left and right moving components of the Fermi field can be thought of as the *chiral* components of a two-component Dirac spinor in one space and one time moving at a "speed of light" equal to the Fermi velocity v_F. The equivalent quantum field theory has the left and right movers interacting through a backscattering process (up to umklapp processes which are crucial to reproduce the correct behavior of the isotropic Heisenberg model[10]). The resulting field theory is the well-known Luttinger-Tomonaga-Thirring model[42] which, by using bosonization, can be shown to be equivalent to the sine-Gordon field theory[9] (if the umklapp terms are kept). The result is that for weak Ising coupling (i.e., strong XY anisotropy) the spectrum of the system is that of fermions with anomalous dimensions (as a result of the backscattering interactions) up to a value of the backscattering coupling constant at which the umklapp processes become marginal operators (and beyond which, in the Ising phase, they are relevant). In the Ising phase there is an energy gap for all excitations.

In the case of the two-dimensional spin system, the AFA of Sec. IV yielded a spectrum of free fermions with a "flux phase" band structure. At half-filling, i.e., total $S_z = 0$, the flux phase band structure has a "Fermi surface" which reduces to four Fermi points located at $(\frac{\pi}{2}, \frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$ and $(-\frac{\pi}{2}, -\frac{\pi}{2})$. Just as in the case of the one-dimensional spin chain, whose fermion description has two Fermi points, the two-dimensional problem can also be mapped onto a Dirac-like problem of Affleck and Marston.[26] The main differences between the problem that we discuss here and the flux phase of Affleck and Marston is that (a) the fermions here are spinless and (b) instead of a Hubbard-Stratonovich field which fluctuates both in amplitude and phase we have just a fluctuating phase on the bonds. The Chern-Simons term is not present in the system discussed by Affleck and Marston.

We now follow the methods and notations of Ref. 34 to obtain an effective continuum theory. Since the detailed derivation is rather tedious we will only highlight the procedure. Our general strategy will be to look for the terms in the action which involve only low energy degrees of freedom. We will only keep terms with the smallest numbers of derivatives in each of the fluctuations since higher derivative terms are irrelevant in the renormalization group sense. We should keep in mind that this procedure will only give an approximate value for the coupling constants of the effective low energy theory since instead of integrating out the high energy degrees of freedom we are simply neglecting them. Hence, even though the form of the effective action will still be correct the values of the parameters (e.g., the anisotropy λ) at which transitions may occur will not coincide with the predictions of the lattice system. In particular the critical anisotropy will not be precisely $\lambda_c = 1$, but only approximately. We will have to use the symmetry properties of the spectrum of excitations at a certain value of the coupling constant of the effective theory to identify the isotropic point.

The starting point is the action of Eq. (3.10). The flux phase that we found in Sec. IV has a spectrum of fermions which become gapless at four Fermi points. The physically important states are those close to the Fermi points. Out of these states, we construct two (two-component) Dirac spinors. We begin by defining first a set of spinor components on the sites of the real square lattice. It is convenient to split the square lattice into four sublattices 1, 2, 3, and 4. Sublattice 1 is the set of sites of the form $\{\mathbf{x} = (2n_1, 2n_2)\}$ (with n_1 and n_2 arbitrary integers). Sublattices 2, 3, and 4 are the sites of the form $\{\mathbf{x} + \hat{\mathbf{e}}_1\}$, $\{\mathbf{x} + \hat{\mathbf{e}}_2\}$, and $\{\mathbf{x} + \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\}$, respectively. The fermion amplitudes on each sublattice will be denoted by $\psi_a(\mathbf{x})$, with $a = 1, \ldots, 4$. Likewise, the gauge fields have to be split into components. This is so because the gauge fields can, and do, couple the different fermionic components. In this fashion we will be left with only slowly varying fields. Thus, with the same notation used for the fermions, out of the components of the gauge field $\mathcal{A}_\mu$ we define four sublattice amplitudes $\mathcal{A}_\mu^a(\mathbf{x}, t)$ $(a = 1, \ldots, 4)$. Since the space components $\mathcal{A}_j$ only enter in the action in exponential hopping amplitudes, it is convenient to define the sublattice amplitudes $W_j^a(\mathbf{x}, t) \equiv \exp[i\mathcal{A}_j^a(\mathbf{x}, t)]$. In the uniform flux phase (i.e., for $\lambda < \lambda_c$) the hopping amplitudes take the expectation values $\bar{W}_j^a$ $(j = 1, 2)$. The AFA equations tell us that, in the flux phase, the oriented product of the amplitudes $\bar{W}_j^a$ around each plaquette must be equal to -1. In Sec. IV we solved this requirement with the choice of Eq. (4.15). For the purposes of taking the continuum limit, we will choose instead $\bar{W}_1^a = i$ and $\bar{W}_2^a = i(-1)^a$. The two configurations are related by a gauge transformation and are equivalent modulo a flux of 2π. The time components have zero average. It is natural to define a set of slow, fluctuating, fields $\mathcal{A}_\mu$ by identifying $W_j^a(\mathbf{x}, t) \to \bar{W}_j^a \exp[i\mathcal{A}_j^a(\mathbf{x}, t)]$ (notice that the $\mathcal{A}_j^a$ is now a fluctuation).

These fluctuating fields are small but not slow, that is the fluctuations involve low-frequency processes with wave vectors which are not necessarily small. In fact, the gauge fields have components with wave vectors which mix all the fermionic components. For this reason, we proceed to define a set of fields which are slow and for which there is a sensible long distance limit. This we do for both fermions and gauge fields. The sublattice fermion amplitudes can be combined into two species of Dirac spinors Ψ_α^r (labeled by $r = 1, 2$, each with two components labeled by $\alpha = 1, 2$), defined by the following linear combinations:

$$
\begin{aligned}
\Psi_1^1 &= \frac{1}{2a_0}(\psi_1 + \psi_2 - i\psi_3 - i\psi_4),\\
\Psi_2^1 &= \frac{1}{2a_0}(-i\psi_1 - i\psi_2 + \psi_3 + \psi_4),\\
\Psi_1^2 &= \frac{1}{2a_0}(-i\psi_1 + i\psi_2 + \psi_3 - \psi_4),\\
\Psi_2^2 &= \frac{1}{2a_0}(\psi_1 - \psi_2 - i\psi_3 + i\psi_4),
\end{aligned}
\tag{5.1}
$$

where a_0 is a lattice spacing. The normalization factor is chosen so that the term in the action which includes the time derivative has coefficient one in the continuum limit. The Dirac structure is a consequence of the spectrum of the flux phase which is linear. Unlike the amplitudes ψ_a which are dimensionless, the continuum Fermi fields Ψ_α^r have dimensions of $(\text{length})^{-1}$. This is the correct canonical dimension for a Dirac field in $2+1$ dimensions. We also need to define a set of gamma matrices which in $2+1$ dimensions are 2×2 matrices which act on the Dirac components of the spinors, labeled by the Greek index α. We choose them to be $\gamma_0 = \sigma_2$, $\gamma_1 = i\sigma_1$, and $\gamma_2 = i\sigma_3$, where σ_i $(i = 1, 2, 3)$ are the 2×2 Pauli matrices. We will also need a second set of Pauli matrices, that we denote by T^b $(b = 1, 2, 3)$, which will act on the species index of the spinors, labeled by the Latin index r. When necessary, we will also use the identity matrix I for each set of indices. In order to simplify the notation, we will avoid the explicit use of the indices and, instead, use the matrices to denote the operators of interest.

