Migdal theorem for the Millis-Monien-Pines model of high-temperature superconductivity

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The validity of the Migdal theorem for the two-dimensional electronic system interacting with the antiferromagnetic excitations with a relaxational dynamics has been investigated. We explicitly show that the first-order vertex correction for this interaction is small for the domains of small and high frequencies.

I. INTRODUCTION

The importance of the antiferromagnetic excitations in high-critical-temperature superconductivity (HTS) has been pointed out by Millis, Monien, and Pines (MMP)[1] in the explanation of NMR experiments. The main point of the MMP phenomenological model is the existence of the relaxational dynamics of the antiferromagnetic excitations, which have an imaginary part of the dynamical susceptibility $\chi(\mathbf{p},\omega)$ with the property that $\mathrm{Im}\chi(\mathbf{p},\omega)\sim\omega$ at all $\mathbf{p}$ measured from the zone corner $(\pi/a,\pi/a)$. The model has been successfully applied by Arfi[2] to calculate the normal optical conductivity for $YBa_2Cu_3O_{6+x}$ and by Brenig and Monien[3] for the Raman scattering theory. Using the plausible arguments given by Millis,[4] Monthoux, Balatski, and Pines[5] built a weak-coupling theory of HTS taking bosons that mediate the two-dimensional electronic quasiparticles and the antiferromagnetic paramagnons described by the dynamical susceptibility from the MMP model. On the other hand, Hertz, Levin, and Beal-Monod[6] have shown the absence of a Migdal theorem[7] for the electron paramagnon, which is essential for obtaining the Eliashberg equations, providing the justification for the BCS theory. However, we have to mention that in the MMP model the antiferromagnetic correlations have a relaxational dynamics which makes the problem more complicated than in the case of Berk-Schrieffer[8] paramagnons. In this paper we analyze the validity of the Migdal theorem for a two-dimensional (2D) MMP model. The results will be discussed in connection with the hydrodynamic behavior of the antiferromagnetic excitations and with the results obtained by Millis.[4]

II. FIRST-ORDER VERTEX CORRECTION

The Migdal theorem states that in the electron-phonon interaction the vertex corrections are small compared to the bare interaction. In the case of the electron-phonon interaction, the first-order vertex correction $\Gamma^{(1)}$ is of order of ω_D/E_F, where ω_D is the characteristic frequency for phonons and E_F is the Fermi energy. It was expected that in the case of paramagnons with the energy ω_{SF} the vertex correction could be of the order ω_{SF}/E_F. However (see Ref. 6), this correction is comparable to the bare vertex for the Berk-Schrieffer paramagnons, and this result justifies the weakening of the superconducting state predicted in Ref. [8]. In order to analyze the validity of the Migdal theorem for a 2D MMP model we can follow the same method used in Ref. 6 and we get at $T=0$ the first-order vertex correction as

$$\Gamma^{(1)}\equiv\Gamma^{(1)}(k_F)=g^3\int\frac{d^2p}{(2\pi)^2}\int\frac{d\omega}{\pi}\chi''(\mathbf{p},\omega)\left[\frac{1/2}{(\varepsilon_{\mathbf{k}+\mathbf{p}}-\omega)^2}+\frac{1/2}{(\varepsilon_{\mathbf{k}+\mathbf{p}}+\omega)^2}-\frac{2\delta(\varepsilon_{\mathbf{k}+\mathbf{p}})}{\omega}\right]_{k=k_F}, \tag{1}$$

where g is the electron-antiferromagnetic excitations bare interaction, $\chi''(\mathbf{p},\omega)$ is the imaginary part of $\chi(\mathbf{p},\omega)$, and $\delta(\varepsilon_{\mathbf{k}+\mathbf{p}})$ is the Dirac function. The angular integration shows that the last term from Eq. (1) vanishes and (1) becomes:

$$\Gamma^{(1)}\sim\frac{1}{2}g^3\int\frac{d^2p}{(2\pi)^2}\int_0^\infty\frac{d\omega}{\pi}\chi''(\mathbf{p},\omega)\times\left[\frac{1}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}-\omega)^2}+\frac{1}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}+\omega)^2}\right]. \tag{2}$$

This equation will be used in the case of the antiferromagnetic paramagnons and diffusive paramagnons.

A. Antiferromagnetic paramagnons

The phenomenological MMP model has the main characteristic of a dynamical susceptibility of the form

$$\chi(\mathbf{p}+\mathbf{Q},\omega)=\frac{\chi_Q(T)}{1+\xi_m^2p^2-i\omega/\omega_{SF}} \tag{3}$$

with the imaginary part

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$$\chi''(\mathbf{p}+\mathbf{Q},\omega)=\chi_Q(T)\frac{\omega/\omega_{SF}}{(1+\xi_m^2p^2)^2+\omega^2/\omega_{SF}^2}\ , \tag{4}$$

where ξ_m is the antiferromagnetic correlation length and $\chi_Q(T)$ is the static susceptibility.

