

## Optical absorption and addition spectra of an $N$ -electron quantum dot: An algebraic Hamiltonian approach

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We calculate analytically the energies of a class of  $N$ -electron quantum dots employing a recently introduced symplectic-group chain. The corresponding algebraic Hamiltonians lie beyond the  $N$ -electron Fock-Darwin Hamiltonian which only describes noninteracting particles in a perfect parabolic confining potential. Nonparabolicity and particle-particle interactions are simulated using quadratic Casimir operators within the symplectic-group chain. We calculate infrared optical absorption energies as a function of magnetic field, and analyze the breakdown of the generalized Kohn theorem. We also calculate the addition spectra associated with transport through the  $N$ -electron quantum dot.

### I. INTRODUCTION

Recent technological advances have meant that quantum devices of ever smaller sizes with a tunable number of electrons can be reliably constructed and investigated experimentally. The current theoretical goal is to provide an understanding and classification of the optical and transport data that have so far been obtained experimentally from quantum dot systems.<sup>1-3</sup> The single-electron spectrum for a parabolic dot has long been known.<sup>4</sup> These single-electron, or Fock-Darwin, states have been used to construct an  $N$ -electron basis for numerically diagonalizing the many-electron Hamiltonian including electron-electron interactions (see, for example, Ref. 5). Such numerical approaches tend to be computationally intensive.

In a previous paper<sup>6</sup> we presented an algebraic model based on the symplectic group  $\text{Sp}(2Nd, R)$  to describe nonrelativistic  $N$ -particle systems in  $d$  dimensions. This model has particular relevance to  $N$ -electron quantum dots, large or small, because of the possibility of expressing simple Hamiltonians describing such systems within the symplectic algebra. The Hamiltonian associated with  $N$  noninteracting particles confined by a two-dimensional parabolic potential in the presence of a perpendicular magnetic field (the so-called  $N$ -particle Fock-Darwin Hamiltonian) is the simplest example of this type since it has a pure oscillator term belonging to the  $\text{Sp}(2, R)$  algebra and an angular-momentum term that is an element of the algebra of  $\text{O}(d)$ . As discussed in Ref. 6, there are various appropriate group-chains of the symplectic model all of which include the chain  $\text{Sp}(2Nd, R) \supset \cdots \supset \text{Sp}(2, R) \times \text{O}(N) \times \text{O}(d)$ . Since we are focusing on quantum systems of identical particles, any  $N$ -electron model Hamiltonian must necessarily be invariant under permutation operations of  $\text{S}(N)$ ; this group is contained in  $\text{O}(N)$ .

The  $N$ -electron quantum dot system must be con-

sidered as an *interacting* many-body quantum-mechanical system. Introducing interaction potentials into the  $N$ -electron Fock-Darwin Hamiltonian presents difficulties as these are not usually expressible as  $\text{Sp}(2Nd, R)$  operators and hence lie outside the algebra. The translationally invariant Coulomb interaction is unfortunately of this nature, although there are some interesting two-body interactions expressible within  $\text{Sp}(2Nd, R)$  which may be appropriate approximations to real interactions. For example, Johnson and Payne<sup>7</sup> introduce a quadrupole-moment interaction term which belongs to an Abelian subalgebra of the  $\text{Sp}(2N, R)$  algebra. If one were to consider such an interaction potential within the  $N$ -particle Fock-Darwin Hamiltonian, then the appropriate group-subgroup chain would be  $\text{Sp}(2Nd, R) \supset \text{Sp}(2N, R) \times \text{O}(d)$  with the subsequent reduction  $\text{Sp}(2N, R) \supset \text{Sp}(2, R) \times \text{O}(N)$ . It is clear that such interaction potentials would mix states at the  $\text{Sp}(2, R) \times \text{O}(N)$  level.

Given the remarkably rapid progress in the manufacture of tailor-made quantum dot systems, it may not be long before quantum device fabrication can actually customize quantum dots with certain types of characteristics; for example a specific confining potential or geometric shape, or even a particular energy-level scheme. In fact, dots could even be designed to represent specific effective Hamiltonians. In this paper, we examine the energy-level structure and states of various algebraic quantum dot Hamiltonians which depend on the invariant (i.e., Casimir) operators of the subgroups of  $\text{Sp}(2Nd, R)$ . We show that matrix elements and energy eigenvalues of such Hamiltonians are straightforward to evaluate, and they have the potential for describing quantum dot properties even in highly anharmonic regimes. This approach using Casimir operators represents a change in philosophy from previous calculations for nonparabolic quantum dots containing  $N$  interacting electrons. Previous work<sup>8,9</sup> considered realistic microscopic perturbations to the  $N$ -electron Hamiltonian, and calculated numerically the shifts of the energy levels using per-

turbation theory (or equivalently, by diagonalizing a truncated matrix). Here we consider the Casimir operators to be *approximate* descriptions of actual perturbations, and then solve for the energy shifts *exactly*. Such an approach, while not common in condensed-matter physics, is well known in nuclear and molecular physics for describing rotational and vibrational band structures.

The paper is organized as follows. In Sec. II we briefly review the important features of the symplectic-group analysis of the  $N$ -electron dot. Full details are given in Refs. 6 and 10. Section III discusses the state labelings and matrix elements within the various group-subgroup chains, and gives the Casimir operators for the relevant groups. Section IV uses these results (in particular the Casimir operators) to generate the energy eigenvalues of a class of Hamiltonians describing  $N$  interacting electrons in a nonparabolic dot. In Sec. V, the associated infrared-absorption energies are calculated as a function of magnetic field, and the breakdown of the generalized Kohn theorem is analyzed. The theoretical infrared-absorption spectrum is qualitatively consistent with recent experimental data. We also calculate the addition spectra associated with transport through the  $N$ -electron quantum dot.

## II. THE SYMPLECTIC-GROUP BASICS

We will now briefly review the essential results needed for this paper; for detailed discussions of the underlying group theory see Refs. 6, 10, and 11. The necessary compact group results are discussed fully in Refs. 12 and 13. The noncompact symplectic group  $\text{Sp}(2n, R)$  will be denoted by  $\text{Sp}(2n)$ . We use the convention of distinguishing irreducible representations (irreps) of the orthogonal group  $\text{O}(N)$  and the symmetric group  $\text{S}(N)$  by square brackets, those of the symplectic group  $\text{Sp}(2n)$  by angular brackets and those of the unitary group  $\text{U}(n)$  by curly brackets. The rank of the group will often be subscripted to make identification of groups easier.

