Experimental observation of solitons in a 1D nonlinear lattice

Wei-zhong Chen
Institute of Modern Physics, Ningbo Normal College, Ningbo, 315211, People's Republic of China
(Received 18 October 1993)

The experimental observations for the nonpropagation solitons with both spatial phase match and mismatch have been reported in a one-dimensional nonlinear lattice that is damped and vertically driven. Theoretically, using the method of multiple-scale solution, we develop a nonlinear Schrödinger equation for the experimental results including the kinks found by Denardo *et al.* Furthermore, under our order estimation the generating conditions for soliton and kink have been discussed. According to the generating conditions, the modes of the self-localized structures are determined closely by the intrinsic properties of the lattice rather than the driven parameters of the shake table.

During the last ten years there has been much interest in experimental observations[1] and in theoretical investigations[2] of nonlinear phenomena in the continuum media. Both propagating and nonpropagating solitons and kinks have been studied. On the other hand, the theoretical studies for the localized structure in discrete system have also been interesting many physicsts[3,4] because of its importance to both solid-state physics and materials science. However, very little experimental work had been done. Fortunately, Denardo *et al.*[5] have carried out on the observations of two kinds of nonpropagating self-localized structures, domain walls which connect standing-wave regions of two different wave numbers and kinks which connect standing-wave regions of the same wave numbers, in a one-dimensional nonlinear coupled lattice. They have a common characteristic in the spatial phase, say, the spatial phases are mismatched, in which every oscillator is a nonzero angle out of phase with its immediate neighbors. The experimental apparatus is quite simple, though the results of the experiment are exciting. They used a discrete lattice of 35 nonlinear coupled pendulums with mass 13.6 g supported by a horizontal rod, and put it on the vertical shake table. Theoretically, a nonlinear Schrödinger equation (NLSE) had been employed to explain the phenomena of kinks in upper cutoff mode in which each oscillator is π out of phase with its immediate neighbors. Although Ref. 5 is devoted only to the two kinds of self-localized structures, the soliton mode one with spatial phase match in which each lattice is no spatial phase difference from its immediate neighbors has been also mentioned by Denardo.[6]

Here we will report our experimental observations which show there exists not only a soliton mode self-localized structure with spatial phase match, but also one with spatial phase mismatch in a nonlinear coupled lattice with lower mass. Figure 1 shows us the steady-state nonpropagating soliton with spatial phase match, while the spatial-phase-mismatched soliton (a typical mode is the upper cutoff mode) is shown in Fig. 2. The data plotted in Fig. 3 are absolute angular displacements of the lattices without considering the spatial phase differences in the mode of mismatch.

The apparatus is quite simple as well as described in Refs. 5 and 6. It is a lattice of 100 pendulums with a sphere in shape and 3.6 g in mass, fixed on the shelf which is attached to the shake table (loudspeaker). Each ball can only move in one degree of freedom and there is no motion of the direction of the array because it is fixed by two strings in a shape similar to the letter V. Furthermore, these balls are coupled nonlinearly by tying the overlapping points between two V strings (Fig. 4). The geometric parameters of the lattice are that the length of each string L is 8.5 cm and the distance between two neighbor balls, a, is 2.5 cm as same as the ones in Ref. 5. For these parameters the free frequency of the uncoupled pendulum f_0 is about 1.7 Hz. To obtain the self-localized structure, the driven frequency of the vertical shake table is adjusted to near twice the free one of the uncoupled pendulum, that is, about 3.4 Hz. At the beginning of the experiment, initial motion is necessary, but there is no requirement for the form of the motion. After about 30 sec of irregular swing, the self-localized structure in embryonic form will appear. For the sake of get-

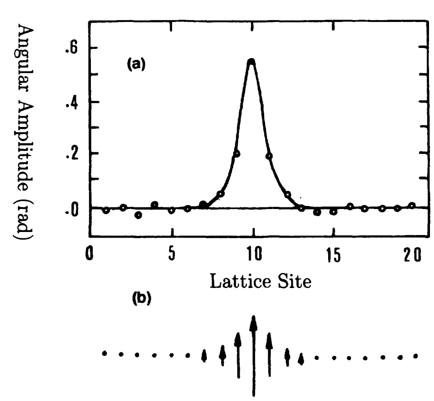

FIG. 1. (a) Experimental data, maximum angular displacements, of the soliton in space phase match mode. The curve is a hyperbolic secant best fit, say, $\theta(x)=\theta_m \mathrm{sech}[k(x-x_0)]$, with $\theta_m=0.551$ rad, $k=0.65$ cm^{-1}, and $x_0=25$ cm: (b) Diagrammatic representation of highly localized symmetric soliton in a uniform lattice. The vertical exciting parameters in this mode are that the driven wave mode is sine; the amplitude A_e is 1.35 mm and the frequency $2f_e$ is 3.5 Hz.