With these definitions, and after rescaling the time coordinate t by the Fermi velocity v_F as $t = x_0/v_F$ (in units of J, we get $v_F = a_0$; here v_F is measured in units in which the XY term of the Hamiltonian has amplitude $\frac{1}{2}$), the free part of the fermion action becomes, in the continuum limit

$$
S_F^{(0)} = \int d^3x\, \bar{\Psi}_r\, i\partial\!\!\!/\, \Psi_r \tag{5.2}
$$

with $\bar{\Psi} = \Psi^*\gamma_0$. This continuum theory is valid for fluctuations with wave vectors smaller than some cutoff $\Lambda \approx \frac{\pi}{a_0}$, where a_0 is the lattice spacing. The fact that this is an effective theory at length scales long compared with the lattice spacing will have important consequences for our analysis. In field theory the choice of cutoff (or regularization) is largely arbitrary. Different choices of regularization usually lead to the same theory. However, for theories of fermions a number of subtleties arise connected with the way regularizations treat the symmetries of the effective continuum theories. In $2+1$ dimensions relativistic massive fermions have a parity anomaly. However, if the fermion mass has a dynamical origin (i.e., if it is induced by fluctuations) the parity anomaly may be lost in some regularization methods (such as dimensional regularization) which set to zero all nonlogarithmic divergencies. In any case, in our problem we are not free to choose an arbitrary regularization scheme to cutoff the divergencies present in various Feynman diagrams of the effective continuum theory. Instead the choice of cutoff will be done in such a way that the symmetries of the *lattice* system are respected. In particular, we will see in the next section that schemes such as dimensional regularization cannot be used for this problem.

Next we define new linear combinations of the gauge fields. We choose the new linear combinations which have a dominant wave vector (q_1, q_2) such that it mixes the fermionic components defined above. We use the notation $A_\mu^{q_1 q_2}$. The important wave vectors are $(q_1, q_2) = (0,0), (\pi, 0), (\pi, \pi), (0, \pi)$. The new fields are

$$A_\mu = \frac{1}{4a_0}(\mathcal{A}_\mu^1 + \mathcal{A}_\mu^2 + \mathcal{A}_\mu^3 + \mathcal{A}_\mu^4), \quad (5.3)$$

$$A_\mu^{\pi 0} = \frac{1}{4a_0}(\mathcal{A}_\mu^1 - \mathcal{A}_\mu^2 + \mathcal{A}_\mu^3 - \mathcal{A}_\mu^4), \quad (5.4)$$

$$A_\mu^{\pi\pi} = \frac{1}{4a_0}(\mathcal{A}_\mu^1 - \mathcal{A}_\mu^2 - \mathcal{A}_\mu^3 + \mathcal{A}_\mu^4), \quad (5.5)$$

$$A_\mu^{0\pi} = \frac{1}{4a_0}(\mathcal{A}_\mu^1 + \mathcal{A}_\mu^2 - \mathcal{A}_\mu^3 - \mathcal{A}_\mu^4), \quad (5.6)$$

where $A_\mu \equiv A_\mu^{00}$ are the smooth fluctuations of the gauge field.

The only fluctuations that are usually kept in these type of analysis are the smooth fields like A_μ. However, we find that some of the other amplitudes are very important. In fact we will find that the fluctuations $A_\mu^{\pi\pi}$ are connected with Néel fluctuations. The fields $A_\mu^{q_1 q_2}$ couple to operators which are bilinear in Fermi fields and whose characteristic wave vector is (q_1, q_2).

In terms of the fermion amplitudes defined in Eq. (5.1) there is a total of 16 operators which are bilinear combinations of the amplitudes Ψ_α^r. They have the form $M = \bar\Psi\Psi$, $M^b = \bar\Psi T^b \Psi$, $M_\mu = \bar\Psi\gamma_\mu\Psi$, and $M_\mu^b = \bar\Psi\gamma_\mu T^b\Psi$. In spin language these fermion bilinears correspond to linear combinations of site occupation numbers and of hopping amplitudes among sites on the four sublattices. Out of the 16 bilinears we will only discuss three of them, M, M^3 and M_0, which correspond to the parity breaking mass operator, the Néel order parameter and the charge density operator, respectively. We find the following identifications:

$$M = \bar\Psi\Psi \leftrightarrow i(\psi_1^\dagger\psi_4 - \psi_4^\dagger\psi_1) + i(\psi_3^\dagger\psi_2 - \psi_2^\dagger\psi_3), \quad (5.7)$$

$$M_0 = \bar\Psi\gamma_0\Psi \leftrightarrow \psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2 + \psi_3^\dagger\psi_3 + \psi_4^\dagger\psi_4, \quad (5.8)$$

$$M^3 = \bar\Psi T^3\Psi \leftrightarrow -\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2 + \psi_3^\dagger\psi_3 - \psi_4^\dagger\psi_4. \quad (5.9)$$

These identifications show that M_0 is the total fermion occupation number averaged over the four sublattices. In the sector with $S_z = 0$ we expect to find $\langle M_0 \rangle = 0$. The operator M^3 is the staggered occupation fermion number which, back in spin language is the Néel order parameter. Finally, M is an operator which induces hopping across the main diagonals of the plaquettes. It is also easy to show that the phase factors present in the definition of M, when combined with the phase factors of the flux phase, indicate that the flux on *every* elementary *triangle* inscribed in each plaquette is equal to $\frac{\pi}{2}$. Thus, a nonzero value of $\langle M \rangle$ in the ground state breaks both T and P. M is the chiral order parameter introduced by Wen, Wilczek, and Zee.[43] We will see in Sec.VI that this operator plays a very important role in the determination of the physically correct (PC) phase diagram.

The next step is to write the action of Eq. (3.10) in terms of the slowly varying fields Ψ_α^r and $A_\mu^{q_1 q_2}$. We discuss first the Fermion part of the effective Lagrangian $\mathcal{L}_F$ in the continuum limit. After expanding the hopping amplitudes up to leading order in fluctuations and after taking the continuum limit we get

$$\mathcal{L}_F = \bar\Psi \, i\not{D} \, \Psi + A_0^{\pi\pi}\bar\Psi T^3 \Psi, \quad (5.10)$$

where $\not{D} \equiv D_\mu\gamma^\mu$ and $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative which represents the coupling to the long wavelength smooth pieces of the Chern-Simons gauge field. We have not included in the final form of the Lagrangian, additional terms of the form $\mathcal{L}_{\rm extra}$,

$$\mathcal{L}_{\rm extra} = A_0^{0\pi}\bar\Psi T^2\gamma_1\Psi + A_0^{\pi 0}\bar\Psi T^1\gamma_2\Psi - A_1^{0\pi}\bar\Psi T^2\gamma_0\Psi - A_2^{\pi 0}\bar\Psi T^1\gamma_0\Psi \quad (5.11)$$

which couple the fermions to fluctuations with wave vectors $(0,\pi)$ and $(\pi, 0)$, since it is possible to show that they are irrelevant in the renormalization group sense. Physically, this can be understood since these operators do not acquire an expectation value in any of the phases of our interest. Using the linear combinations of Eq. (5.1), one can show that these operators are related with both spin density wave order parameters with wave vectors (0π) and $(\pi 0)$ and to Peierls (or dimer) order parameters with the same wave vectors. Neither type of order occurs in this system. From now on we will ignore these terms. Notice that these terms have the same scaling dimension as the ones that are being kept. It is conceivable that there are other situations in which these terms become dominant such as in the vicinity of a dimerization transition. Such phase transitions are possible in a frustrated antiferromagnet. However, the Chern-Simons (or Wigner-Jordan) mapping used here cannot be used as it stands for nonbipartite lattices. The terms in $\mathcal{L}_{\rm extra}$ break the rotational invariance of the continuum theory of Eq. (5.10) down to the allowed symmetries of the point group of the square lattice. Again, such anisotropies will only be relevant in the vicinity of ground states which break such symmetries. Below we will also ignore terms in the bosonic part of the Lagrangian with the same symmetry properties.