The  $\text{Sp}(2n)$  unitary irreps that are considered belong to the family called discrete series (or harmonic series) representations.<sup>11,14</sup> We shall call these uds-irreps. Such irreps are infinite dimensional and will be labeled as  $\langle k/2(\kappa) \rangle_n$  where  $(\kappa)$  denotes the partition, and the constraints  $\bar{\kappa}_1 \leq n$  and  $\bar{\kappa}_1 + \bar{\kappa}_2 \leq k$  apply. The uds constraints imply that the summation is over those standard partitions that label covariant tensor irreps of  $\text{U}(n)$ , and those irrep labels of  $\text{O}(k)$  that are *near-standard* since  $\bar{\kappa}_1$  could be greater than  $k/2$  but is certainly less than  $k$ . These two constraints are important when applying symplectic-group results and provide strong restrictions when applied to physical situations, in particular in one, two, or three dimensions as then either  $n$  or  $k$  can be small integers. The reduction of the basic metaplectic spin irrep  $\langle s; 0 \rangle$  of  $\text{Sp}(2nk)$  to  $\text{Sp}(2n) \times \text{O}(k)$  is

$$\langle s; 0 \rangle_{nk} \downarrow \sum_{\kappa} \left\langle \frac{k}{2}(\kappa) \right\rangle_n \times [\kappa]_k, \quad (1)$$

where the summation is over all the partitions  $(\kappa)$  satisfying the two constraints mentioned above.<sup>15,16</sup> Note that the branching is multiplicity-free. In addition there is just one irrep label of  $\text{O}(k)$  for each irrep label of  $\text{Sp}(2n)$

and vice versa—a fact known in nuclear physics as *complementarity*.<sup>17</sup>

A second result is the general branching rule for  $\text{Sp}(2n) \supset \text{Sp}(2) \times \text{O}(n)$  which is summarized in the following:

$$\left\langle \frac{k}{2}(\kappa) \right\rangle_n \downarrow \sum_m \left\langle \frac{nk}{2}(m) \right\rangle_1 [((m/\mathcal{C}) \circ ({}_s \kappa^k \cdot \mathcal{D})_k)_n / \mathcal{D}]_n, \quad (2)$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are Schur function series,<sup>12,13</sup>  ${}_s \kappa^k$  is the  $\text{O}(k)$  signed sequence,<sup>11,14</sup> and the symbols  $\circ$  and  $\cdot$  denote the Schur function operations of inner and outer multiplication, respectively.<sup>10</sup> Although this result seems complicated, there is some simplification in the evaluation of the expression  $((m/\mathcal{C}) \circ ({}_s \kappa^k \cdot \mathcal{D})_k)_n$ . First, the signed sequence with partitions restricted to at most  $k$  parts becomes a finite sum of partitions. Second,  $(m)$  is only a one-part partition hence  $(m/\mathcal{C}) = (m) - (m-2)$  for  $m \geq 2$ , with the special cases  $(0/\mathcal{C}) = (0)$ ,  $(1/\mathcal{C}) = (1)$ . Third, the inner products are particularly easy since for  $m$  integer  $(m) \circ (\nu) = (\nu)$  where  $(\nu)$  is any partition of weight  $m$ . Hence for given  $m$ , the result of the inner product is to extract from  $({}_s \kappa^k \cdot \mathcal{D})_k$  those terms of weight  $m$  and  $m-2$ . The terms arising from the inner products must also be standard labels of  $\text{U}(n)$  hence the subscript  $n$  in  $((m/\mathcal{C}) \circ ({}_s \kappa^k \cdot \mathcal{D})_k)_n$ . Note that since the inner products are especially simple, we can take the part restrictions inside the inner products.

For general  $n$ , the noncompact symplectic group  $\text{Sp}(2n)$  has  $\text{U}(n)$  as a maximal compact group. In this paper we only need the result for  $\text{Sp}(2) \supset \text{U}(1)$ . Since each  $\text{Sp}(2)$  unitary irrep is infinite dimensional and those of  $\text{U}(1)$  are all one dimensional, it is particularly useful to label the basis states according to this  $\text{U}(1)$  compact subgroup. The  $\text{U}(1)$  content can easily be extracted from each  $\text{Sp}(2)$  irrep,<sup>6,14</sup>

$$\left\langle \frac{Nd}{2}(m) \right\rangle_1 \downarrow \epsilon_{Nd}^{1/2} \cdot \{m \cdot \mathcal{D}_1\}_1 = \sum_j \left\{ \frac{Nd}{2} + m + 2j \right\}_1, \quad (3)$$

where  $(m)$  is a one-row partition of non-negative integers  $m$  and the outer product  $(m \cdot \mathcal{D}_1)$  is evaluated in  $\text{U}(1)$ . The half-power of  $\epsilon_{Nd}$  [the determinantal representation of  $\text{U}(Nd)$ ] implies the presence of a projective representation of  $\text{U}(Nd)$ .<sup>6</sup> If  $Nd$  is odd the  $\text{U}(1)$  irreps are labeled by half-integers corresponding to multivalued representations.

## III. MATRIX ELEMENTS AND CASIMIRS OF THE $N$ -ELECTRON SYSTEM

We now apply the basic character theory results to particular symplectic-group chains which have relevance in the modeling of an  $N$ -particle system in  $d$  dimensions, and in particular the  $d=2$  quantum dots. There are three symplectic-group chains, all three having the common subgroup  $\text{Sp}(2) \times \text{O}(N) \times \text{O}(d)$  (see Fig. 1. of Ref. 6). For these three chains we need to establish the corresponding classification of states. We will start by developing a systematic notation so that the various basis

states and matrix elements can then be related and discussed simultaneously. We consider the states of the  $Nd$ -quantal system to transform under the basic metaplectic spin irrep  $\langle s;0 \rangle$  of  $\text{Sp}(2Nd)$ . This introduces some simplifications due to the uds constraints given in the previous section. We remark that in all three chains the direct product group  $\text{O}(N) \times \text{O}(d)$  separates the total angular momentum of the  $N$ -particle system given by  $\text{O}(d)$  from its permutational symmetry given by  $\text{S}(N) \subset \text{O}(N)$ . Below we present the labeling of states for an  $N$ -particle system in  $d$  dimensions according to the three group-subgroup chains.