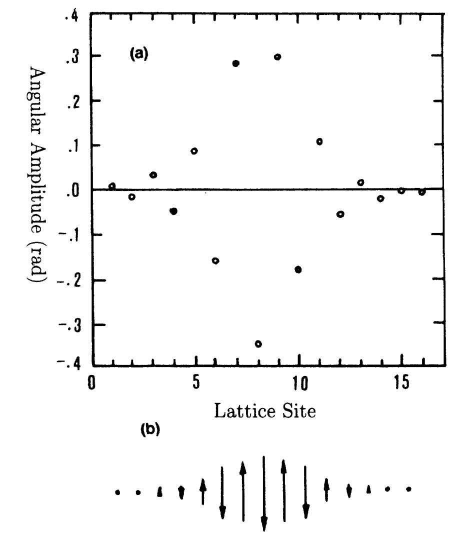

FIG. 2. (a) Experimental data, maximum angular displacements, of the soliton in space phase mismatch mode. (b) Diagrammatic representation of highly localized symmetric soliton in a uniform lattice. The vertical exciting parameters in this mode are that the driven wave mode is sine; the amplitude A_e is 2.55 mm and the frequency $2f_e$ is 4.0 Hz.

ting a clear and robust one, we use the stick to stop a few small irregular motions. The self-localized structure will then develop and eventually reach a steady state.

The experimental data of the steady state are obtained by three steps. Before the experiment, two orthogonal scales in centimeters have been attached to the shake table along the array direction and swing one, respectively. When a steady state is formed, we turn on the television camera just right over the center of the lattice and record completely the pendulum motions by videotape. Finally, we restore the data by a program in which the optical correction has been considered.

In fact, the experimental results do not closely relate to the exciting amplitude of the shake table in our experiments, especially after forming a strong one. However, there is a certain requirement for the driven frequency of the shake table for a special wave model. The driven parameters listed in the following are only two sets of typical values of ones with which the self-localized structure

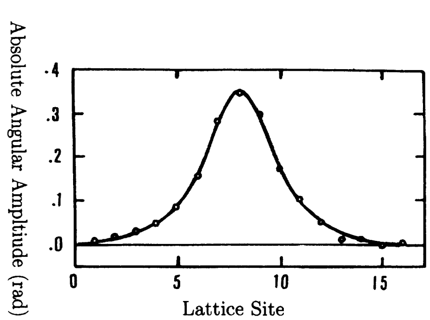

FIG. 3. Experimental data without regard to the difference of spatial phase and the hyperbolic secant curve in the best fit, say, $\theta(x)=\theta_m \operatorname{sech}[k(x-x_0)]$, with $\theta_m=0.355$ rad, $k=0.27\ \mathrm{cm}^{-1}$, and $x_0=20.1$, for the soliton in spatial phase mismatch mode.

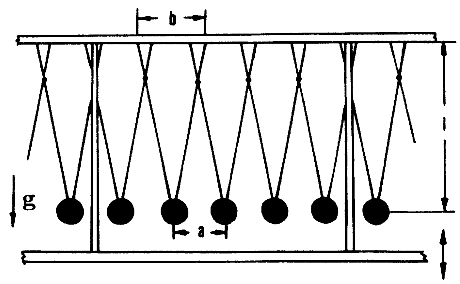

FIG. 4. Schematic drawing of apparatus of nonlinearly coupled lattice. The values of the geometrical parameters are $a=2.5$ cm, $l=8.3$ cm, $b=3.2$ cm, and $m=3.6$ g.

can be obtained.

(i) For the soliton with match model in space phase plotted in Fig. 1, the driven wave model is a sine wave, the driven amplitude A_e is 1.53 mm, and the driven frequency $2f_e$ is 3.5 Hz.

