Landauer resistance of Thue-Morse and Fibonacci lattices and some related issues

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We have reported a comparative study of the Landauer resistance of Thue-Morse (TM) and Fibonacci lattices. Our main objective is to examine the degree of aperiodicity of TM lattices vis-à-vis that of (quasiperiodic) Fibonacci lattices, with the use of relevant Landauer resistances. Our study reveals that TM lattices are more periodic than Fibonacci lattices.

I. INTRODUCTION

For many years, the study of the electronic properties of aperiodic Thue-Morse (TM) lattices has been drawing the attention of a large number of researchers.[1-9] The degree of aperiodicity of TM lattices is quite different from that of Fibonacci lattices. The Fibonacci lattices, which are a class of quasiperiodic (QP) systems, have been studied theoretically quite extensively[10-13] and have also been realized experimentally[14] in the form of systems like GaAs-AlAs. The primary motivation behind the study of TM lattices was to explore the possibility[1] of one-dimensional quasiperiodic systems beyond the realm of Fibonacci lattices.

The Fourier amplitude spectrum of the TM lattices is singular continuous—a fact which indicates that the degree of aperiodicity of such lattices is intermediate between quasiperiodic and disordered systems.[3,15] On the other hand, the electronic spectra of TM lattices indicate that their degree of aperiodicity is intermediate between periodic and QP systems.[1] It is thus apparent that a satisfactory understanding of the degree of aperiodicity of TM lattices requires us to study as many of their properties as is possible, vis-à-vis the corresponding ones of periodic and QP systems. Being guided by this idea, we report in this paper a comparative study of the Landauer resistance (LR) of TM lattices, periodic systems, and QP systems, considering Fibonacci lattices as representatives of the QP system. As we shall see later, our study elucidates meaningfully several facets of the issue regarding the degree of aperiodicity of TM lattices; in particular, we find that TM lattices are likely to be more periodic than Fibonacci lattices as far as the LR is concerned.

The essential features of TM and Fibonacci lattices are incorporated in Sec. II. The derivation of the Landauer resistance for our models requires certain features of transfer matrices for the Kronig-Penney (KP) model on aperiodic lattices, and these features are elucidated in Sec. III. The derivations of the Landauer resistance of TM lattices, Fibonacci lattices, and periodic systems are presented respectively in Secs. IV, V, and VI. Numerical analyses pertinent to our treatment will appear in Sec. VII while, finally, a critical discussion of our results and relevant conclusions are inserted in Sec. VIII.

II. ESSENTIAL FEATURES OF TM AND FIBONACCI LATTICES

TM and Fibonacci lattices are two kinds of aperiodic systems. For our treatment, we realize these systems by placing N rectangular potential barriers along one-dimensional two-tile aperiodic lattices. The separation between centers of two consecutive barriers takes one of the two values c and d of the tiles, which are arranged either in TM sequence or in Fibonacci sequence. The essential features of these two sequences are as described below; it may be noted that the procedure we have followed in constituting both TM and Fibonacci lattices amounts to realizing the KP model on these sequences. In regard to all three cases, namely, periodic systems, TM lattices, and Fibonacci lattices, we have treated the LR for electrons having energy less than that of the height of barriers.

A. TM lattices

For the aperiodic lattice corresponding to the TM sequence, the two tiles c and d, which represent the distance between two consecutive points characterizing the centers of two consecutive barriers, are arranged in the TM sequence S_n given by

$$S_{n+1}=\{S_n,\bar{S}_n\},\quad n\geq 0;\quad S_0=\{c,d\}\ . \tag{1}$$

$\bar{S}_n$ is the complement of S_n obtained by interchanging c and d. Sequence (1) explicitly appears as

$$S_1=\{c,d,d,c\},\quad S_2=\{c,d,d,c,d,c,c,d\},\quad \ldots\ .$$

Denoting the total number of tiles in S_{n+1} and S_n by G_{n+1} and G_n, respectively, we get

$$G_{n+1}=2G_n\ . \tag{2}$$

Equation (2) leads to the following result:

$$G_n=2\times 2^n\ . \tag{3}$$

B. Fibonacci lattices

The arrangement of tiles c and d for Fibonacci lattices follows the Fibonacci sequence J_n which is given[16] by

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$$J_{n+1}=\{J_n,J_{n-1}\}, \quad n\geq 1, \quad J_0=\{c\}, \quad J_1=\{d\} . \tag{4}$$