#### A. $\text{Sp}(2Nd) \supset \text{Sp}(2) \times \text{O}(Nd) \supset \text{Sp}(2) \times \text{O}(N) \times \text{O}(d)$

The first reduction is given by Eq. (1) with the conditions  $n=1$  and  $k=Nd$ . Thus the  $\text{Sp}(2)$  irreps are labeled by  $\langle Nd/2(m) \rangle_1$  with the associated  $\text{O}(Nd)$  irrep labeled by  $[m]_{Nd}$  where  $m$  is a non-negative integer. The  $\text{O}(Nd) \supset \text{O}(N) \times \text{O}(d)$  reduction is a standard compact group reduction.<sup>14</sup>

The states under this group-subgroup chain can then be classified by

$$|(Nd, m)_{1am} \nu \delta jkl \rangle \equiv \left. \begin{array}{c} \langle s;0 \rangle_{Nd} \\ 0 \\ \langle \frac{Nd}{2}(m) \rangle_1 \times [m]_{Nd} \\ a \\ \langle \frac{Nd}{2}(m) \rangle_1 \times [\nu]_N \times [\delta]_d \\ j \quad k \quad l \end{array} \right\} , \quad (4)$$

where  $a=1, \dots, m(m, \nu \delta)$  is the branching multiplicity label index for the reduction  $\text{O}(Nd) \supset \text{O}(N) \times \text{O}(d)$  and  $m(m, \nu \delta)$  denotes the multiple occurrence of  $[\nu] \times [\delta]$  in  $[m]$ . Here  $\nu$  and  $\delta$  are partitions into at most  $\min(N, d)$  parts. The labels  $j, k$ , and  $l$  are arbitrary basis labels of the irrep spaces  $\langle Nd/2(m) \rangle_1$ ,  $[\nu]_N$ , and  $[\delta]_d$ , respectively. For the orthogonal group irreps the bases are finitely indexed whereas for symplectic irreps they are infinitely indexed. The physical interpretation is that, given a Hamiltonian such as the harmonic oscillator Hamiltonian  $H_{\text{osc}}$  which is expressible in terms of the algebra of  $\text{Sp}(2)$ , the dynamical symmetry group of that Hamiltonian is *at least*  $\text{O}(Nd)$  as discussed in Refs. 18 and 19.

#### B. $\text{Sp}(2Nd) \supset \text{Sp}(2N) \times \text{O}(d) \supset \text{Sp}(2) \times \text{O}(N) \times \text{O}(d)$

The first reduction is given by Eq. (1) with  $n=N$  and  $k=d$ . The  $\text{Sp}(2N)$  irreps are then labeled by  $\langle d/2(\delta) \rangle_N$  with the associated  $\text{O}(d)$  irrep labeled by  $[\delta]_d$ , where  $(\delta)$  is at most an  $N$ -part partition whose first two columns satisfy  $\delta_1 + \delta_2 \leq d$ . The labeling of the  $\text{Sp}(2N)$  irreps is less restricted by these two constraints than in the first case. For physical situations ( $d=1, 2, 3$ ), this latter constraint places the strongest restriction. The reduction of the  $\text{Sp}(2N)$  irreps is given by Eq. (2). The states under this

group-subgroup chain can hence be classified by

$$|(d, \delta)_N b m' \nu \delta jkl \rangle \equiv \left. \begin{array}{c} \langle s;0 \rangle_{Nd} \\ 0 \\ \langle \frac{d}{2}(\delta) \rangle_N \times [\delta]_d \\ b \\ \langle \frac{Nd}{2}(m') \rangle_1 \times [\nu]_N \times [\delta]_d \\ j \quad k \quad l \end{array} \right\} , \quad (5)$$

where  $m' = w(\delta) + 2i'$  with  $i'$  a non-negative integer and  $w(\delta)$  is the weight of the partition  $\delta$ ;  $b=1, \dots, m(\delta, m' \nu)$  is the  $\text{Sp}(2N) \supset \text{Sp}(2) \times \text{O}(N)$  branching multiplicity label index. The partition  $\nu$  must be of at most  $\min(N, d)$  parts. This set of states would be more appropriate for Hamiltonians that commute with the angular-momentum group  $\text{O}(d)$ , i.e., conserve total angular momentum, since  $\text{O}(d)$  is extracted from the outset and simple interaction potentials<sup>7</sup> are confined to elements of  $\text{Sp}(2N)$ .

#### C. $\text{Sp}(2Nd) \supset \text{Sp}(2d) \times \text{O}(N) \supset \text{Sp}(2) \times \text{O}(d) \times \text{O}(N)$

The first reduction is given by applying Eq. (1) with  $n=d$  and  $k=N$ . The  $\text{Sp}(2d)$  irreps are labeled by  $\langle N/2(\nu) \rangle_d$  with the associated  $\text{O}(N)$  irrep given by  $[\nu]_N$ , where  $\nu$  is a partition of  $d$  parts satisfying  $\tilde{\nu}_1 + \tilde{\nu}_2 \leq N$ . For large  $N$  the labeling of the  $\text{Sp}(2d)$  irreps is clearly restricted more by the dimension  $d$ . Since  $\nu$  must be a regular partition,  $\tilde{\nu}_1 + \tilde{\nu}_2 \leq 2\tilde{\nu}_1 \leq 2d$ . Therefore,  $\nu$  automatically satisfies these conditions if  $N \geq 2d$ . This would then be the generic large  $N$  case, although there are physically interesting small  $N$  situations when  $N < 2d$  involving two or three particles in two dimensions and two to five particles in three dimensions. The reduction of the  $\text{Sp}(2d)$  irreps follows Eq. (2). The states under this group-subgroup chain can hence be classified by

$$|(N, \nu)_d c m'' \nu \delta jkl \rangle \equiv \left. \begin{array}{c} \langle s;0 \rangle_{Nd} \\ 0 \\ \langle \frac{N}{2}(\nu) \rangle_d \times [\nu]_N \\ c \\ \langle \frac{Nd}{2}(m'') \rangle_1 \times [\nu]_N \times [\delta]_d \\ j \quad k \quad l \end{array} \right\} , \quad (6)$$

where  $m'' = w(\nu) + 2i''$  with  $i''$  a non-negative integer;  $c=1, \dots, m(\nu, m'' \delta)$  is the  $\text{Sp}(2d) \supset \text{Sp}(2) \times \text{O}(d)$  branching label index. The partition  $\delta$  must be at most of  $\min(N, d)$  parts. These states emphasize the  $\text{O}(N)$  symmetry and consequently the permutational symmetry  $\text{S}(N)$ , and will be appropriate for systems of  $N$ -identical particles (e.g.,  $N$  electrons or  $N$  holes as opposed to mixed systems) with an interaction or confinement potential expanded in terms of elements of  $\text{Sp}(4)$ .