(ii) For the soliton with mismatch model in space phase plotted in Fig. 2, the driven wave model is a sine wave, the driven amplitude A_e is 2.50 mm, and the driven frequency $2f_e$ is 4.0 Hz.

The best fits of the hyperbolic secant curves have been plotted in Figs. 1 and 3 for solitons with spatial phase match and mismatch models, respectively. Although there is a slight lack of smoothness of the data in Figs. 1–3 due to the nonuniformities of the pendulums, especially the difference in the pendulum lengths, it is obvious that the distributions of the maximum angular displacements of the lattices are both solitary models.

To explain the experimental phenomena with both match and mismatch modes in theory, we write the Hamiltian for an ideal lattice in one dimension,

$$H=\sum_{j=1}^{N}\left[\frac{1}{2m}p_j^2+\frac{1}{2}K_2 l^2(\theta_{j+1}-\theta_j)^2 + \frac{1}{4}K_4 l^4(\theta_{j+1}-\theta_j)^4+m(g-\ddot{z}_0)(1-\cos\theta_j)\right], \tag{1}$$

with

$$z_0=A_e\cos(2\omega_e t)\ , \tag{2}$$

where l and m are the common length and mass of the lattice, A_e and $2\omega_e$ are the driven amplitude and angular frequency, K_2 and K_4 are the coupled coefficients of linear and cubic nonlinear interactions, p_j is the conjugating moment of the angular coordination θ_j of the jth pendulum, g is the acceleration of gravity, and N is the total number of the pendulums, respectively.

The dynamic equation related to Eq. (1) is that

$$\ddot{\theta}_j+\beta\dot{\theta}_j-c_2^2(\theta_{j+1}+\theta_{j-1}-2\theta_j) -c_4^2[(\theta_{j+1}-\theta_j)^3-(\theta_j-\theta_{j-1})^3] +[\omega_0^2+\eta\cos(2\omega_e t)]\sin\theta_j=0 \quad (j=1,2,\ldots,N)\ , \tag{3}$$

with

$$c_2^2=K_2/m \ , \tag{4}$$

$$c_4^2=K_4 l^2/m \ , \tag{5}$$

$$\eta=4\omega_e^2 A_e/l \ , \tag{6}$$

$$\omega_0^2=g/l \ , \tag{7}$$

where the dot denotes time differentiation and the dumped term $-\beta\dot{\theta}_j$ has been inserted into Eq. (3). Based on the calculations for the values of function and parameters in Eq. (3) under the experimental conditions of ours and Ref. 5, it is suitable to take the following order estimation:

$$\theta_j \sim o(\epsilon) \quad (j=1,2,\ldots,N) \ , \tag{8}$$

$$\eta \sim o(\epsilon^2) \ , \tag{9}$$

$$\beta \sim o(\epsilon^2) \ , \tag{10}$$

$$\omega_e^2-\omega_0^2 \sim o(\epsilon^2) \tag{11a}$$

(for spatial phase match mode), or

$$\omega_e^2-(\omega_0^2+4c_2^2) \sim o(\epsilon^2) \tag{11b}$$

(for spatial phase mismatch mode), where ϵ is a small parameter.

Using the method of the multiple-scale solution,[7] the following solutions for Eq. (3) can be obtained.

(i) For the soliton with space phase match mode,

$$\theta_j(x,t)=\phi(x,t)e^{-i\omega_0 t}+\text{c.c.}+o(\epsilon^2) \ , \tag{12}$$

where $\phi(x,t)$ is a continuous differentiable complex function of its arguments and satisfies the NLSE

$$-2i\omega_0\frac{\partial\phi}{\partial t}+c_2^2 a^2\frac{\partial^2\phi}{\partial x^2}-i\omega_0\beta\phi-\frac{1}{2}\eta\phi^*+\frac{1}{2}\omega_0^2|\phi|^2\phi=0 \ . \tag{13}$$

Obviously, the solutions of Eq. (13) possess only the soliton mode, but not the kink mode, because the sign of the coefficient of the nonlinear term is the same as the one of dispersion term.[8] The solution with spatial phase match mode in first-order approximation is

$$\theta_j(x,t)=\tfrac{1}{2}\theta_{m1}\exp i(\omega_0 t+\delta_1)\operatorname{sech}[k_1(x-x_0)]+\text{c.c.} \ , \tag{14}$$

with

$$\theta_{m1}^2=\frac{1}{2\omega_0^2}\sqrt{\eta^2-4\beta\omega_0} \ , \tag{15}$$

$$k_1^2=\frac{1}{2c_2 a^2}\sqrt{\eta^2-4\beta\omega_0} \ , \tag{16}$$

$$\delta_1=\tfrac{1}{2}\arcsin(2\beta\omega_0/\eta) \ , \tag{17}$$

and x_0 is an arbitrary constant.

