Nonlinear excitations in a Hamiltonian spin-field model in 2 + 1 dimensions

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A spin-field model in two space and one time dimensions is proposed and investigated using a bilinearization technique. This model, which can be considered as a modified version of the Ishimori system, allows a Hamiltonian formulation, a symmetry algebra of the Kac-Moody type with a loop-algebra structure, and the conformal invariance property. Many nonlinear excitations are found, which turn out to be of the helical and roton type, meronlike configurations endowed with a fractional topological charge, radially symmetric solutions, and domain walls. Our results are compared with those of the study of the O(3) nonlinear σ model and the Ishimori system. Finally, the physical role of both our model and the Ishimori system is discussed in the light of certain reduced equations which find applications in the theory of vortex filaments in liquid ^{4}He and in dealing with the two spin-correlation functions of the two-dimensional Ising model in the scaling limit.

I. INTRODUCTION

An important feature of certain nonlinear field equations is the possibility of finding exact configurations (instantons, vortex soliton, merons, domain walls, periodic excitations) which are of interest both in particle and condensed matter physics.[1,2] The topics described by this kind of model run from, say, the problem of quark confinement to the theory of superconductivity. In this context, a basic role is played by the O(3) nonlinear σ model in 2 + 1 dimensions, which presents many analogies with a four-dimensional nonAbelian gauge theory.[3,4] A nonrelativistic dynamical version of this model is the Ishimori system,[5] which has been investigated in recent years by several authors within different frameworks.[6] The Ishimori model is the first example of the integrable spin-one models on the plane. It affords exact solutions classified by an integer topological charge (localized solitons,[7] vortexlike,[5] closed stringlike, and doubly periodic solutions[8]). However, to the best of our knowledge, so far a Hamiltonian (or a Lagrangian) formulation has not been found for this system.

In this paper we study a modified version of the original Ishimori model endowed with a Hamiltonian structure. The model contains two parameters, α^2 and κ^2 (see Sec. II), which correspond, respectively, to the Euclidean ($\alpha^2 = 1$) and the pseudo-Euclidean ($\alpha^2 = -1$) metric, and to the compact ($\kappa^2 = 1$) and the noncompact ($\kappa^2 = -1$) symmetry.

Similarly to what happens for a few integrable nonlinear field equations in 2 + 1 dimensions,[9–11] our model admits an infinite dimensional symmetry algebra of the Kac-Moody type with a loop-algebra structure.[12] Furthermore, we show that it has the conformal invariance property.

Exploiting a bilinearization technique developed by Hirota,[13] we get exact nonlinear configurations of physical significance, such as helicons, rotonlike excitations, solutions provided by a fractional topological charge (meronlike configurations), radially symmetric solutions expressed in terms of the Jacobi elliptic functions, and domain walls. In the Euclidean case ($\alpha^2 = 1$), our procedure allows us to determine the velocity of the excitations obtained. The possibility that our model could describe some phenomenological situation is explored. With this regard, it is noteworthy that for $\alpha^2 = 1$, $\kappa^2 = -1$, our spin system can be related to a reduced equation of the sinh-Gordon type, which describes negative-temperature configurations and is applied in the theory of vortex filaments in ^{4}He.[14]

Section II contains the description of the model and its Hamiltonian formulation. In Sec. III we cast the equations of the model in the Hirota form and show that these, together with the related Hamiltonian, enjoy the conformal invariance property. In Sec. IV we derive classes of exact nonlinear excitations, while in Sec. V some concluding remarks are made.

II. THE MODEL

First, let us consider the spin-field model

$$S_t = [1/(2i)][S, S_{xx} + \alpha^2 S_{yy}] + \beta^2 S_y \phi_x - S_x \phi_y + (\beta^2 + 1) S \phi_{xy}, \tag{2.1a}$$

$$\phi_{xx} + \alpha^2 \beta^2 \phi_{yy} = 2\alpha^2 \beta^2 Q, \tag{2.1b}$$

where $\alpha^2 = \pm 1$, $\beta^2 = \pm 1$, subscripts stand for partial derivatives, $S = S(x,y,t)$ is a 2×2 matrix defined by

$$S = \begin{pmatrix} S_3 & \kappa S_+^* \\ \kappa S_+ & -S_3 \end{pmatrix}, \tag{2.2}$$

$S_+ = S_1 + iS_2$, the asterisk means complex conjugation, $Q = \frac{1}{2}\{\mathrm{Tr}(i/2)S[S_x, S_y]\}$ is a conserved topological charge density, and $\phi = \phi(x,y,t)$ is a real scalar field. The quantities $S_j(x,y,t)$ $(j = 1,2,3)$ are real-valued components of a classical unit "spin" vector $\mathbf{S}(x,y,t)$ belonging to the two-dimensional sphere S^2 $(\kappa^2 = 1)$, or to the pseudosphere $S^{1,1}$ ($\kappa^2 = -1$), i.e.,

$$S_3^2 + \kappa^2(S_1^2 + S_2^2) = 1. \tag{2.3}$$

Equations (2.1) allow the Lax formulation

$$L_1 = \alpha\partial_y + S\partial_x, \tag{2.4a}$$

$$L_2 = \partial_t + 2iS\partial_x{}^2 + i(S_x + \alpha S_y S)\partial_x + (\alpha^3\beta^2 S\phi_x + \phi_y)\partial_x, \tag{2.4b}$$