Summarizing the above, we have specified three group-subgroup chains and classified basis states transforming according to irrep spaces of the groups in each

chain:

$$\begin{aligned} (1) \quad & \cdots \supset \text{Sp}(2) \times \text{O}(Nd) \supset \cdots |(Nd, m)_{1am} \nu \delta jkl \rangle \\ (2) \quad & \cdots \supset \text{Sp}(2N) \times \text{O}(d) \supset \cdots |(d, \delta)_N b m' \nu \delta jkl \rangle \quad (7) \\ (3) \quad & \cdots \supset \text{Sp}(2d) \times \text{O}(N) \supset \cdots |(N, \nu)_d c m'' \nu \delta jkl \rangle . \end{aligned}$$

We have used the labels in parentheses to specify which group-subgroup chain is being used. They are essentially labels constituting the uds-irrep of the intermediate symplectic group, i.e.,  $(k, \kappa)_n \rightarrow \langle k/2(\kappa) \rangle_n$ , or equivalently the rank and irrep label of the intermediate orthogonal group, i.e.,  $(k, \kappa)_n \rightarrow [\kappa]_n$ . Since  $\text{Sp}(2) \times \text{O}(N) \times \text{O}(d)$  is a common subgroup the associated irrep labels must range over the same partition values consistent with the following constraints;  $\bar{\nu}_1 \leq \min(N, d)$  and  $\bar{\nu}_1 + \bar{\nu}_2 \leq N$ ,  $\bar{\delta}_1 \leq \min(N, d)$  and  $\bar{\delta}_1 + \bar{\delta}_2 \leq d$ , and finally  $m' - w(\delta)$  and  $m'' - w(\nu)$  both non-negative even integers. Furthermore, a comparison of the three states yields a correlation between multiplicities

$$m(m, \nu \delta) = m(\nu, m' \delta) = m(\delta, m'' \nu) , \quad (8)$$

from which we can conclude that the multiplicities of the *noncompact* symplectic-group chains depend on multiplicities of the *compact* orthogonal group chain.

Since simple Hamiltonians will often be expressible in terms of operators of the  $\text{Sp}(2Nd)$  algebra and its subalgebras, it is important to be able to determine the action of operators, or more specifically generators, on the basis states. The isotropic harmonic oscillator Hamiltonian  $H_{\text{osc}}$  is a Hermitian operator belonging to the  $\text{Sp}(2)$  algebra. We have seen in Sec. II that  $\text{Sp}(2)$  contains the compact subgroup  $\text{U}(1)$ , and it is easy to extract the  $\text{U}(1)$  content from each  $\text{Sp}(2)$  irrep [see Eq. (3) of Sec. II]. Hence in classifying the states we could use  $\{Nd/2 + m + 2j\}$  to index the  $\text{U}(1)$  irreps or, more simply,  $j$  since  $Nd$  and  $m$  are already explicitly given. If  $H_{\text{osc}}$  is taken as the generator of this  $\text{U}(1)$  group, then it is quite clear that the action of  $H_{\text{osc}}$  is diagonal in all of the three basis states since the irreps of  $\text{U}(1)$  are all one dimensional. Explicitly,

$$H_{\text{osc}} | \cdots m \nu \delta jkl \rangle = | \cdots m \nu \delta jkl \rangle \left\{ \frac{Nd}{2} + m + 2j \right\} \hbar \omega , \quad (9)$$

where  $\omega$  is the oscillator frequency and the energy eigenvalues of  $H_{\text{osc}}$  are given by  $\{Nd/2 + m + 2j\} \hbar \omega$ . The dots imply any of the three group chains discussed above.

As mentioned earlier, two-body interaction terms and nonparabolic confinement terms within model dot Hamiltonians can give rise to operators belonging to algebras other than  $\text{Sp}(2)$ . Consider the special situation of Hamiltonians belonging to the  $\text{Sp}(2) \times \text{O}(N) \times \text{O}(d)$  algebra. Generators from these three algebras act only on the basis labels  $j$ ,  $k$ , and  $l$  of the three different states, with  $\text{Sp}(2)$  generators acting on  $j$ ,  $\text{O}(N)$  on  $k$ , and  $\text{O}(d)$  on  $l$ . For example, if  $L$  is the generator of the total angular momentum  $\text{O}(d)$  algebra, it acts within the irrep space labeled by  $[\delta]_d$  with basis label  $k$ . Hence with the convention of summing over twice-repeated basis labels

$$L | \cdots m \nu \delta jkl \rangle = | \cdots m \nu \delta jkl \rangle [\delta](L)' , \quad (10)$$

where  $[\delta](L)$  is the matrix representation of  $L$  in this basis. It should be noted that  $\text{O}(d)$  contains  $\text{U}(1)$  as a subgroup and the action of any one Hermitian operator of  $\text{O}(d)$  can be diagonalized with respect to a basis of this  $\text{O}(d) \supset \text{U}(1)$  chain.

In practice it is difficult to evaluate matrix elements and diagonalize Hamiltonians with general two-body interaction potentials. In addition, there are a wide variety of additional perturbations to the parabolic confinement potential which could arise experimentally in the dot. For this reason we are presenting a simplifying algebraic approach in this paper. In particular we are going to investigate the effect of certain "algebraic" Hamiltonians, i.e., ones whose interaction potentials depend only on quadratic Casimir operators. Matrix elements, and hence energy levels, of such interaction potentials are especially simple to determine. Specifically, we will analyze here the Casimirs associated with the  $\text{Sp}(2) \times \text{O}(d) \times \text{O}(N)$  direct product group which, as discussed earlier in this section, is common to all three symplectic-group chains. We note that there is only one Casimir for each of the groups  $\text{Sp}(2)$  and  $\text{O}(2)$ , while for  $\text{O}(N)$  there are  $[N/2]$  generalized Casimirs.

We shall first review briefly the main features of Casimir operators. If  $Z_\gamma$  denotes the generators of a group then the quadratic Casimir is defined as

$$C^2 \equiv \sum_{\alpha\beta} g^{\alpha\beta} Z_\alpha Z_\beta , \quad (11)$$

where  $\sum g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha$ . Such Casimirs commute with all the elements of the Lie algebra, i.e.,

$$[C^2, Z_\gamma] = 0 . \quad (12)$$

In general there are additional operators that commute with the operators of the Lie algebra but which cannot be described by a general relation like Eq. (11). Following Wybourne,<sup>19</sup> for every semisimple Lie group there exists a set of functions  $I_n(Z_\alpha)$  of the form

$$I_n(Z_\alpha) = C_{\alpha_1 \beta_n}^{\beta_1} C_{\alpha_2 \beta_n}^{\beta_2} \cdots Z^{\alpha_1} Z^{\alpha_2} \cdots , \quad (13)$$

which commute with every operator of the group, i.e.,

$$[I_n, Z_\alpha] = 0 , \quad (14)$$

and whose eigenvalues characterize the irreducible representations. The minimum number of such operators is equal to the rank of the algebra. Although the Casimir operators commute with every operator of the group, there are cases where their eigenvalues are not sufficient to label the irreducible representations. In general, the functions  $I_n(Z_\alpha)$  are called the *generalized* Casimir operators.