Finally, we need to find the continuum form of the bosonic parts of the action (3.10). We only keep terms to leading order in the lattice spacing for each field. There are two sets of contributions. One set comes from $S_{\rm int}$. When written in terms of the slow components of the gauge fields they contribute with the term $\mathcal{L}_{\rm int}$:

$$\mathcal{L}_{\rm int} = \frac{g}{2}\left(A_2^{\pi\pi} - A_1^{\pi\pi}\right)^2 - \frac{1}{2\bar e^2}\left(\partial_1 A_2 - \partial_2 A_1\right)^2, \quad (5.12)$$

where the effective coupling constants g and $\bar e^2$ are

$$g = \frac{4\lambda}{\pi^2 a_0}, \quad \bar e^2 = \frac{\pi^2}{\lambda a_0}. \quad (5.13)$$

The Maxwell-like term in the Lagrangian $\mathcal{L}_{\rm int}$ has no consequence on the phase diagram since it has one more derivative than the Chern-Simons term and, hence, it is irrelevant at long distances[45] and it will be ignored from now on.

The second set of contributions comes from the Chern-Simons terms $S_{\rm CS}$. We find a contribution to the Lagrangian of the form

$$\mathcal{L}_{\rm CS} = \frac{\theta}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} + \frac{2\theta}{a_0}A_0^{\pi\pi}\left(A_1^{\pi\pi} - A_2^{\pi\pi}\right), \quad (5.14)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength.

Before putting everything together we notice, by inspecting Eqs. (5.12) and (5.14), that the fields $A_1^{\pi\pi}$ and $A_2^{\pi\pi}$ enter in the total Lagrangian in terms of the difference $A_1^{\pi\pi} - A_2^{\pi\pi}$ and that the total Lagrangian is at most quadratic in this difference. Thus, we can integrate out these fields and find the Lagrangian of the form

$$\mathcal{L} = \bar\Psi\, i\not{D}\, \Psi + A_0^{\pi\pi}\bar\Psi T^3\Psi - \frac{1}{2\bar g}\left(A_0^{\pi\pi}\right)^2 + \frac{\theta}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}, \quad (5.15)$$

where $\bar g$ is an effective coupling constant. It is convenient to define a dimensionless coupling constant g_0 such that $\bar g = g_0 a_0$. In terms of the anisotropy λ and of the Chern-Simons coupling θ we get

$$g_0 = \frac{\lambda}{\pi^2\theta^2} = 4\lambda\left(\frac{\delta}{\pi}\right)^2, \quad (5.16)$$

where we have introduced the statistical angle $\delta = 1/2\theta$. Notice that for $\theta \to \frac{1}{2\pi}$ (i.e., in the boson limit of interest here) $\delta \to \pi$ and $g_0 \to 4\lambda$. Conversely, in the fermion limit $\delta \to 0$ or $\theta \to \infty$, the dimensionless coupling constant is weak, $g_0 \to 0$. Thus, in the hard core boson limit, which is the case of interest, we are dealing with a system in which the dimensionless coupling constant is typically of order unity and perturbation theory should not be reliable.

Let us discuss the physical meaning of the terms of the Lagrangian $\mathcal{L}$ of Eq. (5.15). The field $A_0^{\pi\pi}$ couples locally to the Néel order parameter operator $M^3 = \bar\Psi T^3\Psi$. Thus, if $A_0^{\pi\pi}$ picks up an expectation value, so does the order parameter M^3. Clearly such a state has Néel long range order. In Sec. IV we found that beyond some critical value of the anisotropy λ the system is in a Néel state. The field $A_0^{\pi\pi}$ also enters at most quadratically in the Lagrangian $\mathcal{L}$ and it can also be integrated out (in fact, we may regard the field $A_0^{\pi\pi}$ as a Hubbard-Stratonovich field). Indeed, after integrating out $A_0^{\pi\pi}$ the total Lagrangian $\mathcal{L}_{\rm 2D}$ for the two-dimensional (2D) system has the suggestive form

$$\mathcal{L}_{\rm 2D} = \bar\Psi\, i\not{D}\, \Psi + \frac{\bar g}{2}\left(\bar\Psi T^3\Psi\right)^2 + \frac{\theta}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}. \quad (5.17)$$

Thus, we find that the low energy degrees of freedom of the anisotropic quantum antiferromagnet can be described in terms of a theory of two relativistic Fermi fields in $2+1$ dimensions, with a four-Fermi interaction of strength $\bar g$, and which are also coupled with a Chern-Simons gauge field. The Chern-Simons coupling constant is restricted to the value $\theta = \frac{1}{2\pi}$. For the rest of this paper, we will consider this Lagrangian.

In the form of Eq. (5.17), the effective continuum theory for the two-dimensional system is a generalization of the equivalence that exists between the one-dimensional anisotropic spin-$\frac{1}{2}$ quantum Heisenberg antiferromagnet and a theory of a self-interacting relativistic Fermi field, the Luttinger-Thirring model.[9,10] The left and right components of the Fermi fields, Ψ_L and Ψ_R, can be viewed as the two components of a Dirac spinor in $1+1$ dimensions. The Lagrangian $\mathcal{L}_{\rm 1D}$ of the effective field theory for the antiferromagnetic spin chain is[44]

$$\mathcal{L}_{\rm 1D} = \bar\Psi\, i\not\partial\, \Psi + \frac{\bar g_{1D}}{2}\left(\bar\Psi\Psi\right)^2. \quad (5.18)$$

The coupling constant for the 1D system is $\bar g_{\rm 1D} = 2\lambda$ which is dimensionless.

The two-dimensional theory, with Lagrangian $\mathcal{L}_{\rm 2D}$, and the one-dimensional theory with Lagrangian $\mathcal{L}_{\rm 1D}$ differ in a number of ways: (a) in 1D there is only *one* species of Dirac fermions (instead of the two labeled by r in $\mathcal{L}_{\rm 2D}$), (b) there is *no gauge field* in 1D but there is a Chern-Simons gauge field in 2D, (c) $\bar g_{\rm 1D}$ is dimensionless while $\bar g$ has dimensions of length, and (d) the symmetries are different. They both have a self-interacting, quartic, term in fermions which, in both cases, is equal to the square of the Néel order parameter. We will also see that the coupling to the Chern-Simons term is, in this problem, responsible for much more than a change in statistics.