where $[L_1, L_2] = 0$. At this point we observe that condition (2.3) implies

$$(\beta^2 + 1)\phi_{xy} = 0, \tag{2.5}$$

from which (i) $\beta^2 = -1$, where ϕ_{xy} may be different from zero, or (ii) $\beta^2 = 1$, which entails $\phi_{xy} = 0$; in other words, the auxiliary field can be written as

$$\phi(x, y, t) = \phi_1(x, t) + \phi_2(y, t). \tag{2.6}$$

We have that in case (i), Eqs. (2.1) become those describing the Ishimori model.[5] On the contrary, in case (ii) Eqs. (2.1) take the form

$$S_t = [1/(2i)][S, S_{xx} + \alpha^2 S_{yy}] + S_y\phi_x - S_x\phi_y, \tag{2.7a}$$

$$\phi_{xx} + \alpha^2\phi_{yy} = 2\alpha^2 Q. \tag{2.7b}$$

What can we say about the integrability of model (2.7) under the assumption $\phi_{xy} \neq 0$? At present we cannot answer this question exhaustively. Notwithstanding, by carrying out a group analysis of Eqs. (2.7), it turns out that this system has the same symmetry algebra (ignoring any possible annihilation of ϕ_{xy}) of that possessed by the Ishimori model.[12] Such an algebra, which is infinite dimensional, is of the Kac-Moody type with a loop-algebra structure.[15] The existence of symmetry algebras with this characteristic is a property shared with other integrable nonlinear field equations in 2 + 1 dimensions, such as, for instance, the Kadomtsev-Petviashvili equation,[9] the Davey-Stewartson equation,[10] and the three-wave resonant system.[11] Hence, the symmetry property exhibited by model (2.7) (with $\phi_{xy} = 0$ or $\phi_{xy} \neq 0$) offers two possible alternative interpretations: (a) the system (2.7) is integrable, and in this case its Lax pair has in general (for $\phi_{xy} \neq 0$) still to be found, or (b) model (2.7) (for $\phi_{xy} \neq 0$) represents a remarkable example of a (2 + 1)-dimensional nonintegrable system endowed with a Kac-Moody symmetry algebra with a loop-algebra structure. At present, this question remains unanswered.

In the following, we shall study model (2.7) for any value of ϕ_{xy}. This model is a constrained Hamiltonian system described by the Hamiltonian density

$$H = (1/2)\sum_{j=1}^{3}(S_{jx}^2 + \alpha^2 S_{jy}^2) + (1/4)(\alpha^2\phi_x^2 + \phi_y^2), \tag{2.8}$$

where

$$\phi_x = 2\alpha^2(qp_y - \Gamma_y),$$
$$\phi_y = -2(qp_x - \Gamma_x), \tag{2.9}$$

q and p are a pair of canonical variables defined by

$$q = -\arctan(S_2/S_1), \quad p = S_3, \tag{2.10}$$

and $\Gamma = \Gamma(x, y, t)$ is a differentiable function determined by the compatibility condition $\phi_{xy} = \phi_{yx}$, which yields

$$\Gamma_{xx} + \alpha^2\Gamma_{yy} = \partial_x(qp_x) + \alpha^2\partial_y(qp_y). \tag{2.11}$$

The quantities (2.9) satisfy Eq. (2.7b).

Then, Eqs. (2.7) can be written as

$$q_t = -\delta H/\delta p,$$
$$p_t = \delta H/\delta q. \tag{2.12}$$

We remark that for $\alpha^2 = 1$, Eq. (2.11) reads

$$\nabla \cdot \mathbf{v} = 0, \tag{2.13}$$

where

$$\mathbf{v} = \nabla\Gamma - q\nabla p. \tag{2.14}$$

Consequently, our model (for $\alpha^2 = 1$) can be regarded as an incompressible "spin fluid," in which the velocity is given by (2.14). This result is not surprising, because q and p are really a pair of Clebsch variables usually employed in hydrodynamical problems.[16]

III. THE HIROTA REPRESENTATION

In what follows, it is convenient to adopt the stereographic projection representation, that is

$$S_+ = 2\zeta/(1+\kappa^2|\zeta|^2), \quad S_3 = (1-\kappa^2|\zeta|^2)/(1+\kappa^2|\zeta|^2), \tag{3.1}$$

where $\zeta = \zeta(x, y, t)$ is an arbitrary differentiable complex function. Inserting (3.1) into (2.7), we obtain

$$i\zeta_t + \zeta_{xx} + \alpha^2\zeta_{yy} - 2\kappa^2(\zeta^*\zeta_x^2 + \alpha^2\zeta^*\zeta_y^2)/(1 + \kappa^2|\zeta|^2) - i(\zeta_y\phi_x - \zeta_x\phi_y) = 0, \tag{3.2a}$$

$$\phi_{xx} + \alpha^2\phi_{yy} = 4i\alpha^2\kappa^2(\zeta_y\zeta_x^* - \zeta_y^*\zeta_x)/(1 + \kappa^2|\zeta|^2)^2. \tag{3.2b}$$