As in Ref. 6 we will denote the coordinate and momentum components of the  $r$ th particle ( $r = 1, \dots, N$ ) by  $x_{ri}$  and  $p_{ri}$  ( $i = 1, \dots, d$ ), respectively, with the associated operators  $X_{ri}$  and  $K_{ri}$  obeying the Heisenberg algebra.<sup>6</sup> The contracted forms of the symplectic-group generators of interest are  $K \equiv K_{ri} K_{ri}$ ,  $Q \equiv X_{ri} X_{ri}$ ,  $L \equiv X_{r1} K_{r2} - X_{r2} K_{r1}$ , and  $T \equiv X_{ri} K_{ri} + K_{ri} X_{ri}$ , as in Ref. 6.

The quadratic Casimirs of  $O(k)$  are well known and are given by  $C^2(O_k) = \frac{1}{2}L \cdot L$  with eigenvalue  $\sum_i \mu_i(\mu_i + k - 2i)$  when acting on the  $O(k)$  irrep space labeled by  $[\mu]$ . For example,

$$C^2(O_d)|\dots m\nu\delta jkl\rangle = |\dots m\nu\delta jkl\rangle \times \left[ \sum_i \delta_i(\delta_i + d - 2i) \right], \quad (15)$$

$$C^2(O_N)|\dots m\nu\delta jkl\rangle = |\dots m\nu\delta jkl\rangle \times \left[ \sum_i \nu_i(\nu_i + N - 2i) \right], \quad (16)$$

and similarly  $C^2(O_{Nd})$  acting on irrep space  $[m]_{Nd}$  gives  $m(m + Nd - 2)$ . The  $Sp(2)$  Casimir is of the form

$$C^2(Sp_2) = \frac{1}{8}[\{Q, K\} - \frac{1}{2}T^2] \quad (17)$$

(see for example Ref. 19) but it can be shown that this Casimir is related to that of  $O(Nd)$ ,

$$C^2(Sp_2) = \frac{1}{4}C^2(O_{Nd}) + \frac{Nd}{4} \left[ \frac{Nd}{4} - 1 \right] I. \quad (18)$$

This means that  $C^2(Sp_2)$  acting on the  $\langle N(m) \rangle_2$  irrep space yields the eigenvalue

$$\frac{1}{8} \left[ m \left( m + \frac{Nd}{2} - 2 \right) + \frac{Nd}{2} \left( \frac{Nd}{2} - 2 \right) \right].$$

The interesting aspect of this is that it is quadratic in both  $N$  and  $m$ . We shall use these Casimir results in the following section.

#### IV. ENERGY EIGENVALUES OF ALGEBRAIC HAMILTONIANS

In this section we will focus our attention on two-dimensional quantum dots, i.e.,  $d=2$ . Our starting Hamiltonian will essentially be the  $N$ -particle Fock-Darwin Hamiltonian  $H_{FD}$  describing  $N$  noninteracting electrons in a two-dimensional isotropic harmonic oscillator in a homogeneous magnetic field with circular gauge  $A_{\pi} = \frac{1}{2}B_{ij}X_{rj}$  where  $B_{ij}$  is skew-symmetric with  $B_{12}$  equal to the magnetic-field strength. This is given by<sup>6</sup>

$$H_{FD} = \frac{\hbar^2}{2m}K + \frac{m\omega^2}{2}Q - \frac{\hbar\omega_c}{2}L, \quad (19)$$

where  $\omega = \sqrt{\omega_0^2 + \omega_c^2}/4$ . We will be considering perturbations to  $H_{FD}$  constructed from the previously introduced quadratic Casimirs. As  $H_{FD}$  belongs to the  $Sp(2) \times O(N) \times O(2)$  algebra,<sup>6</sup> it is not important at this stage which chain of groups, and hence basis states, we take. As a consequence we shall label our states as

$$|\dots m\nu\delta jkl\rangle, \quad (20)$$

where the dots indicate whichever chain of groups is desirable. As discussed in Sec. III,  $m$  is a non-negative integer,  $\nu$  is a two-part partition where the sum of the first two columns is less than  $N$ , and the partition denoted by  $\delta$  is either  $[M]$  (where  $M$  is a non-negative integer) or  $[1^2]$ .

The total angular momentum of the system is obtained from the  $O(2)$  group labeled by  $\delta$ . Recall that the standard  $O(2)$  irrep labels are one-row partitions;  $[M]$  with  $M \geq 1$ ,  $[0]$  and  $[0^*]$  [which is equivalent to  $[1^2]$  in  $O(2)$ ]. Furthermore, except for the one-dimensional irreps  $[0]$  and  $[0^*]$ , the irreps  $[M]$  are two-dimensional and hence each total angular-momentum quantum number  $M$  is doubly degenerate. The component of angular momentum can be found by the reduction to a  $U(1)$  subgroup given by

$$[M]_2 \downarrow \{0; M\}_1 + \{M; 0\}_1 = \{+M\}_1 + \{-M\}_1, \quad (21)$$

with the special cases

$$[0]_2 \downarrow \{0; 0\}_1 = \{0\}_1, \quad [0^*]_2 \downarrow \{0; 0\}_1 = \{0\}_1. \quad (22)$$

The  $U(1)$  labels distinguish clockwise from anticlockwise rotations.

To label the basis of the  $Sp(2)$  irrep spaces, we exploit the reduction to the subgroup  $U(1)$ ,

$$\langle N(m) \rangle_1 \downarrow \epsilon_1^N \cdot \{m \cdot D\}_1 = \sum_{j=0}^{\infty} \{N+m+2j\}_1. \quad (23)$$

In general for a given  $Sp(2)$  irrep  $\langle N(m) \rangle_1$ , the range of angular momentum will be bound by  $0 \leq M \leq m$  decreasing from  $m$  in steps of two, with the possibility of multiplicities equal to zero or greater.

As explained in the previous paper,<sup>6</sup> the presence of  $O(N)$  is important as this contains the subgroup  $S(N)$  and means that spin states of permutation symmetry conjugate to that of the spatial states can be coupled to produce totally antisymmetric states in accordance with the Pauli principle. However the spin states are restricted to permutational symmetry type  $[\sigma] = [\sigma_1, \sigma_2]$  with  $\sigma_1 + \sigma_2 = N$ . Only spatial permutational symmetry types  $[2^{\sigma_2}, 1^{\sigma_1 - \sigma_2}]$  are, therefore, permitted and as a consequence certain spatial states are absent. Although for simplicity we will essentially ignore these permutation considerations, in general the  $O(N) \supset S(N)$  reduction can be given in an  $N$ -explicit way using the form

$$[v]_N \downarrow \sum_{\rho} m(\nu, \rho) [N-r, \rho]_N, \quad (24)$$

where  $r = w(\rho)$  is the weight of  $(\rho)$ ,  $m(\nu, \rho)$  is the multiplicity of  $(\rho)$  in  $(\nu)$ , and the partitions  $(\rho)$  are obtained by a compact Schur function formula involving plethysms<sup>20</sup> as discussed in Ref. 6. We emphasize that this formula gives results that are valid for any  $N$ .