(ii). For the soliton with space phase mismatch mode,

$$\theta_j(x,t)=(-)^j\psi(x,t)e^{-i\omega_{\max}t}+\text{c.c.}+o(\epsilon^2) \ , \tag{18}$$

where $\psi(x,t)$ satisfies the NLSE

$$-2i\omega_{\max}\frac{\partial\psi}{\partial t}-c_2^2a^2\frac{\partial^2\psi}{\partial x^2}-i\omega_{\max}\beta\psi-\frac{1}{2}\eta\psi^* + \left[\frac{1}{2}\omega_0^2-48c_4^2\right]|\psi|^2\psi=0 \tag{19}$$

and

$$\omega_{\max}=\sqrt{\omega_0^2+4c_2^2} \ . \tag{20}$$

It is interesting that the coefficient of the nonlinear term in Eq. (19) is a difference between two positive parts, say, $\frac{1}{2}\omega_0^2-48c_4^2$. The former, $\frac{1}{2}\omega_0^2$, comes from the nonlinear relation between the radian displacement and the horizontal one, that is, between θ_j and $\sin\theta_j$, as well as in the mode of case (i). The latter, $48c_4^2$, however, originates from the cubic nonlinear interactions between immediate neighbors, which has the same order as the former in this case, but is one order higher than the former in the case (i). So in space phase mismatch mode, when the sign of the coefficient of the nonlinear term is opposite to the one of the dispersion term, that is,

$$mg-98K_4 l^3>0 \ , \tag{21}$$

the self-localized structure should be in the kink model,

$$\theta_j(x,t)=(-)^j\tfrac{1}{2}\theta_{m2}\exp i(\omega_{\max}t+\delta_2) \times\tanh[k_2(x-x_0)]+\text{c.c.} \ , \tag{22}$$

with

$$\theta_{m2}^2=\frac{2}{96c_4^2-\omega_0^2}\sqrt{\eta^2-4\beta\omega_{\max}} \ , \tag{23}$$

$$k_2^2=\frac{1}{2c_2^2a^2}\sqrt{\eta^2-4\beta\omega_{\max}} \ , \tag{24}$$

$$\delta_2=\frac{\pi}{2}-\frac{1}{2}\arcsin(2\beta\omega_{\max}/\eta) \ . \tag{25}$$

And when the sign of the coefficient of the nonlinear term is the same as the one of the dispersion term, that is,

$$mg-98K_4 l^3<0 \ , \tag{26}$$

the self-localized structure should be in the soliton model,

$$\theta_j(x,t)=(-)^j\tfrac{1}{2}\theta_{m3}\exp i(\omega_{\max}t+\delta_3) \times\operatorname{sech}[k_3(x-x_0)]+\text{c.c.} \ , \tag{27}$$

with

$$\theta_{m3}^2=\frac{2}{\omega_0^2-96c_4^2}\sqrt{\eta^2-4\beta\omega_{\max}} \ , \tag{28}$$

$$k_3=k_2 \ , \tag{29}$$

$$\delta_3=\tfrac{1}{2}\arcsin(2\beta\omega_{\max}/\eta) \ . \tag{30}$$

Furthermore, from Eqs. (15), (16), (28), and (29), c_4^2 can be determined by

$$c_4^2=\frac{\omega_0^2}{96}(1+\gamma^2) \ , \tag{31}$$

where

$$\gamma = \frac{k_3 \theta_{m1}}{k_1 \theta_{m3}} . \tag{32}$$

So the cubic nonlinear coupled constant

$$K_4 = \tfrac{1}{96}\omega_0^2(1+\gamma^2) . \tag{33}$$

The value of K_4 roughly equals 8.8×10^{-7} N/m^3 in this experimental lattice.