Explicitly, sequence (4) appears as

$$J_2=\{d,c\}, \quad J_3=\{d,c,d\}, \quad \ldots .$$

Using sequence (4), we obtain

$$F_{n+1}=F_n+F_{n-1} , \tag{5}$$

where F_n is the total number of tiles in the sequence J_n. Equation (5) leads to the following result:

$$\sigma=\frac{1}{\sigma}+1 , \tag{6}$$

where

$$\sigma=\lim_{n\to\infty}\frac{F_n}{F_{n-1}} . \tag{7}$$

Equation (6) yields two solutions $\sigma_\pm$ of σ, namely, $\sigma_\pm=\frac{1}{2}[1\pm\sqrt{5}]$. The root σ_+ is the so-called golden mean associated with Fibonacci lattices.

III. TRANSFER MATRICES FOR THE KP MODEL ON APERIODIC LATTICES

For our treatment of the Landauer resistance of both TM and Fibonacci lattices, we require some features of transfer matrices relevant to the KP model on these sequences. In this section, we discuss the transfer matrices generally in the context of aperiodic lattices, and take up the cases of TM and Fibonacci lattices in two subsequent sections.

The Hamiltonian H for a system of N barriers with their centers at x_n is given by

$$H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\sum_{n=1}^{N}V(x-x_n) . \tag{8}$$

$V(x-x_n)$ takes the constant value V_0 for $(x_n-b/2)\leq x\leq(x_n+b/2)$, and it is zero elsewhere. The Hamiltonian H given by (8) is a continuous Hamiltonian. This kind of Hamiltonian seems to be capable of taking care of realistic features better than the so-called discrete Hamiltonian based on the tight-binding approximation.[17]

Now the Schrödinger equation for Hamiltonian H yields the following wave function for the zero-potential region between the nth and $(n+1)$th barriers:

$$\psi_n=A_n\exp[ik(x-x_n-b/2)]+B_n\exp[-ik(x-x_n-b/2)] , \quad (x_n+b/2)<x<(x_{n+1}-b/2) , \tag{9}$$

where $k^2=2mE/\hbar^2$, and E is the energy eigenvalue of the electron. Introducing two (2×2) transfer matrices $M(n)$ and M_N we can obtain

$$\begin{bmatrix}A_n\\B_n\end{bmatrix}=M(n)\begin{bmatrix}A_{n-1}\\B_{n-1}\end{bmatrix} , \tag{10}$$

$$\begin{bmatrix}A_N\\B_N\end{bmatrix}=M_N\begin{bmatrix}A_0\\B_0\end{bmatrix} , \tag{11}$$

$$[M(n)]_{11}=[M(n)]^*_{22}=\left[\cosh\beta b+\frac{i}{2k\beta}(k^2-\beta^2)\sinh\beta b\right]\times\exp[ik(\Delta x_n-b)] , \tag{12}$$

$$[M(n)]_{12}=[M(n)]^*_{21}=\left[-\frac{i}{2k\beta}(k^2+\beta^2)\sinh\beta b\right]\times\exp[-ik(\Delta x_n-b)] , \tag{13}$$

$$\Delta x_n=(x_n-x_{n-1}), \quad \beta^2=\frac{2m(V_0-E)}{\hbar^2}, \quad E<V_0 ,$$

$$\det M(n)=1 ,$$

$$M_N=M(N)M(N-1)\cdots M(1) , \tag{14}$$

$$(M_N)_{11}=(M_N)^*_{22} , \tag{15}$$

$$(M_N)_{12}=(M_N)^*_{21} , \tag{16}$$

$$\det M_N=1 . \tag{17}$$

If c and d are interchanged in the two-tile aperiodic sequence, the axis of symmetry of potential barriers is shifted to new positions $\bar{x}_n$; consequently, the matrices $M(n)$ and M_N change, respectively, to $\overline{M}(n)$ and $\overline{M}_N$ as given below,

$$\overline{M}(n)=M(n) \text{ with } \Delta x_n=\overline{\Delta x_n}, \quad \overline{\Delta x_n}=\bar{x}_n-\bar{x}_{n-1} , \tag{18}$$

$$\overline{M}_N=\overline{M}(N)\overline{M}(N-1)\cdots\overline{M}(1) . \tag{19}$$

IV. LANDAUER RESISTANCE OF TM LATTICES

For the purpose of deriving the Landauer resistance of TM lattices, we require the analogs of various aspects of the transfer matrices discussed in Sec. III, in regard to TM lattices. We first discuss these analogs.