Using the above results the states under the group-subgroup chain

$$Sp(4N) \supset \dots \supset Sp(2) \times O(2) \times S(N) \supset U(1) \times U(1) \times E$$

(with  $E$  the identity group) can then be classified as

$$\left| \begin{array}{c} \langle s; 0 \rangle_{2N} \\ \dots \\ \langle N(m) \rangle_1 \times [M]_2 \times [v]_N \\ d \\ \{N+m+2j\}_1 \times \{\bar{M}\}_1 \times [N-r, \rho]_N \\ k \end{array} \right|, \quad (25)$$

where the dots signify the intermediate chain of groups,  $d=1, \dots, m(\nu, \rho)$  is the  $O(N) \supset S(N)$  branching label,  $k=1, \dots, |\rho|$  is the basis multiplicity label of the  $S(N)$  irrep  $[N-r, \rho]_N$  of dimension  $|\rho|$  with  $r=w(\rho)$ , and  $\bar{M}=\pm M$  determines the component of angular momentum. As discussed above in connection with the partition  $\delta_2$ ,  $[M]_2$  must be substituted by  $[1^2]_2$  to obtain the complete set of states.

Having established the state classification and determined the action of the Casimirs, we turn to the energy spectrum of the  $N$ -electron algebraic Hamiltonians. The eigenenergies of a single electron in a parabolic potential subject to a magnetic field (i.e., the single-electron Fock-Darwin Hamiltonian) are given by<sup>4</sup>

$$E(j, l) = (1 + |l| + 2j)\hbar\omega - \frac{1}{2}l\hbar\omega_c, \quad (26)$$

where  $l$  is any integer while  $j$  is a non-negative integer. The action of the noninteracting  $N$ -electron Hamiltonian  $H_{\text{FD}}$  on the above states gives the  $N$ -electron energies

$$E_{\text{FD}}(N, m, j, \bar{M}) = (N + m + 2j)\hbar\omega - \frac{1}{2}\bar{M}\hbar\omega_c, \quad (27)$$

which clearly depends on additional quantum numbers. The noninteracting  $N$ -electron contribution increases linearly with  $m$ . For electrons the positive angular-momentum states  $\{+M\}$  have lower energy with respect to the associated negative angular-momentum states  $\{-M\}$  of the same quantum numbers. The set of states with  $m=M$ ,  $\bar{M}=+M$ , and  $j=0$  corresponds to the so-called lowest Fock-Darwin level. In the high-field limit or zero confining potential regime,  $\omega \rightarrow \omega_c/2$ , the  $N$ -electron energies approach

$$E_{\text{FD}}(N, m, j, \bar{M}) \rightarrow \frac{1}{2}(N + m + 2j - \bar{M})\hbar\omega_c. \quad (28)$$

Unlike the single-electron Landau levels there is no complete cancellation of the energy contributions coming from positive angular momentum for the  $N$ -electron states unless  $\bar{M}=M=m$ . In the low-field limit,  $\omega_c \rightarrow 0$ ,  $\omega \rightarrow \omega_0$  and hence the  $N$ -electron energies are given by

$$E_{\text{FD}}(N, m, j, \bar{M}) \rightarrow (N + m + 2j)\hbar\omega_0 \quad (29)$$

and become independent of angular-momentum values  $M$  for  $N > 1$ . Comparing with the well-known single-electron energy curves as a function of magnetic field, the form of the  $N$ -electron energy spectrum not surprisingly remains identical for any  $N$ . Essentially one needs only to make the substitutions of  $1 \rightarrow N$  and  $l \rightarrow m$  in the  $\hbar\omega$  term. The only differences are that each energy curve can now correspond to various values of  $N$ , and there are in addition many possible many-electron configurations for each  $N$ .

parameter perturbing Hamiltonian  $H_C$  which depends linearly on the quadratic Casimirs associated with the  $\text{Sp}(2) \times \text{O}(2) \times \text{O}(N)$  direct product group, as discussed in Sec. III. The perturbing Hamiltonian is a polynomial of  $X$  and  $P$  of order 4 and is given by

$$H_C = \hbar\omega_1 I + \hbar\omega_2 C^2(\text{Sp}_2) + \hbar\omega_3 C^2(\text{O}_2) + \hbar\omega_4 C^2(\text{O}_N). \quad (30)$$

Note that more complex terms can in fact be included,

such as product terms of the Casimirs which generate perturbations that are polynomials of  $X$  and  $P$  of order 8. The Casimirs comprising  $H_C$  correspond to anharmonic terms. For example, consider  $N$  electrons initially in the lowest Fock-Darwin level: the  $\text{O}(2)$  Casimir  $C^2(\text{O}_2)$  provides an energy perturbation that is basically proportional to the square of the sum of average moments of inertia of the individual electrons. [The average moment of inertia for a single electron of angular momentum  $m_r$  in the lowest Landau level is proportional to  $(m_r + 1)$ .] The total Hamiltonian  $H_{\text{FD}} + H_C$  is, therefore, an anharmonic (i.e., nonlinear)  $N$ -electron oscillator. Note that the perturbing Hamiltonian  $H_C$  includes product terms involving more than one particle index. Hence in addition to describing one-body perturbations (i.e., nonparabolicity) of the external parabolic confining potential seen by each electron, there are terms in  $H_C$  that will actually simulate two-body interactions between the electrons. Moreover, these electron-electron interaction terms are not in general translationally invariant. Such nontranslationally invariant interactions can actually arise in experimental quantum dots due to the lack of translational invariance of the quantum dot structure in the two-dimensional  $x$ - $y$  plane, and the presence of image charges in the surrounding dielectrics. (This is in contrast with the situation for an infinite, translationally invariant two-dimensional electron gas.) To our knowledge, all theoretical work to date on quantum dots has assumed the electron-electron interaction to be translationally invariant.

Using the results of Sec. III we can easily write down the eigenvalues of Eq. (30),

$$\begin{aligned} E_C(N, m, \nu, M) = & \hbar\omega_1 + [(N + m)(N + m - 2) \\ & + N(N - 2)]\hbar\omega_2 + M^2\hbar\omega_3 \\ & + [\nu_1(\nu_1 + N - 2) \\ & + \nu_2(\nu_2 + N - 4)]\hbar\omega_4, \end{aligned} \quad (31)$$

where  $\nu_1$  and  $\nu_2$  are the parts<sup>6</sup> of the partition  $\nu$ . In the fields of nuclear, atomic, and molecular physics, the coefficients of the Casimirs ( $\omega_1, \omega_2$ , etc.) would be obtained by fitting the theoretical energy spectrum  $E_{\text{FD}} + E_C$  to available experimental vibrational-rotational spectra. Such a procedure can equally well be carried out for quantum dots.