In order to get exact field configurations for model (2.7), we shall cast Eqs. (3.2) into the Hirota form.[13] This can be done by putting $\zeta = g/f$, where $f = f(x, y, t)$ and $g = g(x, y, t)$ are two complex functions. Then, by using the operators defined by

$$D_x^j D_y^k D_t^m u(x,y,t)\cdot w(x,y,t)=(\partial_x-\partial_{x'})^j(\partial_y-\partial_{y'})^k(\partial_t-\partial_{t'})^m u(x,y,t)w(x',y',t')|_{x=x',y=y',t=t'}, \tag{3.3}$$

Eqs. (3.2) become

$$(|f|^2-\kappa^2|g|^2)(iD_t-D_x^2-\alpha^2 D_y)(f^*\cdot g)-f^*g(iD_t-D_x^2-\alpha^2 D_y^2)(f^*\cdot f-\kappa^2 g^*\cdot g)=0, \tag{3.4a}$$

$$\phi_{xx}+\alpha^2\phi_{yy}=[(4i\alpha^2\kappa^2)/\Delta^2][D_y(g\cdot f)D_x(g^*\cdot f^*)-D_y(g^*\cdot f^*)D_x(g\cdot f)], \tag{3.4b}$$

with $\Delta=|f|^2+\kappa^2|g|^2$. A special solution to Eq. (3.4b) is

$$\phi_x=-(2i\alpha^2/\Delta)D_y(f^*\cdot f+\kappa^2 g^*\cdot g),$$
$$\phi_y=(2i/\Delta)D_x(f^*\cdot f+\kappa^2 g^*\cdot g). \tag{3.5}$$

At this stage, let us assume that $f=f(z,z^*,t)$ and $g=g(z,z^*,t)$, where $z=x+iy$. Then, introducing the operators $\partial_z=(1/2)(\partial_x-i\partial_y)$, $\partial_{z^*}=(1/2)(\partial_x+i\partial_y)$, the compatibility condition $\phi_{xy}=\phi_{yx}$ coming from (3.5) implies the constraint

$$2(\alpha^2+1)\{(ff^*_{zz^*}+\kappa^2 gg^*_{zz^*}-\text{c.c.})\Delta-[(ff^*_z+\kappa^2 gg^*_z)(ff^*_{z^*}+\kappa^2 gg^*_{z^*})-\text{c.c.}]\}$$

$$-(\alpha^2-1)\{[f(f^*_{zz}+f^*_{z^*z^*})+\kappa^2 g(g^*_{zz}+g^*_{z^*z^*})-\text{c.c.}]\Delta+[(f^*f_{z^*}+\kappa^2 g^*g_{z^*})^2+(f^*f_z+\kappa^2 g^*g_z)^2-\text{c.c.}]\}=0. \tag{3.6}$$

On the other hand, Eq. (3.4a) provides

$$\begin{aligned}&(|f|^2-\kappa^2|g|^2)\{i(f^*_t g-f^*g_t)-2(1+\alpha^2)(gf^*_{zz^*}+f^*g_{zz^*}-f^*_z g_{z^*}-f^*_{z^*}g_z)-(1-\alpha^2)\\&\times[g(f^*_{zz}+f^*_{z^*z^*})+f^*(g_{zz}+g_{z^*z^*})-2f^*_z g_z-2f^*_{z^*}g_{z^*}]\}-f^*g\{i[ff^*_t-f^*f_t-\kappa^2(gg^*_t-g^*g_t)]\\&-2(1+\alpha^2)[ff^*_{zz^*}+f^*f_{zz^*}-f^*{}_zf_{z^*}-f^*_{z^*}f_z-\kappa^2(gg^*_{zz^*}+g^*g_{zz^*}-g^*{}_zg_{z^*}-g^*_{z^*}g_z)]\\&-(1-\alpha^2)[f(f^*_{zz}+f^*_{z^*z^*})-2f^*{}_zf_z-2f^*_{z^*}f_{z^*}+f^*(f_{zz}+f_{z^*z^*})\\&-\kappa^2(gg^*_{zz}+gg^*_{z^*z^*}-2g^*_z g_z-2g^*_{z^*}g_{z^*}+g^*g_{zz}+g^*g_{z^*z^*})]\}=0.\end{aligned} \tag{3.7}$$

Now, in order to establish the conformal invariance property of the model (2.7), it is convenient to consider Eq. (3.2b). This is satisfied by

$$\phi_x=-4i\alpha^2\kappa^2[\zeta\zeta^*_y/(1+\kappa^2|\zeta|^2)+\Lambda_y],$$
$$\phi_y=4i\kappa^2[\zeta\zeta^*_x/(1+\kappa^2|\zeta|^2)+\Lambda_x], \tag{3.8}$$

where $\Lambda=\Lambda(x,y,t)$ is a differentiable function which is connected with $\Gamma(x,y,t)$ given by (2.11) through the relations

$$\Lambda_x=\frac{\kappa^2}{2}\{p/(1+p)+i[qp_x-(1-p)q_x-\Gamma_x]\},$$
$$\Lambda_y=\frac{\kappa^2}{2}\{p/(1+p)+i[qp_y-(1-p)q_y-\Gamma_y]\}. \tag{3.9}$$

This can be easily seen by putting $\zeta=[\kappa^2(1-p)/(1+p)]^{1/2}\exp(-iq)$ into Eqs. (3.8) and taking account of (2.9). An explicit expression for Λ can be obtained by comparing the quantities (3.8) with (3.5). We get

$$\Lambda=(1/2)\kappa^2\ln\left(\frac{\Lambda_0 f^*}{f(1+\kappa^2|\zeta|^2)}\right), \tag{3.10}$$

Λ_0 being an arbitrary function of time.