For TM lattices, the centers of potential barriers are distributed according to the TM sequence given by (1). The total number N of barriers then becomes a TM number G_n. We now introduce the symbols Q_n and $\overline{Q}_n$, which are the forms taken for TM lattices by M_N and $\overline{M}_N$, respectively. With the help of (1), we can obtain

$$Q_{n+1}=\overline{Q}_nQ_n , \tag{20}$$

$$\overline{Q}_{n+1}=Q_n\overline{Q}_n . \tag{21}$$

We can write $(Q_n)_{11}$ and $(\overline{Q}_n)_{11}$ as follows:

$$(Q_n)_{11}=R_n+iI_n , \tag{22}$$

$$(\overline{Q}_n)_{11}=\overline{R}_n+i\overline{I}_n . \tag{23}$$

R_n and $\overline{R}_n$ are half the trace of Q_n and $\overline{Q}_n$, respectively. Application of the trace commutative law to Q_n and $\overline{Q}_n$ yields

$$R_n=\overline{R}_n . \tag{24}$$

With the help of (20), (21), and (24), we can obtain the well-known trace map[4]

$$R_{n+1}=4R^2_{n-1}R_n-4R^2_{n-1}+1, \quad n\geq 1 . \tag{25}$$

Using (20)–(24), we can derive the following equation for I_{n+1}:

$$I_{n+1}=4R_nR_{n-1}I_{n-1}-2R_{n-1}(I_{n-1}-\bar{I}_{n-1}),\quad n\geq 1\ . \tag{26}$$

The equation for $\bar{I}_{n+1}$ can be obtained from (26) by interchanging I_{n-1} and $\bar{I}_{n-1}$.

Now, as is well known, the Landauer resistance for a chain of barriers is defined[18,19] as the ratio of reflection coefficient to transmission coefficient. This definition leads to the LR being equal to the square of the modulus of $(Q_{n+1})_{12}$ for a chain of barriers.[19,20] Using this fact, we have

$$\rho^T_{n+1}=|(Q_{n+1})_{12}|^2\ . \tag{27}$$

ρ^T_{n+1} is the LR for the TM sequence S_{n+1}. Using (20) and (22)–(26) and the fact that $\det Q_{n+1}=1$, Eq. (27) can be reduced to

$$\rho^T_{n+1}=4R^2_{n-1}(P_n+C_n),\quad n\geq 1\ , \tag{28}$$

where

$$P_n=2(R_n-1)[2R^2_{n-1}(R_n-1)+1]+(I_{n-1}-\bar{I}_{n-1})^2\ , \tag{29}$$

$$C_n=4R_nI_{n-1}[\bar{I}_{n-1}+I_{n-1}(R_n-1)]\ . \tag{30}$$

V. LANDAUER RESISTANCE OF FIBONACCI LATTICES

To derive the Landauer resistance of Fibonacci lattices, we need the analogs of various aspects of the transfer matrices discussed in Sec. III, in regard to such lattices. For Fibonacci lattices, the centers of potential barriers are distributed according to the Fibonacci sequence given by (4); the total number of barriers N now becomes a Fibonacci number F_n. We introduce the symbol W_n to denote M_N for the Fibonacci lattice. Using (4), we obtain

$$W_{n+1}=W_{n-1}W_n\ . \tag{31}$$

It is now easy to establish the following relations:

$$Y_{n+1}=2Y_nY_{n-1}-Y_{n-2}\ , \tag{32}$$

$$Z_{n+1}=2Y_nZ_{n-1}+Z_{n-1}\ , \tag{33}$$

where

$$(W_n)_{11}=Y_n+iZ_n\ . \tag{34}$$

In analogy with (27), the Landauer resistance ρ^F_{n+1} of Fibonacci lattices appears as

$$\rho^F_{n+1}=|(W_{n+1})_{12}|^2\ . \tag{35}$$

Using (31)–(33), we can reduce (35) to the following form:

$$\rho^F_{n+1}=4Y^2_n(Y^2_{n-1}+z^2_{n-1})+(Y^2_{n-2}+z^2_{n-2})-4Y_n(Y_{n-1}Y_{n-2}-z_{n-1}z_{n-2})-1\ . \tag{36}$$