## V. OPTICAL ABSORPTION AND ADDITION SPECTRA

### A. Infrared optical absorption

Here we consider the infrared optical-absorption spectrum for the  $N$ -electron quantum dot. First for the  $N$ -electron Fock-Darwin (i.e., unperturbed) Hamiltonian  $H_{\text{FD}}$ , we see that the possible absorption energies  $\Delta E_{\text{FD}} \equiv E_{\text{FD}}(\text{final}) - E_{\text{FD}}(\text{initial})$  are given by

$$\Delta E_{\text{FD}} = (\Delta m + 2\Delta j)\hbar\omega - \frac{1}{2}\Delta\bar{M}\hbar\omega_c, \quad (32)$$

which is independent of  $N$ . So far Eq. (32) allows for multiphoton absorption (i.e., arbitrary  $\Delta\bar{M}$ ). For single-photon processes, the incoming photon polarized in the

$xy$  plane can be considered as carrying either  $+1$  or  $-1$  of angular-momentum component along the  $z$  axis (i.e., either right- or left-handed circular polarization). From the constraints given earlier we see that  $\Delta\bar{M} = \pm 1$  implies that  $\Delta m$  is an odd integer. The possible transitions associated with  $\Delta m + 2\Delta j = 1$  and  $\Delta\bar{M} = \pm 1$  correspond to photon absorption at frequencies  $\omega \pm \frac{1}{2}\omega_c$ , in agreement with the generalized Kohn theorem.<sup>5,21</sup>

Now consider the effect on the absorption spectrum of the additional perturbation  $H_C$  introduced above. The  $N$ -electron energies are given by Eq. (31), yielding energy shifts  $\Delta E_C \equiv E_C(\text{final}) - E_C(\text{initial})$  of

$$\begin{aligned} \Delta E_C = & (2N + m' + m - 2)\Delta m \hbar\omega_2 + (M' + M)\Delta M \hbar\omega_3 \\ & + [\Delta v_1(v'_1 + v_1 + N - 2) + \Delta v_2(v'_2 + v_2 + N - 4)] \\ & \times \hbar\omega_4, \end{aligned} \quad (33)$$

where the primed (unprimed) variables correspond to the final (initial)  $N$ -electron state. The two absorption lines at  $\omega \pm \frac{1}{2}\omega_c$  for the  $N$ -electron Fock-Darwin Hamiltonian have been replaced by sets of lines located near (but not at)  $\omega \pm \frac{1}{2}\omega_c$ . Another way of viewing this is that the center-of-mass motion has become coupled to the relative motion. Again from the constraints given earlier, we note that if  $\Delta\bar{M} = \pm 1$  then this necessarily induces only odd-integer changes in both  $\Delta m$  and  $\Delta w(\nu) = w(\nu') - w(\nu)$ . [Recall  $w(\nu)$  is the weight of partition  $\nu$ .]

For simplicity we will only discuss low-order excitations  $\Delta m = \pm 1$  and  $\Delta w(\nu) = \pm 1$  associated with single photon absorption  $\Delta\bar{M} = \pm 1$ . First consider the case where both the initial and final states lie in the lowest Fock-Darwin level ( $m = M, \bar{M} = +M, j = 0$ );  $\Delta m = \Delta M = 1$  and, in addition,  $\Delta v_1 = \pm 1$  with  $\Delta v_2 = 0$  or vice versa. The Kohn theorem absorption line at  $\omega - \frac{1}{2}\omega_c$ , therefore, becomes replaced by a set of lines shifted from  $\omega - \frac{1}{2}\omega_c$  by

$$\begin{aligned} \Delta E_C = & (2N + 2M - 1)\hbar\omega_2 \\ & + (2M + 1)\hbar\omega_3 \pm (2v_i + N - 2i \pm 1)\hbar\omega_4. \end{aligned} \quad (34)$$

In particular, taking  $\Delta v_i \geq 0$  means that the single absorption line at  $\omega - \frac{1}{2}\omega_c$  has been replaced by two lines, as shown schematically in Fig. 1. (Note that the heights of the lines in Fig. 1 do not represent predicted intensities. The actual intensity of each line will depend on the dipole matrix element squared, which is not considered explicitly in this paper.) So far we have considered the case where the initial and final states lie in the lowest Fock-Darwin level. Relaxing this restriction is straightforward since Eq. (33) is independent of the  $j$  quantum label. In this case  $\Delta m = \Delta M = -1$  and  $\Delta j = 1$ ; the Kohn theorem absorption line at  $\omega + \frac{1}{2}\omega_c$  becomes replaced by a set of lines which are shifted from  $\omega + \frac{1}{2}\omega_c$  by

$$\begin{aligned} \Delta E_C = & -(2N + 2M - 3)\hbar\omega_2 \\ & - (2M - 1)\hbar\omega_3 \pm (2v_i + N - 2i \pm 1)\hbar\omega_4. \end{aligned} \quad (35)$$

For  $\Delta v_i \geq 0$  these two lines are also shown schematically in Fig. 1. Depending on the values of the parameters  $\omega_1$ ,

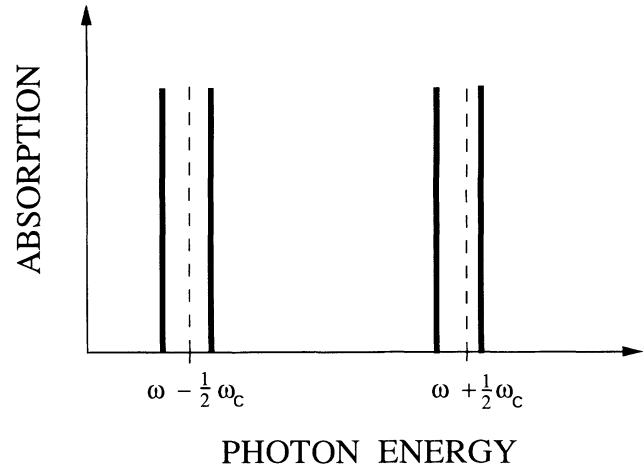


FIG. 1. Schematic infrared-absorption spectrum (solid lines) for  $N$  electrons in a quantum dot at finite magnetic field. The quantum dot Hamiltonian is described in the text. Dashed lines are the Kohn theorem absorption frequencies corresponding to  $N$  electrons in a perfectly parabolic quantum dot. The height of the absorption lines is not related to predicted intensities.

$\omega_2, \omega_3$ , the particle number  $N$ , etc., either or both of the two new lines (solid lines in Fig. 1) can be above or below the corresponding Kohn theorem frequency (dashed line in Fig. 1). This finding of two absorption lines near each Kohn theorem frequency is actually qualitatively consistent with recent experimental absorption spectra obtained from nearly parabolic quantum dots (e.g., see Fig. 6 of Ref. 1).