VI. CRITICAL BEHAVIOR AND PHASE DIAGRAM

In this section we discuss the phase diagram of the effective field theory derived in Sec.V with the Lagrangian $\mathcal{L}_{\rm 2D}$ of Eq. (5.17). Given the apparent similarity between this theory and its analog in one dimension $\mathcal{L}_{\rm 1D}$ of Eq. (5.18), one might think that the phase diagrams may also be quite similar. However, a closer analysis shows that this cannot possibly be correct.

The two theories have different symmetries as well as different scaling properties. In both cases, the order parameter of the Néel state is odd under a sublattice exchange or, what is the same, it is odd under a global shift of the field configuration by one lattice spacing. This is a global discrete symmetry which can be spontaneously broken by the ground state, even in one space dimension. In the case of the one-dimensional chain, the Néel order parameter, i.e., the difference of the spinless fermion occupancy between the two sublattices, is proportional to $\bar\Psi\Psi$. This operator is odd under the transformation $\Psi \to \gamma_5\Psi$ which changes the relative sign of the right and left moving amplitudes of the fermions. This symmetry is known as a discrete chiral symmetry. The Lagrangian $\mathcal{L}_{\rm 1D}$ is even under this chiral symmetry. As a result, it is possible to show that the operator $\bar\Psi\Psi$ does not acquire an expectation value to all orders in perturbation theory in the coupling constant $\bar g_{\rm 1D}$. Also, because this symmetry is present, renormalization effects do not induce fermion mass terms in the Lagrangian, which are proportional to $\bar\Psi\Psi$ and thus brake the symmetry explicitly.

The (1+1)-dimensional system has very special scaling properties. In space-time dimensions $D = 2$, the coupling constant $\bar{g}_{1D}$ is dimensionless. Then, the standard field-theoretic analysis tells us that the theory defined by the Lagrangian $\mathcal{L}_{1D}$, the Gross-Neveu model,[44] is renormalizable. If the number of fermionic species is N, this theory is asymptotically free with a β function[46] $\beta(\bar{g}_{1D}) = \left(\frac{N-2}{2\pi}\right)\bar{g}_{1D}^2$. For $N \geq 2$ this β function is strictly positive and the system has a dynamically generated energy gap. For the case of interest for the *one-dimensional* quantum Heisenberg antiferromagnet, $N = 1$ and the coupling constant is still dimensionless. However, in this case, the β function vanishes to all orders in perturbation theory and the system has a line of fixed points. In other words, the four-fermion operator $(\bar{\Psi}\Psi)^2$ is *marginal.* Bosonization studies show[9,47,48] that there is an operator which is irrelevant at small coupling but that at a critical value of the coupling constant (namely for a critical anisotropy) it becomes relevant. The effect of this operator, which represents umklapp processes in the lattice theory, is to end the line of fixed points at a multicritical point which is in the Kosterlitz-Thouless universality class. An explicit computation of the critical exponents shows[9] that the correlation functions at this multicritical point have the symmetries of the isotropic Heisenberg antiferromagnet.

In the case of the two-dimensional system, the order parameter is $\bar{\Psi}T^3\Psi$. Unlike one dimension, there is no γ_5 Dirac matrix in $D = 2 + 1$. Instead, the order parameter is now odd under an operator which effectively exchanges the two species of fermions. However, a mass term proportional to the operator $\bar{\Psi}\Psi$ is *even* under the exchange of fermionic species (or sublattices, which is the same) but it is *odd* under parity (P) and time reversal (T) transformations. In contrast, $\bar{\Psi}T^3\Psi$ does not break these symmetries. The reason is that, for *each* species, the operator $\bar{\Psi}\Psi$ changes sign under P and T and, hence, $\bar{\Psi}T^3\Psi$ changes sign too. However, this effect is equivalent to a redefinition of the sublattices and, thus, it breaks neither parity nor time reversal. For this reason, the operator $\bar{\Psi}T^3\Psi$ is usually called a parity invariant mass term. Thus, renormalization effects cannot induce a parity invariant mass term in perturbation theory since it is a symmetry breaking field. We will show below that the phase transition found in Sec. IV represents the spontaneous breaking of the symmetry $\bar{\Psi}T^3\Psi \to -\bar{\Psi}T^3\Psi$ to a Néel phase in which $\langle\bar{\Psi}T^3\Psi\rangle \neq 0$.

Unlike the (1+1)-dimensional theory, the Lagrangian $\mathcal{L}_{2D}$ of Eq. (5.17) has, in addition to the discrete symmetry $\bar{\Psi}T^3\Psi \to \bar{\Psi}T^3\Psi$, a continuous gauge (local) symmetry:

$$\Psi(x) \to e^{i\phi(x)}\Psi(x),$$
$$A_\mu(x) \to A_\mu(x) + \partial_\mu\phi(x). \tag{6.1}$$

With the rules that we are using here, the gauge field has scaling dimension one and the Chern-Simons term has scaling dimension three and it is marginal. The Chern-Simons coupling constant θ is dimensionless and the theory is renormalizable. In contrast, the four-fermion operator has naive scaling dimension four and it is irrelevant. The four-fermion coupling constant $\bar{g}$ has naive scaling dimension of length. The dimensionless coupling constant g_0 of Eq. (5.16) has a *negative* β function, $\beta(g_0) = -g_0 + O(g_0^2)$, and $g_0^* = 0$ is an infrared stable fixed point of the renormalization group flow. In contrast, the Chern-Simons coupling θ has a vanishing β function. Thus, at least naively, it appears that this renormalization group flow has a line of fixed points parametrized by θ.

Chen and Li[49] have recently put forward arguments in favor of a scenario in which theories of relativistic fermions coupled to Chern-Simons gauge fields have a line of fixed points parametrized by θ. In particular they argue that the fluctuations of the gauge fields induce anomalous dimensions in the fermion operators and, as a result, the four-Fermi operators become less irrelevant as θ grows larger. However, their analysis is based in dimensional regularization. It is well known that this regularization scheme sets to zero all singular Feynman diagrams except for logarithmically divergent terms. Since it is an analytic regularization method, it also sets to zero all contributions which cannot be continued in dimension. Thus, fermion mass terms are not induced in dimensional regularization.

At the infrared stable fixed point $\bar{g} = 0$, the Lagrangian $\mathcal{L}_{2D}$ is manifestly scale invariant. The only parameter left is the Chern-Simons coupling constant θ which is dimensionless. Hence, at least in the absence of fluctuations, it is at a fixed point and it has no scale. However, it is easy to convince oneself that fluctuations lead to divergent corrections of the classical (that is, mean-field) values of the observables. The presence of divergent contributions in the perturbation series requires the use of a cutoff or, more generally, of a regulator. Any cutoff introduces a microscopic scale in the problem and therefore it breaks the apparent scale invariance. Thus, in the presence of a cutoff, dimensionful terms can be induced by renormalization effects. On dimensional grounds, an induced mass term will have to be proportional to the cutoff since there is no other scale left. The theory that we are studying has a natural cutoff, the lattice spacing, which acts as a natural scale. Hence, in this case, it is physically incorrect to use dimensional regularization. Furthermore, the Chern-Simons term breaks both T and P. Thus, it is expected that renormalization effects should generate all possible terms which break the same symmetries. The parity-odd fermion mass term $\bar{\Psi}\Psi$ breaks precisely the same symmetries. Hence, unlike the (1+1)-dimensional case there is no symmetry that prohibits these terms to be induced by renormalization.