This equation will be approximated as

$$\chi''(\mathbf{p},\omega)\simeq\frac{\omega}{\omega_{SF}}\chi_Q/(1+\xi_m^2p^2)^2 \tag{5a}$$

for small frequencies and as

$$\chi''(\mathbf{p},\omega)\simeq\chi_Q\frac{\omega_{SF}}{\omega} \tag{5b}$$

for high frequencies.

In the evaluation of the vertex correction we will take

$$\tilde{\omega}=\omega_{SF}(1+p^2\xi_m^2) \tag{6}$$

where

$$\omega_{SF}=\frac{\Gamma}{\pi\sqrt{\beta}}\cdot\left[\frac{a}{\xi_m}\right]^2 ,$$

Γ and β being parameters which have been determined from the NMR data. Using the relations (5a) and (5b) the vertex correction (2) becomes

$$\Gamma^{(1)}\sim g^3\int\frac{d^2p}{(2\pi)^2}\int_0^{\tilde{\omega}}\frac{d\omega}{\pi}\frac{\chi_Q}{\omega_{SF}}\frac{\omega}{(1+\xi_m^2p^2)^2}\left[\frac{1}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}-\omega)^2}+\frac{1}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}+\omega)^2}\right]$$
$$+g^3\int\frac{d^2p}{(2\pi)^2}\int_{\tilde{\omega}}^{\infty}\frac{d\omega}{\pi}\chi_Q\frac{\omega_{SF}}{\omega}\left[\frac{1}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}-\omega)^2}+\frac{1}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}+\omega)^2}\right]=\Gamma_1^{(1)}+\Gamma_2^{(1)} . \tag{7}$$

The first term from (7) represents the small frequencies contribution and is given by

$$\Gamma_1^{(1)}\sim\frac{g^3\chi_Q}{(2\pi)^2\pi\omega_{SF}}\int_0^{2k_F}p\,dp\,\frac{1}{(1+\xi_m^2p^2)^2}\int_0^{2\pi}d\theta\left[\int_0^{\tilde{\omega}}d\omega\frac{\omega}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}-\omega)^2}+\int_0^{\tilde{\omega}}d\omega\frac{\omega}{(\varepsilon_{\mathbf{k}_F+\mathbf{p}}+\omega)^2}\right] . \tag{8}$$

The ω integration in (8) can be performed analytically and we obtain

$$\Gamma_1^{(1)}\sim\frac{g^3\chi_Q}{(2\pi)^2\pi\omega_{SF}}\int_0^{2k_F}p\,dp\frac{1}{(1+\xi_m^2p^2)^2}\int_0^{2\pi}d\theta\left[\ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}}-\tilde{\omega}|+\ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}}+\tilde{\omega}|-2\ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}}|\right.$$
$$\left.+\frac{\varepsilon_{\mathbf{k}_F+\mathbf{p}}}{\varepsilon_{\mathbf{k}_F+\mathbf{p}}-\tilde{\omega}}+\frac{\varepsilon_{\mathbf{k}_F+\mathbf{p}}}{\varepsilon_{\mathbf{k}_F+\mathbf{p}}+\tilde{\omega}}-2\right] , \tag{9}$$

where

$$\varepsilon_{\mathbf{k}_F+\mathbf{p}}=\frac{k_F^2}{2m}+\frac{p^2}{2m}+\frac{pk_F}{m}\cos\theta .$$

The angular integration in (9) can be performed using the formula

$$\int_0^{2\pi}d\theta\ln|a+b\cos\theta|=\begin{cases}2\pi\ln|\frac{1}{2}(a+\sqrt{a^2-b^2})|, & a>|b| ,\\ 0, & |a|<|b| ,\end{cases} \tag{10}$$

$$\int\frac{\alpha+\beta\cos\theta}{a+b\cos\theta}d\theta=\frac{\beta\theta}{b}+\frac{\alpha b-a\beta}{b}\int\frac{d\theta}{a+b\cos\theta} , \tag{11}$$

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\begin{cases}\frac{2\pi}{\sqrt{a^2-b^2}}, & a>|b| ,\\ 0, & |a|<|b| .\end{cases} \tag{12}$$

Then

$$\int_0^{2\pi}d\theta\ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}}-\tilde{\omega}|=\begin{cases}2\pi\ln|\frac{1}{2}(a+\sqrt{a^2-b^2})|, & a>|b|,\\ 0, & |a|<|b| ,\end{cases}$$

$$a=\frac{k_F^2}{2m}+\frac{p^2}{2m}-\omega_{SF}-\omega_{SF}\xi_m^2p^2,\quad b=\frac{pk_F}{m} .$$

The condition $a>|b|$ implies

$$\frac{k_F^2}{2m}+\frac{p^2}{2m}-\omega_{SF}-\omega_{SF}\xi_m^2p^2>\frac{pk_F}{m} \tag{13}$$

or

$$\left[\frac{1}{2m}-\omega_{SF}\xi_m^2\right]p^2-\frac{k_F}{m}p+\left[\frac{k_F^2}{2m}-\omega_{SF}\right]>0 , \tag{13a}$$

where

$$\left[\frac{1}{2m}-\omega_{SF}\xi_m^2\right]=\frac{1}{k_F^2}E_F\left[1-\frac{\omega_{SF}}{E_F}k_F^2\xi_m^2\right]>0$$

because $\omega_{SF}/E_F\ll1$.