An interesting case of the above analysis concerns the infrared optical absorption from fractional quantum Hall states in quantum dots. The infrared-absorption frequencies of such states in a parabolic quantum dot are, by the generalized Kohn theorem, identical to those of a noninteracting electron gas and are independent of filling fraction. This will not be the case however in a nonparabolic dot. Assuming the electrons to be spin polarized for filling fractions  $1/n$  where  $n = 3, 5, \dots$ , only spatially antisymmetric states can occur. The partition  $\nu$  becomes a one-row partition given by  $\nu = (nM_0)$  where  $M_0 = \frac{1}{2}N(N-1)$ ,<sup>22</sup> and  $m = M = nM_0$ . The frequencies of the set of lines near each Kohn theorem absorption frequency given above, therefore, depend explicitly on  $n$  and hence on the filling fraction  $1/n$ .

Since the single absorption lines predicted by the generalized Kohn theorem only occur for an exactly parabolic potential, satellite peaks are obviously the rule rather than the exception. For a quantum dot at finite temperature, the absorption spectrum will consist of all possible transition lines (i.e., satellite peaks) due to the many available initial states, weighted by the Boltzmann factor describing the probability of occupancy of the initial state.

## B. Transport addition spectra

Transport measurements through quantum dots have been attracting much attention recently (see, for example,

Ref. 3). The series of conductance peaks found experimentally as a function of applied gate and source-drain bias voltages show an extremely rich structure. Deducing the particular number  $N$  of electrons on the dot from a given set of conductance peaks is proving difficult since the experiments often operate with  $N$  of the order of tens whereas theoretical calculations are greatly limited by numerical complexity to treating  $N \leq 6$ .

The previously obtained algebraic expressions for the electron energies of an  $N$ -electron dot allow us to give analytic expressions for the spectrum of conductance peaks with applied voltage. Consider the tunneling at low source-drain bias voltage. As the gate voltage is varied, an extra electron can jump on to the  $N$ -electron dot when the chemical potential (i.e., the energy of the incoming  $N+1$  electron) is equal to  $\Delta E_{N \rightarrow N+1} = E(N+1) - E(N)$ , where  $E(N+1)$  and  $E(N)$  are the ground-state energies of the  $N+1$  and  $N$ -electron dot, respectively. The  $N$  dependence of  $\Delta E_{N \rightarrow N+1}$  yields the so-called ‘‘addition spectrum.’’ As an illustration, we consider the case where the  $N$  and  $N+1$  electron dot ground states lie in the lowest Fock-Darwin level (i.e.,  $m = M, j = 0$ ) and also correspond to definite filling fractions  $1/n$  [i.e.,  $M = nM_0$ , where  $M_0 = \frac{1}{2}N(N-1)$ ]. It can easily be shown from the energy expressions in Eqs. (27) and (31) that

$$\begin{aligned} \Delta E_{N \rightarrow N+1} = & N^3 n^2 \hbar(\omega_2 + \omega_3 + \omega_4) + N^2 n \hbar(\omega_2 + \frac{3}{2}\omega_4) \\ & + N \hbar[n(\omega - \frac{1}{2}\omega_c + \omega_2 - \frac{3}{2}\omega_4) + 4\omega_4] - 2\hbar\omega_2. \end{aligned} \quad (36)$$

For a perfect parabolic quantum dot without interactions (i.e.,  $\omega_2, \omega_3$ , and  $\omega_4$  all equal to zero) Eq. (36) reduces to  $\Delta E_{N \rightarrow N+1} = Nn\hbar(\omega - \frac{1}{2}\omega_c)$  as expected.

Figure 2 shows the resulting addition spectrum  $\Delta E_{N \rightarrow N+1}$  as a function of  $N$  at the one-third filling fraction (i.e.,  $n = 3$ ). The cubic dependence of Eq. (36) is in contrast with the perfect parabolic, noninteracting system where  $\Delta E_{N \rightarrow N+1}$  increases linearly with  $N$ . Such a cubic dependence suggests  $\Delta E_{N \rightarrow N+1}$  could even have a maximum as a function of  $N$ , which seems to be in contradiction with simple charging ideas where the energy to add an extra electron is always positive. However, we note that we have not included the full effects of the electron-electron Coulomb interaction in our  $N$ -electron energies. A (large) classical charging term that increases

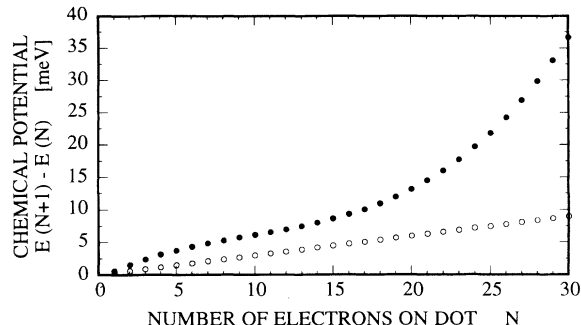


FIG. 2. Theoretical addition spectrum  $\Delta E_{N \rightarrow N+1} = E(N+1) - E(N)$  for transport through a quantum dot containing  $N$  electrons at filling fraction  $\frac{1}{3}$  (i.e.,  $n = 3$ ). The quantum dot Hamiltonian is described in the text. Parameters for the solid circles are  $\hbar\omega_2 = 2.66 \times 10^{-1}$  meV,  $\hbar\omega_3 = -7.07 \times 10^{-2}$  meV, and  $\hbar\omega_4 = -1.95 \times 10^{-1}$  meV. The dot confinement potential  $\hbar\omega_0 = 1.00$  meV and  $\hbar\omega_c = 10.0$  meV (i.e.,  $B = 6$  T for GaAs). As a comparison, the open circles correspond to the parameters  $\omega_2, \omega_3$ , and  $\omega_4$  being set to zero.

linearly with  $N$  could be added to the values  $\Delta E_{N \rightarrow N+1}$  in Fig. 2, restoring a monotonic  $N$  dependence which deviates only slightly from a linear form by the cubic equation Eq. (36).

## VI. CONCLUSIONS

In conclusion, we have presented an analytic approach to the problem of calculating the many-body energy spectrum of  $N$ -interacting electrons in a nonparabolic quantum dot. This approach can be used to describe perturbations to the noninteracting  $N$ -electron gas in a parabolic confining potential without resorting to computationally intensive perturbation theory or matrix diagonalization. Future work will explore both the parameter space of such algebraic Hamiltonians and higher-order Casimir terms in a similar way to work carried out in molecular and nuclear physics, with a view to fitting existing experimental spectra.

## ACKNOWLEDGMENTS

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