Unlike the (1+1)-dimensional theory, nonperturbative tools such as bosonization are not available for the study of relativistic systems in $2 + 1$ dimensions. In order to proceed further we will use a semiclassical approximation, like the one of Sec. IV, but going beyond the leading order. In $1+1$ dimensions this approach would not be sufficient since this approximation misses the marginality of the interaction. However, in $2 + 1$ dimensions the four-fermion interaction is irrelevant (in weak coupling) and this approximation reproduces this result correctly. In order to determine the induced fermion mass M_{ind}, we will compute the leading self-energy correction to the fermion propagator due to fluctuations of the gauge field. Since the gauge field couples in the same way to both species of fermions, the only possible induced mass term is $\bar{\Psi}\Psi$ which is even under the exchange of species. Since this term is odd under T and P it can only arise from fluctuations of the Chern-Simons gauge field.

The partition function at zero temperature is

$$\mathcal{Z} = \int \mathcal{D}\bar{\Psi}\, \mathcal{D}\Psi\, \mathcal{D}A_\mu\, \mathcal{D}A_0^{\pi\pi}\, e^{i\int d^3x\, \mathcal{L}(\bar{\Psi},\Psi,A_\mu,A_0^{\pi\pi})}, \quad (6.2)$$

where $\mathcal{L}$ is the Lagrangian of Eq. (5.15), which we reproduce here for clarity

$$\mathcal{L} = \bar{\Psi}\, i\not{D}\, \Psi + A_0^{\pi\pi}\bar{\Psi}T^3\Psi - \frac{1}{2\bar{g}}\left(A_0^{\pi\pi}\right)^2 + \frac{\theta}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}. \quad (6.3)$$

This form of the theory makes the semiclassical approximation more transparent. We now follow the methods outlined in Sec. IV and find an effective action S_{eff} for the Bose fields, which here are A_μ and $A_0^{\pi\pi}$, after integrating out the fermions:

$$S_{\text{eff}} = -i\, \text{Tr}\ln\left[i\not{D} + A_0^{\pi\pi}T^3\right] - \int d^3x\, \frac{1}{2\bar{g}}\left(A_0^{\pi\pi}\right)^2 + \int d^3x\, \frac{\theta}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}. \quad (6.4)$$

The stationary points of this action satisfy the equations

$$\frac{\delta S_{\text{eff}}}{\delta A_\mu(x)} = i\, \text{Tr}\left[S(x,x)\gamma_\mu\right] + \frac{\theta}{2}\epsilon_{\mu\nu\lambda}F^{\nu\lambda} = 0, \quad (6.5)$$

$$\frac{\delta S_{\text{eff}}}{\delta A_0^{\pi\pi}(x)} = i\, \text{Tr}\left[S(x,x)T^3\right] - \frac{1}{\bar{g}}A_0^{\pi\pi}(x) = 0, \quad (6.6)$$

where $S(x,x')$ is the (Feynman) fermion propagator

$$S(x,x') = \left\langle x\right|\frac{1}{i\not{\partial} + A_\mu\gamma^\mu + A_0^{\pi\pi}T^3}\left|x'\right\rangle. \quad (6.7)$$

In order to simplify the notation we have dropped all the indices that are attached to the fermions. The traces run over both spinor (Dirac) and "flavor" (species) indices.

Equation (6.6) has a solution of the form $A_\mu = 0$ and $A_0^{\pi\pi} = M$. Please recall that the gauge field A_μ is the long wavelength fluctuation of the (lattice) Chern-Simons gauge field around the flux phase. M is the Néel mass and it is given by

$$\frac{M}{\bar{g}} = i\int \frac{d^3p}{(2\pi)^3}\text{Tr}\frac{1}{\not{p} - MT^3 + i\epsilon}T^3, \quad (6.8)$$

which is the continuum analog of the gap equation of Sec. IV. After performing the integral (with a cutoff on the space components of the momentum $\Lambda \approx \frac{\pi}{2a_0}$) we find that the mass M is the solution of

$$\frac{M}{\bar{g}} = \frac{\Lambda}{\pi}\left(\sqrt{1+\frac{M^2}{\Lambda^2}} - \frac{|M|}{\Lambda}\right)M. \quad (6.9)$$

This *gap equation* has the solution

$$|M| = \frac{\Lambda}{2}\left(\frac{\bar{g}}{\bar{g}_c} - \frac{\bar{g}_c}{\bar{g}}\right)\Theta(\bar{g} - \bar{g}_c), \quad (6.10)$$

where $\Theta(t)$ is the Heavyside function and $\bar{g}_c = \frac{\pi}{\Lambda}$ is the critical coupling constant of the effective continuum theory. Using this value of $\bar{g}_c$ and Eq. (5.16) we get an estimate for the critical anisotropy λ_c from this continuum theory given approximately by $\lambda_c \approx \frac{1}{2}$ which should be compared with the (lattice) AFA value $\lambda_c \approx 0.4$ of Sec. IV. This different value of λ_c, which is a nonuniversal quantity, reflects the approximations made in taking the continuum limit. In particular, it depends on the precise relation between the momentum cutoff Λ and the lattice spacing a_0.

Let us denote the full fermion propagator at momentum p_μ ($\mu = 0,1,2$) by $S(p)$ and by $G^{\mu\nu}(p)$ the propagator of the Chern-Simons gauge field. At this level of approximation, the fermion propagator, at three-momentum p_μ, is given by

$$S_0(p) = \frac{\not{p} + MT^3}{p^2 - M^2 + i\epsilon}, \quad (6.11)$$

where, once again, we have dropped all indices. The bare propagator of the gauge field at three-momentum p_μ, in the Lorentz gauge $\partial_\mu A^\mu = 0$, is

$$G_0^{\mu\nu}(p) = \frac{i}{\theta}\epsilon^{\mu\nu\lambda}\frac{p_\lambda}{p^2 + i\epsilon}. \quad (6.12)$$

Thus, in every Feynman diagram, each propagator of the gauge field contributes with a weight proportional to $1/\theta = 2\delta$, where δ is the statistical angle. Therefore, this is an expansion in powers of the statistics and it is accurate only near the fermion limit $\delta \to 0$ or $\theta \to \infty$. The value of δ of interest for the Heisenberg antiferromagnet is $\delta = 2\pi$ which is not small. A more serious problem is that the physical properties of systems of this sort must be periodic in the statistics. Namely, all amplitudes for any physical observables must not change under the replacements $\delta \to \delta + 2\pi k$, where k is an *even* integer (periodicity) and $\delta \to 4\pi - \delta$ (symmetry around bosons). We will use parturbation theory around fermions and demand that it holds around each *period*. The extrapolation to the boson point $\delta = 2\pi$ should yield qualitatively correct results.