Another interesting thing is the soliton-kink mode translation. If the critical condition

$$mg - 98K_4 l^3 = 0 \tag{34}$$

is held, in other words, for a given l and K_4 if the mass of pendulum reaches the critical value

$$m_c = 98K_4 l^3/g , \tag{35}$$

then the nonlinear term disappears in Eq. (19) and becomes a linear Schrödinger equation for which it is known that there is a plane-wave solution only. Thus, according to Eq. (35) or Eqs. (21), (26), and the fact that the main difference in experimental conditions between Ref. 5 and here is in the mass of the pendulum, 13 and 3.6 g, respectively, we find a satisfying and explicit answer for why there is only a kink mode self-localized structure rather than a soliton mode in Ref. 5 and there is only a soliton mode rather than a kink mode in this paper. Also, the experimental value of critical mass m_c^{expt} should be larger than 3.5 g and smaller than 13.6 g. Indeed, it is easy to obtain the theoretically estimating one $m_c^{\text{theor}} \approx 5.0$ g by substituting the values of K_4, l into Eq. (35).

Therefore, when the order estimations, Eqs. (8)–(11), are satisfied well enough, the wave mode of the self-localized structure which can be observed in an experiment is determined by the intrinsic parameters of the lattice, for example, the length of the pendulum l and positions of the coupled points, and the mass of the pendulum rather than the vibrating parameters such as the driven amplitude A_e and the driven frequency $2f_e$. But the effect of driven parameters, frequency especially, on the results of experiments determines whether there is success in the order estimation which determines whether there is success in perturbation technique. In our experiments the driven frequency can be a variable within a small range for each kind of solitary self-localized structure and there is no self-localized structure beyond these ranges. It is, however, far less sensitive for changing the driven amplitude than doing the driven frequency especially as it becomes a strong and robust one. Also, comparing k_1 with k_2 in Eqs. (16) and (24), respectively, we can easily answer why the soliton in space phase match mode is thinner than the one in space phase mismatch mode in our experiments.

In summary, we have reported two experimental phenomena of self-localized structures, spatial match and mismatch mode solitons, in a one-dimensional nonlinear lattice. Theoretically, if our order estimations, Eqs. (8)–(11), are satisfied well enough, then observations of both space phase mismatch kinks in Ref. 5 and space phase match and mismatch solitons reported in this paper can be explained well by the solutions of two multiple-scale reductions of Eqs. (3), (13), and (19). Based on the theoretical frame of the NLSE under order estimation, one could predict that the forms of the self-localized structures are determined closely by the intrinsic properties of the lattice rather than the driven parameters of the shake table, and there is no kink mode self-localized structure with spatial phase match mode in this lattice under vertical vibration. Of course, the correctness of the self-localized structure mode conditions [Eqs. (21), (26), and (34)] as well as the above predictions need further to be confirmed by experiments. We are now in the process of several experiments for these purposes which are very encouraging, and the results will be reported elsewhere.

This work was supported by the Natural Science Foundation of Zhejiang Province, People's Republic of China. The author would like to thank Professor Guo-Xiang Huang for providing the Denardo thesis and his helpful discussions and Professor Sen-yue Lou for his helpful discussions.

[1]Junru Wu, Robert Keolian, and Isadore Rudnick, Phys. Rev. Lett. **52**, 1421 (1984); B. Denardo, W. Wright, and S. Putterman, *ibid.* **61**, 1518 (1990).

[2]A. Larraza and S. Putterman, J. Fluid Mech. **148**, 443 (1984).

[3]J. B. Page, Phys. Rev. B **41**, 7835 (1990); A. J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (1988); Phys. Rev. B **39**, 3374 (1989).

[4]S. R. Bickham, A. J. Sievers, and S. Takeno, Phys. Rev. B **45**, 10 334 (1992); S. R. Bickham and A. J. Sievers, *ibid.* **43**, 2339 (1991); V. M. Burlakov, S. A. Kiselev, and V. N. Pyrkov, *ibid.* **42**, 4921 (1990).

[5]B. Denardo *et al.*, Phys. Rev. Lett. **68**, 1730 (1992).

[6]B. Denardo, Ph.D. thesis, University of California, Los Angeles, 1990.

[7]G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974), pp. 466; G-x Huang and Z-x Xu, Chin. Phys. Lett. **12**, 659 (1992).

[8]For example, G. L. Lamb, *Elements of Soliton Theory* (Wiley, New York, 1980).