In complex form, the Hamiltonian density (2.8) reads

$$\begin{aligned}H=\ &(1-\alpha^2)\{2\kappa^2(1+\kappa^2|\zeta|^2)^{-2}(\zeta_z\zeta^*_z+\zeta_{z^*}\zeta^*_{z^*})-4[(\zeta\zeta^*_z)/(1+\kappa^2|\zeta|^2)+\Lambda_z]^2-4[(\zeta\zeta^*_{z^*})/(1+\kappa^2|\zeta|^2)+\Lambda_{z^*}]^2\}\\&+(1+\alpha^2)\{2\kappa^2(1+\kappa^2|\zeta|^2)^{-2}(\zeta_z\zeta^*_{z^*}+\zeta_{z^*}\zeta^*_z)-8[(\zeta\zeta^*_z)/(1+\kappa^2|\zeta|^2)+\Lambda_z][(\zeta\zeta^*_{z^*})/(1+\kappa^2|\zeta|^2)+\Lambda_{z^*}]\}.\end{aligned} \tag{3.11}$$

A direct calculation shows that (3.11) is invariant under the conformal transformation

$$\zeta=\frac{a\zeta'+b}{-\kappa^2 b^*\zeta'+a^*} \tag{3.12a}$$

and

$$\Lambda=\Lambda'+\kappa^2\ln(a-\kappa^2 b\zeta^{*\prime}), \tag{3.12b}$$

where a,b are arbitrary complex constants such that

$|a|^2 + \kappa^2|b|^2 = 1$. Equations (3.12) reflect the SU(2) ($\kappa^2 = 1$) and the SU(1,1) ($\kappa^2 = -1$) symmetry, respectively. Furthermore, we can see straightforwardly that (3.10) is related to the Liouville equation

$$\partial_{\zeta\zeta^*}\rho = \exp(2\rho), \tag{3.13}$$

where $\rho = 2(\kappa^2\Lambda + i\theta + \rho_0)$, with $\theta = \arg f$ and $\Lambda_0{}^2 = -\exp(-2\rho_0)$. This notable property reveals the conformal invariant nature of the model (2.7).[17]

IV. NONLINEAR EXCITATIONS

Explicit field configurations of model (2.7) can be obtained exploiting Eqs. (3.6) and (3.7). To be definite, here we limit ourselves to consider the case $\alpha^2 = 1$, $\kappa^2 = 1$. We shall deal with only a few examples of special nonlinear excitations of physical significance. An easy way to scrutinize Eqs. (3.6) and (3.7) is to look for static field configurations of the form

$$f = f_0 \exp F(z, z^*, t), \quad g = g_0 \exp G(z, z^*, t), \tag{4.1}$$

or

$$f = A(z) + B^*(z^*), \quad g = C(z) + D^*(z^*), \tag{4.2}$$

where f_0 and g_0 are constants. These simple *Ansätze* lead to interesting exact solutions (e.g., helicons, rotons, meronlike configurations, and radially symmetric solutions) which are pertinent to planar magnetic systems.[18]

A. Helical-type configurations (helicons)

Keeping in mind (4.1), let us choose $f_0 = A_0$, $g_0 = -A_0$, $F = az^N + (a^* - iA_0)z^{*N}$, $G = F^*$, where A_0, N, and a are arbitrary constants (the first two are real, while the latter is complex). Then Eqs. (3.6) and (3.8) are identically satisfied. The spin components read [see (3.1)]

$$S_1 = -\cos[A_0(z^N + z^{*N})], \quad S_2 = -\sin[A_0(z^N + z^{*N})], \quad S_3 = 0. \tag{4.3}$$

We point out that for $N = 1$, (4.3) reproduces just the static helical configuration arising in helimagnets,[19] i.e.,

$$S_1 = -\cos 2A_0x, \quad S_2 = -\sin 2A_0x, \quad S_3 = 0. \tag{4.4}$$

On the other hand, for $N \neq 1$ the quantities (4.3) furnish static configurations of the model (2.7) which may be regarded as a generalization of the helical structure (4.4). As far as we know, these solutions are new and their possible physical interpretation has to be explored.