VI. LANDAUER RESISTANCE FOR PERIODIC LATTICES

The Landauer resistance for a periodic system can be derived by suitably manipulating the transfer matrices M_N of Sec. III. Many authors[19,21] have reported earlier the formulas for Landauer resistance of a finite periodic system consisting of rectangular barrier-type potentials with barrier width b and we quote them below. The formulas for the LR ρ^P for such systems appear as

$$\rho^P=|q(12)|^2[U_r(\cos\theta)]^2,\quad |g|<2\ , \tag{37}$$

$$\rho^P=|q(12)|^2[h_r(\emptyset)]^2,\quad |g|\geq 2\ , \tag{38}$$

$$q(12)=\frac{-i(k^2+\beta^2)}{2k\beta}\sinh\beta b\ , \tag{39}$$

$$U_r(\cos\theta)=\frac{\sin(r+1)\theta}{\sin\theta},\quad r=N-1\ , \tag{40}$$

$$h_r(\emptyset)=\frac{\sinh(r+1)\emptyset}{\sinh\emptyset},\quad r=N-1\ , \tag{41}$$

$$\cos\theta=\tfrac{1}{2}g,\quad |g|<2\ , \tag{42}$$

$$\cosh\emptyset=\tfrac{1}{2}g,\quad |g|\geq 2\ , \tag{43}$$

$$g=2\left[\cos ka\cosh\beta b+\frac{k^2-\beta^2}{2k\beta}\sin ka\sinh\beta b\right]\ . \tag{44}$$

VII. NUMERICAL ANALYSIS

We have computed the Landauer resistance of TM lattices, Fibonacci lattices, and periodic lattices, as a function of energy. The number of barriers for TM lattices corresponds to a set of TM numbers, while the number of barriers for Fibonacci lattices corresponds to a set of Fibonacci numbers. In regard to TM lattices, we have used (28). For Fibonacci lattices, we have taken recourse to (36), while for periodic systems, we have used (37) and (38). We have constructed the periodic system with a finite number of barriers. The number of barriers in these systems is either a Fibonacci number or a TM number, and their periodicities correspond to the ratio of total length of the relevant Fibonacci or TM lattices to the total number of barriers. The way we have chosen our periodic systems makes quite meaningful the comparison between results related to them and those related to Fibonacci and TM lattices associated with them. Our numerical results are presented in Figs. 1–3.

VIII. RESULTS, DISCUSSION, AND CONCLUSIONS

As mentioned in the Introduction, the objective behind our paper is to compare the degree of aperiodicity of Fibonacci lattices with that of TM lattices as far as the LR is concerned. In regard to our objective, we have evaluated the LR of Fibonacci and TM lattices, their respective values of N being chosen in such a way that they are as close to each other as is possible; these LR's are shown in Figs. 1 and 2. Further, we have evaluated the LR ρ^P of certain periodic lattices, which we call equivalent periodic lattices (EPL's). The width and height of the potential

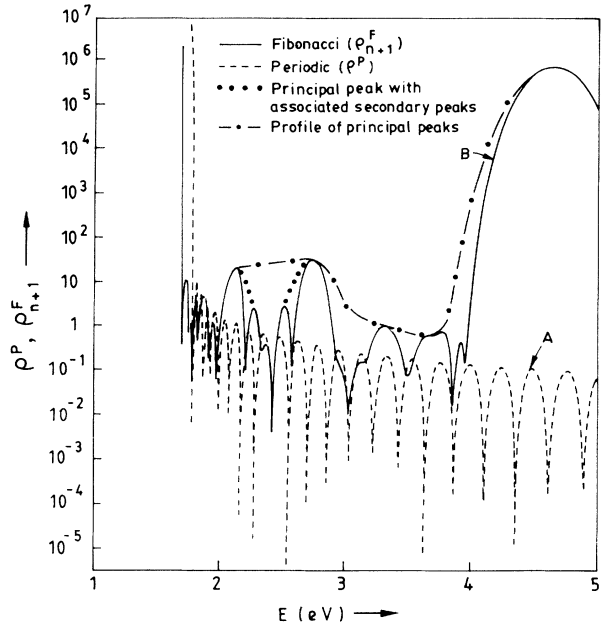

FIG. 1. Variation of Landauer resistances ρ^P and P^F_{n+1} with energy E. Curve A corresponds to periodic lattices and curve B corresponds to Fibonacci lattices. Parameters are for A, $N=55$, $f=1.381\,818\,2$ Å, $b=0.5$ Å, $V_0=5$ eV; for B, $N=55$, $c=1$ Å, $d=2$ Å, $b=0.5$ Å, $V_0=5$ eV.

barriers in EPL's are the same as those of the (aperiodic) Fibonacci and TM lattices, and their periodicity f is taken as $f=L/N$, where L and N are respectively the total length and total number of barriers of the aperiodic lattice. The Landauer resistances of the EPL's are also shown in Figs. 1 and 2.