The exact fermion propagator $S(p_\mu)$ obeys the Dyson equation

$$S(p)^{-1} = S_0(p)^{-1} - \Sigma(p), \quad (6.13)$$

where $\Sigma(p)$ is the fermion self-energy. Due to the symmetry of the bare theory, $\Sigma(p)$ has the form

$$\Sigma_{rr'}(p) = \Sigma_1(p) + \Sigma_2(p)T^3. \quad (6.14)$$

To leading order in δ, $\Sigma(p)$ is given by

$$\Sigma(p) = \int \frac{d^3k}{(2\pi)^3}\,\gamma^\mu S_0(p-k)\,\gamma^\nu iG_0^{\mu\nu}(k) \; + O(\delta^2). \quad (6.15)$$

Explicitly, we find

$$\Sigma(p) = i \int \frac{d^3k}{(2\pi)^3}\, \gamma^\mu \frac{\not p - \not k + MT^3}{(p-k)^2 - M^2 + i\epsilon} \gamma^\nu \frac{i}{\theta} \epsilon_{\mu\nu\lambda} \times \frac{ik^\lambda}{k^2 + i\epsilon} + O(\delta^2). \tag{6.16}$$

By counting powers of the momentum of integration we see that this contribution has an ultraviolet, linear divergence. By expanding the self-energy in powers of the external momentum p_μ, we see that the (ultraviolet) linear divergence only affects the term at zero external momentum $p = 0$ and that all contributions at nonzero external momentum are finite. This ultraviolet divergence is an artifact of ignoring the fact that we are working with an effective field theory and that the quantum antiferromagnet, from which this field theory is derived, is defined on a lattice and it does have a cutoff. Thus, this divergence has to be cutoff at values of the internal momentum of the order of $\Lambda \approx \pi/2a_0$, where a_0 is the lattice spacing. After some algebra we find

$$\Sigma(p) = \frac{i}{\theta} \int_0^1 dx \int \frac{d^3q}{(2\pi)^3} \frac{(q_\lambda + xp_\lambda)\left[(p_\rho(1-x) - q_\rho)\, 2g^\rho_\lambda - 2MT^3\gamma_\lambda\right]}{\left[q^2 + p^2x(1-x) - M^2x\right]^2}, \tag{6.17}$$

where we have used the covariant notation $q^2 = q_0^2 - \mathbf{q}^2$ and an $i\epsilon$ prescription is assumed.

After an integration over the frequency variable q_0 and over the spacial components of the momentum of integration $\mathbf{q}$ (with a cutoff Λ) we get

$$\begin{aligned}\Sigma(p) = \int_0^1 dx \Bigg\{ & (xMT^3\not p)\, \frac{1}{4\pi\theta}\left[\frac{1}{\sqrt{M^2x - p^2x(1-x)}} - \frac{1}{\sqrt{\Lambda^2 + M^2x - p^2x(1-x)}}\right] \\ & - \frac{1}{2\pi\theta}\left[\sqrt{\Lambda^2 + M^2x - p^2x(1-x)} - \sqrt{M^2x - p^2x(1-x)}\right] \\ & + \frac{1}{4\pi\theta}\left(M^2x - 2x(1-x)p^2\right)\left[\frac{1}{\sqrt{M^2x - p^2x(1-x)}} - \frac{1}{\sqrt{\Lambda^2 + M^2x - p^2x(1-x)}}\right]\Bigg\}. \end{aligned} \tag{6.18}$$

Since we are only interested in the computation of the effective (or renormalized) mass, it will be sufficient for our purposes to compute the integral of Eq. (6.18) at $p = 0$. In this limit we find that $\Sigma_2(0) = 0$ and we get a value for $\Sigma(0)$ which is independent of the fermionic species. In this limit, and after some algebra, Eq. (6.18) becomes

$$\Sigma_{rr'}(0) = -\frac{\Lambda}{2\pi\theta}\left(\sqrt{1 + \frac{M^2}{\Lambda^2}} - \frac{M}{\Lambda}\right)\delta_{rr'}. \tag{6.19}$$

This contribution to the self-energy of the fermion plays the role of an effective or *induced* mass and we will denote it by $\Sigma(0) \equiv M_{\text{ind}}$. Due to the symmetry of the bare fermion propagator, this (divergent) induced mass is *the same* for all fermionic species which thereby acquire the *same* mass. This result also tells us that the induced mass is proportional to $-1/\theta$. The fact that this sign is *opposite* to the sign θ will play a fundamental role in our analysis.

Let us now use these results to compute the mass of *each* species of fermions up to corrections of order δ^2. Our calculation tells us that the total fermion propagator $S(p_\mu)$ at zero external momentum has the form

$$S^{-1}(0) = -MT^3 - \Sigma(0) \tag{6.20}$$

from where we find that the effective masses M_i ($i = 1, 2$) for each of the species are

$$M_i = -(-1)^i M - \frac{\Lambda}{2\pi\theta}\left(\sqrt{1 + \frac{M^2}{\Lambda^2}} - \frac{M}{\Lambda}\right), \tag{6.21}$$

where M is the solution of the gap equation (6.9) and it is a function of the coupling constant $\bar{g}$. Hence, the effective masses M_i are also functions of the coupling constant. In Fig. 1 we show the qualitative form of the functions $M_i(\bar{g})$ for the entire range of couplings. Given the relation between $\bar{g}$ and the anisotropy λ, the curves of Fig. 1 will help us to determine the phase diagram.

Thus, while the spectrum of the semiclassical theory consists of two massless fermions for all $\bar{g} \leq \bar{g}_c$

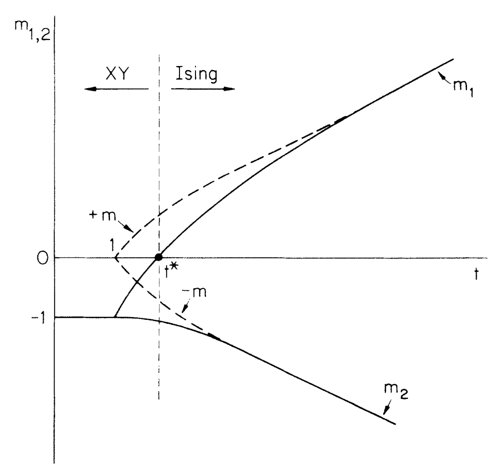

FIG. 1. Fermion mass gaps. The full curves show the mass of the fermion species $m_i = M_i/\Lambda$ ($i = 1, 2$) against the (normalized) coupling constant $t = \bar{g}/\bar{g}_c$. The broken curves are the fermion masses $m = M/\Lambda$, predicted by the AFA. The phase transition occurs at $t^* = \sqrt{3}$ where the mass m_1 vanishes. The range of the Ising and XY phases is also shown.

and two massive fermions (but whose masses have opposite signs) for $\bar{g} > \bar{g}_c$, the quantum fluctuations of the gauge fields make the fermion spectrum generically massive for *all* values of $\bar{g}$ (and, hence for all λ). However, Eq. (6.21) shows that, for θ positive and $\bar{g} \leq \bar{g}_c$, $M_1 = M_2 = -\frac{\Lambda}{2\pi\theta} < 0$. In fact, for $\theta > 0$, M_2 is always negative. However, M_1 goes through zero and changes sign at some critical value of the coupling constant $\bar{g}^*(\theta)$

$$\bar{g}^*(\theta) = \bar{g}_c \sqrt{1 + \frac{1}{\pi\theta}}. \tag{6.22}$$

For the value of $\theta = 1/2\pi$, of interest for the Heisenberg antiferromagnet, we get $\bar{g}^* = \sqrt{3}\bar{g}_c$.