The energy density related to (4.3) is given by

$$H = 2A_0^2N^2r^{2N-2}, \tag{4.5}$$

with $z = r\exp(i\theta)$, while the topological charge density vanishes because $\phi_z = \phi_{z^*} = 0$. We notice also that Eqs. (2.7) allow a dynamical helical configuration which is obtained by putting $f_0 = -g_0 = A_0$, $F = i(z + z^* + c_0t + \theta_0)$, $G = i[(1 + A_0)(z + z^*) + \gamma_0t + \delta_0]$ into (4.1), where $c_0 = 2A_0^2 - 8$, $\gamma_0 = -2(A_0 + 2)(3A_0 + 2)$, and A_0, θ_0, δ_0 are real constants. The associated spin-field configuration is

$$S_1 = -\cos[2A_0x + (\gamma_0 - c_0)t + \beta_0],$$
$$S_2 = -\sin[2A_0x + (\gamma_0 - c_0)t + \beta_0], \quad S_3 = 0, \tag{4.6}$$

where β_0 is a constant.

The energy density is

$$H = H_M + H_\phi = 2A_0^2 + 4(A_0 + 2)^2, \tag{4.7}$$

where H_M and H_ϕ denote, respectively, the first and the second terms in Eq. (2.8), while the topological charge density turns out to be zero. This is due to the fact that $\phi_x = 0$, $\phi_y = 4(A_0 + 2)$. Furthermore, since for $A_0 = -2$ we have $\gamma_0 = c_0 = 0$, in this case the quantities (4.6) reproduce the static solution (4.4), whose density energy is consistent with that emerging from (4.7). In other words, the presence of a nonconstant auxiliary field, which depends linearly on the variable y (for $A_0 \neq -2$), corresponds to switching on a time dependence in the helical structure (4.4).

B. Rotonlike configurations

Looking at (4.2), let us set $A = -D = -bz^N$, $B = C = az^N$, where a, b are complex constants and N is a real number. This choice is consistent with Eqs. (3.6) and (3.7). The components of the spin field are

$$S_1 = \frac{(a^2 - b^2)z^{2N} + (a^{*2} - b^{*2})z^{*2N}}{\Delta|z|^{2N}},$$

$$S_2 = \frac{-i[(a^2 + b^2)z^{2N} - (a^{*2} + b^{*2})z^{*2N}]}{\Delta|z|^{2N}},$$

$$S_3 = \frac{-2[abz^{2N} + a^*b^*z^{*2N}]}{\Delta|z|^{2N}}, \tag{4.8}$$

where $\Delta = 2(|a|^2 + |b|^2)$. For $a = 1$, $b = i$, and using polar coordinates, (4.8) takes the simple form $\mathbf{S} = (\cos N\theta, 0, \sin N\theta)$, which resembles the type of solutions admitted by the plane rotator model.[20] The topological charge density associated with (4.8) vanishes, while the Hamiltonian density is given by $H = 2N^2/r^2$. The auxiliary field ϕ is such that $\phi_x = \phi_y = 0$.

C. Meronlike configurations

An interesting class of nonlinear excitations are obtained by (4.2) with $A = -D = -bz^{-N/2}$, $B = C = az^{N/2}$, where N is a positive integer and a, b are two complex constants such that $ab = a^*b^*$. A subclass of these configurations possesses one half unit of topological charge. First let us discuss the solution arising for $N=1$. Putting for simplicity $a=-b=1$, we get

$$\zeta=(z/z^*)^{1/2}\frac{|z|-1}{|z|+1}, \tag{4.9}$$

which corresponds to the spin components

$$S_1=\frac{r^2-1}{r^2+1}\cos\theta,$$

$$S_2=\frac{r^2-1}{r^2+1}\sin\theta,$$

$$S_3=\frac{2r}{r^2+1}, \tag{4.10}$$

in polar coordinates. The auxiliary field ϕ can be derived from (3.5). We have

$$\phi_x=\frac{4}{1+r^2}\cos\theta,$$

$$\phi_y=\frac{4}{1+r^2}\sin\theta, \tag{4.11}$$

which yields

$$\phi=4\arctan r, \tag{4.12}$$

apart from a constant of integration. As $r\to\infty$, the configuration (4.10) satisfies the boundary condition

$$S_1=\cos\theta,\quad S_2=\sin\theta,\quad S_3=0, \tag{4.13}$$

which represents a solution of the planar rotator model.[20] The contributions to the energy density H coming from the magnetic part and the field ϕ are $H_M=\frac{1}{2r^2}$ and $H_\phi=\frac{4}{(1+r^2)^2}$, respectively. On the other hand, the topological charge density is given by

$$Q=2\frac{1-r^2}{r(1+r^2)^2}. \tag{4.14}$$

The total energy diverges logarithmically, while the total topological charge vanishes, i.e.,

$$\begin{aligned}Q_T&=[1/(4\pi)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q\,dx\,dy\\&=\int_0^{\infty}\frac{1-r^2}{(1+r^2)^2}\,dr=0.\end{aligned} \tag{4.15}$$