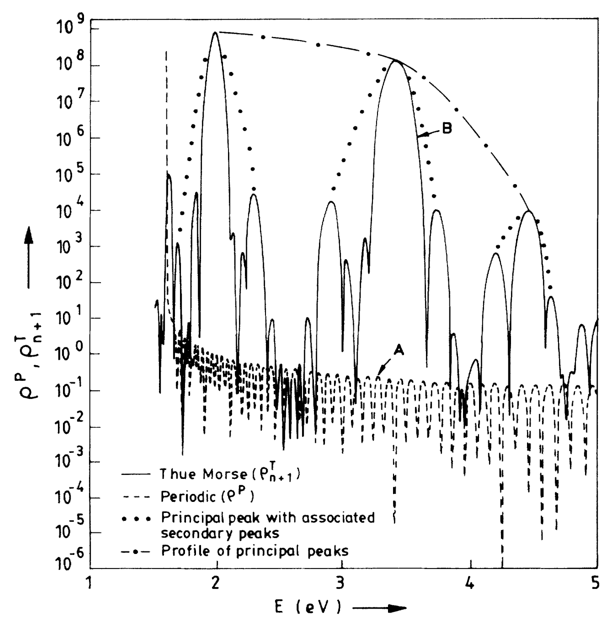

FIG. 2. Variation of Landauer resistances ρ^P and ρ^T_{n+1}, with energy E. Curve A corresponds to periodic lattices and curve B corresponds to TM lattices. Parameters are for A, $N=128$, $f=1.5$ Å, $b=0.5$ Å, $V_0=5$ eV; for B, $N=128$, $c=1$ Å, $d=2$ Å, $b=0.5$ Å, $V_0=5$ eV.

For the purpose of judging the degree of aperiodicity of Fibonacci and TM lattices, we take recourse to comparing the LR for these lattices with that of the periodic lattices. First, we note the following prominent features of the LR of periodic lattices.

(1) ρ^P oscillates with energy.

(2) The amplitude of oscillation of ρ^P decreases with energy, while the width $(\Delta E^j_m)_P$ between the $(j+1)$th and jth minima increases with energy. Graph III of Fig. 3 shows that this width varies linearly with energy E.

(3) With increase of N, the rapidity of oscillation of ρ^P increases, the features (2) being present for all values of N.

We now come to discuss the features of ρ^T_{n+1} and ρ^F_{n+1}, which are, respectively, the LR's of the TM lattice and the Fibonacci lattice, in the context of the above-mentioned features of ρ^P. Looking at Figs. 1 and 2, we note the following features of ρ^F_{n+1} and ρ^T_{n+1}.