Thus, we conclude that the quantum fluctuations yield the following spectrum for the fermions. For $\bar{g} \leq \bar{g}^*$ the two species of fermions have masses which, for general values of $\bar{g}$ have different absolute values but have the *same* sign. This sign is *opposite* to the sign of θ. In contrast, for $\bar{g} > \bar{g}^*$ the two fermionic species have masses with different absolute values *and* opposite signs. Precisely *at* $\bar{g}^*$, the mass of one of the species of fermions passes through zero. We should regard this phenomenon as a *phase transition* and $\bar{g}^*$ as a critical point. From the arguments presented above we should expect that the critical value $\bar{g}^*$ should correspond to a critical value of λ, which we will denote by λ^*, and that there should be a phase transition at λ^* in the anisotropic quantum Heisenberg antiferromagnet.[31]

Given that the fluctuations make such important effects already at the level of the leading corrections to the AFA, it is natural to inquire what the effects are of even higher-order corrections. It is clear that nothing special happens at $\bar{g}_c$ and that it does not correspond to a phase transition which has been shifted to $\bar{g}^*$. The apparent discontinuity in the derivative of the masses at $\bar{g}_c$ is an artifact of the leading order calculation. Higher-order terms will smooth out this spurious effect. The actual value of the critical coupling will also be renormalized by higher-order terms.

We now will argue that the phase transition at $\bar{g} = \bar{g}^*$ should be identified with the *isotropic* Heisenberg antiferromagnet. The argument in support of this identification relies on the counting of massless excitations of the system. In turn, we will identify these massless excitations with the Goldstone modes of the antiferromagnet.

At this level of approximation, the spectrum of fermions is massive for generic values of the anisotropy with one of the species becoming massless just at the critical coupling. Let us now investigate the spectrum of the bosonic excitations, A_μ and $A_0^{\pi\pi}$, as a function of $\bar{g}$. Since the fermions are massive, it is possible to integrate them out off the partition function and to find an effective action for the Bose fields which is local at length scales long compared with the inverse of the mass. From the work of Deser, Jackiw, and Templeton,[29] we know that the long distance effective Lagrangian for the gauge field, $\mathcal{L}_{\text{ind}}(A_\mu)$, induced by the fluctuations of a fermion of mass M, is of the form

$$\mathcal{L}_{\text{ind}}(A_\mu) \approx -\frac{1}{4\kappa^2} F^{\mu\nu} F_{\mu\nu} + \frac{\text{sgn}(M)}{4\pi} \epsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda}, \tag{6.23}$$

where we recognize the last term as a Chern-Simons term and the sign of its coupling constant is *equal* to the sign of the fermion mass M. The parameter κ is proportional to $|M|$. A similar analysis implies that the fluctuations of $A_0^{\pi\pi}$ are always massive. However, since the fermions are actually massive at $\bar{g}_c$, the fluctuations of the Néel order parameter field $A_0^{\pi\pi}$ *never* become critical.

The total effective action for the fluctuating gauge field is the sum of three contributions: (a) the (bare) Chern-Simons term with coupling constant θ, and (b) two additional Chern-Simons terms each with the value of Eq. (6.23). Thus, the *total* Chern-Simons coupling constant θ_{eff} is

$$\theta_{\text{eff}}(\bar{g}) = \theta + \frac{\text{sgn}[M_1(\bar{g}) + M_2(\bar{g})]}{4\pi}. \tag{6.24}$$

We find

$$\theta_{\text{eff}}(\bar{g}) = \begin{cases} \theta - \frac{2}{4\pi} & \text{if } \ \bar{g} \leq \bar{g}^*, \\ \theta & \text{otherwise.} \end{cases} \tag{6.25}$$

Thus, in the phase in which the effective fermion masses have the *same* sign, the fluctuations of the fermions act so as to reduce (or *screen*) the bare value of the Chern-Simons coupling constant θ. This is the "Lenz law of statistics" referred to in the Introduction.

For the particular case of interest for the antiferromagnet, $\theta = 1/2\pi$, the screening is *complete* and $\theta_{\text{eff}} = 0$ for the case of bosons. Thus, the Chern-Simons term is cancelled out from the effective action of the gauge field which now represents massless excitations. This cancelation of the Chern-Simons term from the effective action of the gauge field is the anyon superfluid scenario of Refs. 20–22. In the anyon superfluid, the transverse massless gauge field is interpreted as the Goldstone boson of the superfluid state. We identify this regime with the XY phase of the antiferromagnet.

Conversely, for $\bar{g} > \bar{g}^*$, we get $\theta_{\text{eff}} = \theta$ and all collective modes are massive. In this phase the Néel order parameter is nonzero and *all* excitations have a gap. This is the *Ising* phase of the antiferromagnet.

The phase diagram is now *almost* complete. Two issues still need to be resolved: (1) the nature of the transition point at $\bar{g}^*$ and (2) are the fermion states really part of the spectrum for any value of the coupling constant?

These two problems actually are not independent from each other. Let us first consider the fate of the fermions. For $\bar{g} > \bar{g}^*$, the fermions have nonzero masses with opposite sign. In this phase, the effective Chern-Simons coupling constant is equal to $1/2\pi$. Hence, in this phase, by the standard argument of statistical transmutation, Chern-Simons gauge field turns the massive *fermions* into massive *bosons*. However, for $\bar{g} < \bar{g}^*$, the long distance effective action for the gauge field does not include a Chern-Simons, which cancels out but, instead, the leading term has a Maxwell form. Thus, the actual physical mass (or gap) of the fermion will be significantly renormalized by the fluctuations of the transverse massless collective mode (the gauge field). The result, exactly as in the case of the semion superfluid,[20] is that due to the quantum fluctuations of the collective mode, the fermion acquires a logarithmically divergent mass and, therefore, it disappears from the physical spectrum.

The presence of infrared divergent corrections to the fermion self-energy in the range $\bar{g} < \bar{g}^*$ has important consequences for the Néel order. Equation (6.6) relates the expectation value of the Néel order parameter $\bar{\Psi}T^3\Psi$ to the expectation value of the field $A_0^{\pi\pi}$ (this equation is valid beyond the saddle-point approximation provided that the field $A_0^{\pi\pi}$ is replaced by its exact expectation value). At the level of the saddle point, the expectation value of $A_0^{\pi\pi}$ is equal to $M\bar{g}$. Higher-order corrections involve fermion self-energy insertions in the low order diagrams. For $\bar{g} > \bar{g}^*$, these corrections are finite in the infrared. However, for $\bar{g} < \bar{g}^*$, these corrections are infrared divergent (due to the fermion self-energy insertions) and negative (since they have to give a value smaller than the classical result). The precise computation of these effects to all orders is difficult and beyond the domain of perturbation theory. The presence of these infrared divergent contributions, already in the leading corrections, suggests that the Néel order is unstable for $\bar{g} < \bar{g}^*$ and that the exact expectation value of this operator has to vanish in that regime. An alternative picture of this effect can be seen by noticing that, as the coupling constant $\bar{g}$ is decreased from large values (that is from the classical Néel regime) and one of the masses becomes small, tunneling processes between the two Néel states become increasingly favorable. In particular, the magnitude of the energy per unit length of a Néel *domain wall* is set by the mass of the fermionic excitations. Thus, in the regime in which the fermionic excitations have a finite mass, the energy per unit length of the domain wall is finite. Below $\bar{g}^*$, the infrared divergencies in the fermion self-energy will force the energy per unit length of the wall to vanish. Consequently, the domain walls will condense in this regime and will destroy the Néel long range order. Precisely at $\bar{g}^*$, the energy per unit length of the domain wall is still finite since there is still Néel order. Thus, the domain wall energy should drop to zero with a jump at $\bar{g}^*$. The domain wall condensation as a mechanism for the destruction of long range order is well known in (1+1)-dimensional systems, where the solitons play the role of the domain wall. Hence, we conclude that the Néel order parameter should drop to zero discontinuously at $\bar{g}^*$ and to vanish for all $\bar{g} < \bar{g}^*$.