Starting from (4.10), we can build up a static solution $\mathbf{T}$ to the model (2.7) endowed with a fractional topological charge $Q_T=+1/2$, namely,

$$\mathbf{T}=\sigma(1-r)\mathbf{S}+\sigma(r-1)\mathbf{S}_0, \tag{4.16}$$

where σ denotes the step function, $\mathbf{S}$ is given by (4.10), and $\mathbf{S}_0\equiv(0,0,1)$. A configuration having an opposite topological charge, $Q_T=-1/2$, is

$$\mathbf{T}'=\sigma(r-1)\mathbf{S}+\sigma(1-r)\mathbf{S}_0. \tag{4.17}$$

Now some comments are in order. The above situation bears some analogies with those offered by other field configurations with $Q_T=\pm1/2$ (merons and torons), discovered in the two-dimensional O(3) nonlinear σ model and in the four-dimensional non-Abelian gauge theory.[3,4,21] We recall that merons are localized field configurations which possess one half unit of topological charge (at zero and at infinity) and have logarithmically divergent action. They play a basic role in quark confinement. To be precise, let us limit ourselves to consider the O(3) nonlinear σ model. Then the singular meron solution takes the form[3]

$$\zeta=(z/z^*)^{1/2}, \tag{4.18}$$

which is singular at $z=0$ and at $z\to\infty$. This corresponds to a zero-size meron at $z=0$ and at infinity. The topological charge density Q associated with (4.18) vanishes identically. An arbitrary number of singular merons of the type (4.18) can be constructed as well. Anyway, these can be smeared out to get meron configurations which satisfy the equations of motion of the O(3) nonlinear σ model everywhere except on circles each of which contains one half unit of topological charge. Furthermore, merons can be regarded as point charges and the instantons in the O(3) nonlinear σ model can be thought as dipoles made up from meron pairs. Coming back to our model, we remark that (4.18) is a solution to Eqs. (3.2) also. But we have not been able to find instanton configurations by means of Hirota's technique. In other words, the choice $\zeta=z$, which leads to the simpler instanton configuration in the O(3) nonlinear σ model, now does not work, in the sense that Eqs. (3.2) are not satisfied. This fact excludes the possibility of applying the procedure followed in Ref. 3 to smear out the singular meron (4.18) or the singular configuration of the meron type (3.9). Notwithstanding, resorting to (4.9), we can build up solutions to the model (2.7), such as (4.16) and (4.17), provided by a fractional topological charge. Unlike what happens for the O(3) nonlinear σ model, we cannot interpret our fractional charge configurations as poles from which the instanton dipole is formed. With regard to this point, keeping in mind solutions (4.10) and (4.12) we observe that the field ϕ contributes to the total energy by a finite term. In other words, ϕ presents a kink (instantonlike) behavior (it obeys the sine-Gordon equation $\phi_{uu}=\sin\phi$), in opposition to the spin components (4.10).

Multiple meronlike configurations can be constructed starting from

$$\zeta=(z/z^*)^{N/2}\frac{|z|^N-1}{|z|^N+1}, \tag{4.19}$$

where N is a positive integer. In polar coordinates, the spin components read

$$S_1=\frac{r^{2N}-1}{r^{2N}+1}\cos N\theta,$$

$$S_2 = \frac{r^{2N}-1}{r^{2N}+1}\sin N\theta,$$

$$S_3 = \frac{2r^N}{1+r^{2N}}, \tag{4.20}$$

while the auxiliary field ϕ can be derived by

$$\phi_x = \frac{4Nr^{N-1}}{1+r^{2N}}\cos\theta,$$

$$\phi_y = \frac{4Nr^{N-1}}{1+r^{2N}}\sin\theta. \tag{4.21}$$

The energy density and the topological charge density are given by $H = N^2/2r^2 + (4N^2r^{2N-2})/(1+r^{2N})^2$ and $Q = 2N^2(r^{2N-2} - r^{3N-2})/(1+r^{2N})^2$, respectively. Thus, the total energy diverges logarithmically and the total topological charge is zero. N-fractional charge configurations, with total topological charge $Q_T^N = \pm N/2$, can be easily written keeping in mind (4.16) and (4.17), where now the spin components (4.10) are replaced by (4.20).

D. Radially symmetric solutions

A class of radially symmetric solutions to model (2.7) can be found assuming that

$$\zeta = [a(z)/a^*(z^*)]^{1/2}\psi(|z|), \tag{4.22}$$

where $a(z)$ and $\psi(|z|)$ are, respectively, a complex and a real function to be determined in such a way that Eqs. (3.2) [or, equivalently, (3.6) and (3.7)] are satisfied. In doing so, it is convenient to put $g = [a(z)]^{1/2}\psi(|z|)$ and $f = [a^*(z^*)]^{1/2}$. Then, Eqs. (3.6) and (3.7) provide

$$(1+\psi^2)\left(\psi_{rr} + \frac{1}{r}\psi_r\right) + (1-\psi^2)|a_z/a|^2\psi = 2\psi{\psi_r}^2, \tag{4.23}$$

where $a(z) = cz^N$, $r = |z|$, c is an arbitrary complex constant, and N is a real number. Performing the change of variables

$$\psi = \tan\frac{\gamma}{4},$$

$$u = \ln r, \tag{4.24}$$

Eq. (4.23) transforms into the equation for the pendulum, that is

$$\gamma_{uu} + N^2\sin\gamma = 0. \tag{4.25}$$

Equation (4.25) affords the solution

$$\gamma = 2\arcsin[k\,\mathrm{sn}(Nu,k)], \tag{4.26}$$

where sn( ) is a Jacobian elliptic function of modulus k $(0 \le k \le 1)$.