(i) ρ^T_{n+1} and ρ^F_{n+1} show oscillations with energy; the

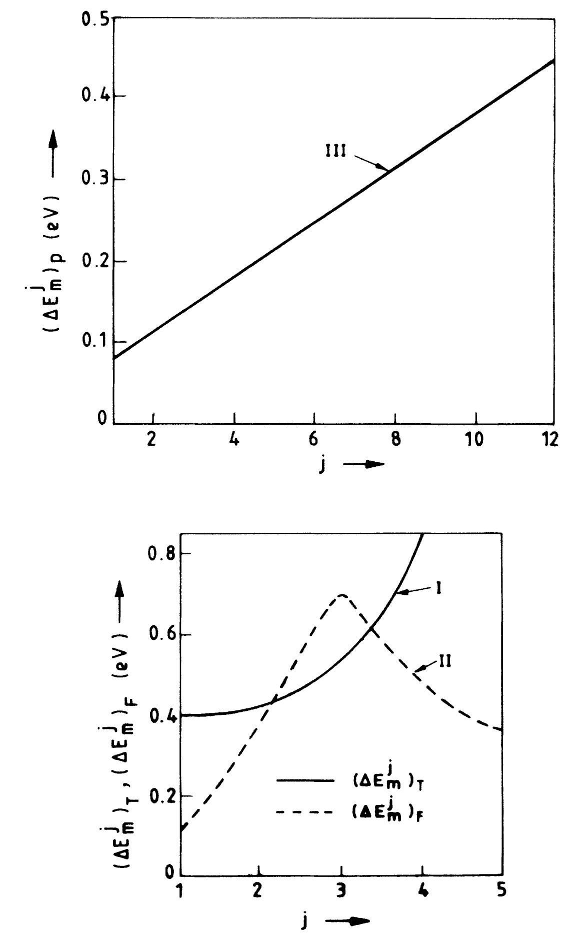

FIG. 3. Variation of widths of the energy range between the $(j+1)$th and jth minima of the Landauer resistance of the TM lattice, Fibonacci lattice, and periodic system. I, TM lattice; II, Fibonacci lattice; III, periodic system. For I, $N=32$, and the other parameters are the same as those for curve B in Fig. 2. For II, $N=34$, and the other parameters are the same as those for curve B in Fig. 2. For III, $N=32$, $f=1.5$ Å, $b=0.5$ Å, and $V_0=5$ eV.

qualitative features of their oscillations differ in many respects.

(ii) It appears that ρ_{n+1}^{T} shows one kind of maxima which are larger than another kind of maxima; we call the former class of maxima principal maxima and the latter class of maxima secondary maxima; the secondary maxima occur in the vicinity of principal maxima. The maxima of ρ_{n+1}^{F} do not afford any clear division into principal maxima and secondary maxima.

(iii) It is seen that successive principal maxima of ρ_{n+1}^{T} decrease with energy (Fig. 2); the maxima of ρ_{n+1}^{F} do not show systematic decrease with energy (Fig. 1).

(iv) The width $(\Delta E_m^j)_T$ of the energy range between the $(j+1)$th and jth minima of ρ_{n+1}^{T} (graph I of Fig. 3) increases with energy, while the corresponding behavior of the Fibonacci lattice (graph II of Fig. 3) is of a different nature.

Considering the features (iii) and (iv) of ρ_{n+1}^{T} and ρ_{n+1}^{F} in the context of features (2) of ρ^{P}, we can say that, as far as LR is concerned, the degree of aperiodicity of the TM lattice is less than that of Fibonacci lattices; this finding of ours corroborates the observation reported by Riklund, Severin, and Liu[1] on the basis of electronic spectra.

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[1]R. Riklund, M. Severin, and Y. Liu, Int. J. Mod. Phys. B **1**, 121 (1987).
[2]Z. Cheng, R. Savit, and R. Merlin, Phys. Rev. B **37**, 4375 (1988).
[3]M. Kolar, M. K. Ali, and F. Nori, Phys. Rev. B **43**, 1034 (1991).
[4]Y. Avishai and D. Berend, Phys. Rev. B **45**, 2717 (1992).
[5]J. X. Zhong, J. Q. You, and J. R. Yan, J. Phys. Condens. Matter **4**, 5959 (1992).
[6]M. G. Qin, H. R. Ma, and C. H. Tsai, J. Phys. Condens. Matter **2**, 1059 (1990).
[7]C. S. Ryu, G. Y. Oh, and M. H. Lee, Phys. Rev. B **48**, 132 (1993).
[8]J. M. Luck, Phys. Rev. B **39**, 5834 (1989).
[9]J. X. Zhong, J. R. Yan, and J. Q. You, J. Phys. Condens. Matter **3**, 6293 (1991).
[10]Q. Niu and F. Nori, Phys. Rev. Lett. **57**, 2057 (1986).
[11]J. A. Ashraff and R. B. Stinchcombe, Phys. Rev. B **37**, 5723 (1988).
[12]Y. Avishai and D. Berend, Phys. Rev. B **41**, 5492 (1990).
[13]J. Kollar and A. Suto, Phys. Lett. A **117**, 203 (1986).
[14]R. Merlin, K. Bejema, Roy Clarke, F. Y. Juang, and P. K. Bhattacharya, Phys. Rev. Lett. **55**, 1768 (1985).
[15]G. Y. Oh, C. S. Ryu, and M. H. Lee, J. Phys. Condens. Matter **4**, 8187 (1992).
[16]M. Kohmoto, Int. J. Mod. Phys. B **1**, 31 (1987).
[17]M. Baake, D. Joseph, and P. Kramer, Phys. Lett. A **168**, 199 (1992).
[18]R. Landauer, Philos. Mag. **21**, 863 (1970).
[19]P. Erdos and R. C. Herndon, Adv. Phys. **31**, 65 (1982).
[20]C. L. Roy and C. Basu, Phys. Rev. B **45**, 14 293 (1992).
[21]C. L. Roy and C. Basu, Phys. Status Solidi B **177**, 315 (1993).