In contrast, the operator which creates (or removing) one fermion *and* one flux quantum simultaneously, is gauge invariant and it has a *finite* mass. In anyon superfluidity this state is usually called the vortex and it has the statistics of an anyon. In our problem, again by statistical transmutation, it is a massive *boson.* The mass of this state should scale with M_1. Hence, it should vanish *exactly* at $\bar{g}^*$. In the lattice Chern-Simons theory this state is created by an operator which changes both charge and flux. For the case of the antiferromagnet, the operator which creates this state is S^+. Thus, we argue that the massless fermion of our spectrum is actually the extra Goldstone boson of the antiferromagnet. Since this state becomes massless only at $\bar{g}^*$, we identify this phase transition with the *isotropic* quantum antiferromagnet. This point is then viewed as a limiting point and it has the attributes of both phases. In particular, it has a nonvanishing value for the Néel order parameter and two Goldstone bosons.[50] We also note, in passing, that in one dimension a strikingly similar picture of the nature of the isotropic point (from the point of view of the symmetry analysis) was developed by Luther and Peschel.[9]

However, in the case of our problem, unlike the case of the one-dimensional spin chain, there are no nonperturbative methods available, such as bosonization, that will allow for an exact treatment of the long distance behavior. The topological invariance of the Chern-Simons theory strongly suggests that no further renormalizations occur and that we have successfully characterized the infrared stable fixed points. A detailed analysis of the phase transition at $\bar{g}^*$, particularly the determination of its universality class, critical exponents, etc., requires a more sophisticated analysis of the effective theory than the one performed here. The phase transition at $\bar{g}_c$ is in the universality class of theories of self-interacting relativistic fermions (of the Gross-Neveu type). Our analysis shows that this fixed point is unstable and does not represent the long distance behavior of the system. It is interesting to note that the removal of the phase transition at $\bar{g}_c$ and its replacement by the transition at $\bar{g}^*$ where the Néel order parameter drops to zero *discontinuously*, is strongly reminiscent of the physics of fluctuation induced first-order transitions.

We can also give a renormalization group picture for the arguments presented above. The AFA or, equivalently, the semiclassical theory of this section, does not represent faithfully all the relevant fluctuations of the system. In particular, the *infrared stable* fixed point associated with the XY phase is simply not present in the semiclassical theory. The quantum fluctuations of the gauge field contain the appropriate relevant operator, the parity breaking induced mass term. Once this operator is generated, the flow of the coupling constants is drastically changed as we explore the low energy regime. In particular, the effective action of the gauge fields acquires a finite renormalization of its Chern-Simons coupling constant which tends to screen the statistics. The fermions are also affected by this flow since the physical low energy mass (or energy gap) of the spectrum is altered by the fluctuations of the bosons. In the XY phase, the excitations of the gauge fields are massless and their quantum fluctuations suppress the fermions from the spectrum. In the Ising phase, their fluctuations turn the fermions into bosons. Clearly at both stable fixed points, parity *and* time reversal are not broken (for $\theta = 1/2\pi$).

We conclude this section with a few remarks. First, our arguments show that, for all values of the anisotropy, there are no states in the spectrum with the quantum numbers of a fermion. All the states are bosonic. This is not an accident since the system is not in a spin liquid state for all values of the anisotropy. Nevertheless, given the analogies between the AFA and the flux phases of the theories of frustrated antiferromagnets, our results should be viewed as an indication that, once fluctuations are fully taken into account, the flux phases could become more like the standard phases of antiferromagnets. We have presented qualitative arguments which show that the phase transition at $\bar{g}^*$ can be viewed, in some sense, as a first-order transition since the order parameter must have a jump at that point. A more rigorous proof of this statement still needs to be constructed. Nevertheless, the arguments presented above show clearly the physical mechanism behind this phenomenon. A direct computation of the spin correlation function $\langle S^+(x)S^-(x')\rangle$ should demonstrate quite explicitly the presence of an additional massless state at the critical point. Work on this problem is currently in progress.

VII. CONCLUSIONS

In this paper we have presented a field theory for the anisotropic quantum antiferromagnet on a square lattice, based on the Chern-Simons or Wigner-Jordan approach. We discussed in detail the phase diagram, as a function of anisotropy, at the level of the average field approximation. We found a continuous, second-order phase transition at a critical anisotropy from a fluxlike phase to an Ising phase. We showed that this phase transition is spurious and that the massless flux phase cannot possibly describe the the XY regime of the quantum antiferromagnet. We used a semiclassical theory, based in the method of spin coherent states, to derive an anisotropic nonlinear σ model for this system and derived a phase diagram for the system, valid in the limit of large spin. We considered the role of fluctuations around the AFA and showed that they induce relevant operators, not included in the AFA, which drive the low energy behavior of the system. We derived an effective field theory of self-interacting fermions coupled to Chern-Simons gauge fields and showed that its fluctuations contain all the necessary relevant operators to yield a correct phase diagram. In particular they induce P and T symmetry breaking fermion mass terms which should necessarily be present for arbitrary values of the Chern-Simons coupling constant. We gave a set of arguments which indicate that the fluctuations of the gauge field drive the theory away from the universality class of theories of self-interacting relativistic fermions (of the Gross-Neveu type). We identified the infrared stable fixed points and showed that the spectrum of the system at these fixed points coincides with the expected spectrum of the phases of the antiferromagnet discussed in Sec. II.

We conclude with a comment on the accuracy of the AFA. The AFA has become a standard tool and it is widely used in a variety of fields of condensed matter, most prominently in theories of the fractional quantum Hall effect. The difficulties that we found in applying the AFA to the quantum Heisenberg antiferromagnet show that this approximation can be unreliable if the resulting fermion spectrum is massless. In such situation fluctuations may (and generally will) induce relevant operators which will necessarily generate gaps in the fermionic spectrum. This is not a problem for theories of the incompressible states of the FQHE but could well be the case for the compressible even denominator states.

ACKNOWLEDGMENTS

A.G.R. thanks the Department of Physics of the University of Illinois for its kind hospitality. This work was done in part while A.G.R. was a member of the Department of Physics of the University of Tennessee at Knoxville. This work was supported in part by the National Science Foundation through Grant Nos. NSF DMR-9015771 at the University of Tennessee and NSF STC-9120000 at the University of Chicago (A.G.R.), NSF DMR-91-22385 at the University of Illinois at Urbana-Champaign (E.F.), by support from the American Association of University Women (A.L.), and by the U.S. Department of Energy through Contract No. DE-AC05-84OR21400 administered by Martin Marietta Energy Systems Inc.

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