By virtue of (4.24) and (4.26), and choosing $c = 1$ for simplicity, from (4.22) we get

$$\zeta = (z/z^*)^{N/2}\left(\frac{1-\mathrm{dn}(Nu,k)}{1+\mathrm{dn}(Nu,k)}\right)^{1/2}, \tag{4.27}$$

where the identity $\mathrm{dn}^2(\,) - 1 = k^2\mathrm{sn}^2(\,)$ has been used. Substitution from (4.27) into (3.1) yields

$$S_1 = k\,\mathrm{sn}(Nu,k)\cos N\theta, \quad S_2 = k\,\mathrm{sn}(Nu,k)\sin N\theta,$$
$$S_3 = \mathrm{dn}(Nu,k). \tag{4.28}$$

These quantities represent a radially symmetric solution to the model (2.7) written in terms of the Jacobi elliptic functions.[22] For $k = 1$, the solutions (4.28) reduce to the meronlike configurations (4.20), while for $k = 0$ they take the values $S_1 = 0, S_2 = 0, S_3 = 1$. The auxiliary field ϕ corresponding to (4.27) (with $N = 1$) can be derived from (3.5) by taking $g = \{[1-\mathrm{dn}(u,k)]/z^*\}^{1/2}$ and $f = \{[1+\mathrm{dn}(u,k)]/z\}^{1/2}$. We find

$$\phi_x = (2/r)\mathrm{dn}(u,k)\cos\theta, \ \phi_y = (2/r)\mathrm{dn}(u,k)\sin\theta, \tag{4.29}$$

from which

$$\phi = 2\arcsin[\mathrm{sn}(u,k)] + \pi, \tag{4.30}$$

where the constant of integration has been chosen equal to π just to reproduce (4.11). The topological charge density induced by (4.29) is

$$Q = -\frac{k^2}{r^2}\mathrm{sn}(u,k)\mathrm{cn}(u,k) = \frac{1}{r^2}\frac{d}{du}\mathrm{dn}(u,k), \tag{4.31}$$

which entails a vanishing total topological charge $Q_T = 0$. Concerning the energy density, we obtain

$$H = H_M + H_\phi = k^2/(2r^2) + (1/r^2)\mathrm{dn}^2(u,k). \tag{4.32}$$

Therefore, the contribution of H_M to the total energy diverges logarithmically. This feature is the same which occurs for $k = 1$.

At this stage, it should be instructive to compare the configurations (4.28) with those found by Takeno and Homma in the study of the three-dimensional continuous Heisenberg model for the case of cylindrical symmetry.[23] These authors parametrize the spin-field components, s_j, by two angles of rotation β and θ as follows:

$$s_1 = \sin\beta\cos\theta, \ s_2 = \sin\beta\sin\theta, \quad s_3 = \cos\beta, \tag{4.33}$$

and investigate stationary solutions to the equations of motion arising from the continuum limit of the discrete model. The equations under consideration are

$$2\cos\beta\nabla\beta\cdot\nabla\theta + \sin\beta\nabla^2\theta = 0, \tag{4.34a}$$

$$\nabla^2\beta - \sin\beta\cos\beta(\nabla\theta)^2 = 0, \tag{4.34b}$$

$$(1+|\zeta|^2)\nabla^2\zeta = 2\zeta^*(\nabla\zeta)^2, \tag{4.34c}$$

where the stereographic variable

$$\zeta = \tan(\beta/2)e^{i\theta} \tag{4.35}$$

can be exploited to express the quantities s_j in the same way as (3.1) (with $\kappa^2 = 1$).

Looking for cylindrical symmetric solutions to Eqs. (4.34), and requiring that θ obeys the Laplace equation, one has that β satisfies the sine-Gordon equation

$$\beta_{uu} - n^2\sin\beta\cos\beta = 0, \tag{4.36}$$

rather than the pendulum equation (4.25), where $n = \pm1, \pm2, \ldots$, and $r = (x^2 + y^2)^{1/2}$. This equation is solved by $\cos\beta = k\,\mathrm{sn}(nu, k)$; then, the corresponding spin-field components s_j of the continuous Heisenberg model take the form (for $n = 1$)

$$s_1 = \mathrm{dn}(u,k)\cos\theta, \quad s_2 = \mathrm{dn}(u,k)\sin\theta, \quad s_3 = k\,\mathrm{sn}(u,k). \tag{4.37}$$

In the limit $k \to 0$, the solution (4.37) reduces to that of the planar rotator model, while for $k \to 1$ it becomes the one-vortex string configuration

$$s_1 = \mathrm{sech}u\cos\theta, \quad s_2 = \mathrm{sech}u\sin\theta, \quad s_3 = \tanh u. \tag{4.38}$$

In the variable β, (4.38) corresponds to the solution of the kink-type

$$\beta = 2\arctan e^u. \tag{4.39}$$

As $r \to \infty$, (4.38) obeys the boundary condition $s_1 = 0$, $s_2 = 0$, $s_3 = 1$, rather than (4.13). The energy density for solution (4.37) reads

$$H = 4/(r^2+1)^2. \tag{4.40}$$

Therefore, the total energy related to the one-vortex string (4.38) is finite. This is essentially due to the form of the kink-type of the solution (4.39). We notice also that the configurations (4.28) [and, for $k = 1$, Eqs. (4.10)] can be obtained formally from (4.37) [(4.38)] by interchanging $k\,\mathrm{sn}(u,k)$ with $\mathrm{dn}(u,k)$ ($\tanh u$ with $\mathrm{sech}u$). Furthermore, as we have already observed, the behavior of the auxiliary field ϕ (4.12) is that of a kink, just like β [see (4.39)]. Hence, it is worthwhile, but expected, that the energy density (4.40) coincides with the energy density H_ϕ arising from (4.11).

V. CONCLUSIONS

We have shown that model (2.7), which can be considered as a modified version of the Ishimori system, possesses many interesting properties which suggest possible connections with physical problems. Indeed, model (2.7) is endowed with a Hamiltonian, has a conformal invariant nature, and admits exact configurations which have been explicitly obtained for $\alpha^2 = 1$ and $\kappa^2 = 1$. These turn out to be helicons, rotonlike and meronlike excitations, and radially symmetric solutions. Furthermore, we recall that model (2.7) allows time-dependent domain wall configurations as well. These are found for $\alpha^2 = -1$ and $\kappa^2 = \pm1$. For example, the form of the domain wall corresponding to $\kappa^2 = 1$ is

$$S_1 = \cos(2a_1x + a_0)\mathrm{sech}\chi, \quad S_2 = \sin(2a_1x + a_0)\mathrm{sech}\chi, \\ S_3 = \tanh\chi, \tag{5.1}$$

where $\chi = 2a_1y + 8a_1a_2t + \alpha_0$, and a_0, a_1, a_2, α_0 are (real) constants. The auxiliary field ϕ is

$$\phi = 4a_2x - 2\ln\cosh\chi + \phi_0(t), \tag{5.2}$$

where $\phi_0(t)$ is an arbitrary function of integration. We observe that ϕ is of the form (2.6). Furthermore, for $a_2 = 0$ the spin configuration (5.1) coincides with the static domain wall solution to the continuous Heisenberg model in two spatial dimensions.[24] As we have seen via a group analysis carried out following the same procedure already applied in Ref. 12 for the Ishimori system, model (2.7) has a symmetry algebra of the Kac-Moody type with a loop-algebra structure. However, this important feature, which is partaken with other nonlinear field equations in 2+1 dimensions,[9–11] does not allow us to affirm that the model is generally (for $\phi_{xy} \neq 0$) integrable. The problem of the integrability of model (2.7) is open and deserves further investigation. One could ask what might be the physical meaning of model (2.7) and what is that of the Ishimori system. This question can be in part answered by making some reductions of the original equations. For the Ishimori model, a method based on the use of the symmetry algebra associated with the system leads to reduced equations of physical significance, such as, for instance, the isotropic Heisenberg model and the Landau-Lifshitz model with uniaxial anisotropy.[12] The role of the anisotropy parameter is played by the constant α^2 appearing in the Ishimori equations. Concerning our model, interesting possibilities of physical applications emerge from certain reduced equations derived through the Hirota technique. Precisely, limiting ourselves to the cases $\alpha^2 = 1$, $\kappa^2 = \pm1$, and using the Hirota formulation for both the models, we have started from the trial function (4.22) and have obtained $a(z) = cz^N$ for all the cases, where c is a complex constant and N is a real number. In dealing with model (2.7), when $\alpha^2 = 1$, $\kappa^2 = 1$ (compact case) we have already seen that the function $\psi(|z|)$ is linked to the pendulum equation (4.25). Conversely, when $\alpha^2 = 1$, $\kappa^2 = -1$ (noncompact case) the equation corresponding to (4.25) is of sinh-Gordon type, namely

$$\gamma_{uu} + N^2\sinh\gamma = 0, \tag{5.3}$$

where $\psi = \tanh(\gamma/4)$ ($u = \ln r$). Equation (5.3), which can be solved analytically, is relevant from a physical point of view because it may be employed to formally describe negative-temperature configurations. In fact, it can be considered as the one-dimensional version of the sinh-Poisson equation, which describes two-dimensional steady-state distributions of elements with a logarithmic interaction potential and finds application in the theory of vortex filaments in liquid ^{4}He.[14] On the other hand, as far as the Ishimori model is concerned, for $\alpha^2 = 1$, $\kappa^2 = 1$, form (4.22) leads to the equation

$$\gamma_{uu} + N^2 e^{4u} \sin\gamma = 0, \tag{5.4}$$

while for $\alpha^2 = 1$, $\kappa^2 = -1$ we get

$$\gamma_{uu} + N^2 e^{4u} \sinh\gamma = 0. \tag{5.5}$$

Both Eqs. (5.4) and (5.5) can be reduced to special cases of the third Painleve' transcendent,[25] which is connected with the two-spin correlation functions of the two-dimensional Ising model in the scaling limit.[26] This situation bears some analogies with that considered in Ref. 23 in relation to the axisymmetric solutions to Eqs. (4.34). To conclude, we point out that by means of Eq. (2.14) we are able to evaluate (for $\alpha^2 = 1$) the velocity of all the configurations determined by the Hirota technique.

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