The condition (13) and $\omega_{SF}/E_F\ll1$ requires that p be in the interval

$$p\in[0,p_1]\cup[p_2,2k_F] ,$$

where

$$p_1 \simeq k_F \left[1 - \xi_m k_F \left[\frac{\omega_{SF}}{E_F} \right]^{1/2} \right]$$

and (14)

$$p_2 \simeq k_F \left[1 + \xi_m k_F \left[\frac{\omega_{SF}}{E_F} \right]^{1/2} \right]$$

are the roots of the equation corresponding to (13a). However, the length of the interval $[p_1, p_2]$ is (much) smaller than $[0, 2k_F]$.

The second angular integral is

$$\int_0^{2\pi} d\theta \ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}} + \tilde{\omega}| = 2\pi \ln|\tfrac{1}{2}(c + \sqrt{c^2 - b^2})| \qquad (15)$$

for $p \in [0, 2k_F]$ and

$$c = \frac{k_F^2}{2m} + \frac{p^2}{2m} + \omega_{SF}(1 + \xi_m^2 p^2) .$$

In a similar way:

$$\int_0^{2\pi} d\theta \ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}}| = 2\pi \ln|\tfrac{1}{2}(d + \sqrt{d^2 - b^2})| \qquad (16)$$

for $p \in [0, 2k_F]/\{k_F\}$. Here

$$d = \frac{k_F^2}{2m} + \frac{p^2}{2m} .$$

The fourth angular integral is

$$\int_0^{2\pi} d\theta \frac{\varepsilon_{\mathbf{k}_F+\mathbf{p}}}{\varepsilon_{\mathbf{k}_F+\mathbf{p}} - \tilde{\omega}} = \int_0^{2\pi} d\theta \frac{d + b\cos\theta}{a + b\cos\theta} = \begin{cases} 2\pi + \omega_{SF}(1 + \xi_m^2 p^2) \dfrac{2\pi}{\sqrt{a^2 - b^2}}, & a > |b|, \\ 2\pi, & |a| < |b| . \end{cases} \qquad (17)$$

The first line of (17) implies that

$$p \in [0, p_1] \cup [p_2, 2k_F] .$$

The integral

$$\int_0^{2\pi} d\theta \frac{\varepsilon_{\mathbf{k}_F+\mathbf{p}}}{\varepsilon_{\mathbf{k}_F+\mathbf{p}} + \tilde{\omega}} = 2\pi - \omega_{SF}(1 + \xi_m^2 p^2) \frac{2\pi}{\sqrt{c^2 - b^2}} \qquad (18)$$

has no restrictions in the interval $[0, 2k_F]$.

The low-frequency first-order vertex correction is then given by:

$$\begin{aligned}
\Gamma_1^{(1)} \sim \frac{g^3 \chi_Q}{2\pi\pi\omega_{SF}} \Bigg[& \int_0^{p_1} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \ln\left|\frac{1}{2}(a + \sqrt{a^2 - b^2})\right| + \int_{p_2}^{2k_F} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \ln\left|\frac{1}{2}(a + \sqrt{a^2 - b^2})\right| \\
& + \int_0^{2k_F} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \ln\left|\frac{1}{2}(c + \sqrt{c^2 - b^2})\right| \\
& - \int_0^{2k_F} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \ln\left|\frac{1}{2}(d + \sqrt{d^2 - b^2})\right| \\
& + \int_0^{p_1} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \left[1 + \omega_{SF}(1+\xi_m^2 p^2) \frac{1}{\sqrt{a^2 - b^2}} \right] + \int_{p_1}^{p_2} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \\
& + \int_{p_2}^{2k_F} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \left[1 + \omega_{SF}(1+\xi_m^2 p^2) \frac{1}{\sqrt{a^2 - b^2}} \right] \\
& + \int_0^{2k_F} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \left[1 - \omega_{SF}(1+\xi_m^2 p^2) \frac{1}{\sqrt{c^2 - b^2}} \right] - 2\int_0^{2k_F} p\, dp \frac{1}{(1+\xi_m^2 p^2)^2} \Bigg] . \qquad (19)
\end{aligned}$$

The p integration may be estimated and the leading contribution to the first-order vertex correction is

$$\Gamma_1^{(1)} \sim \frac{g^3 \chi_Q N(0)}{\pi} \frac{\omega_{SF}}{E_F} \left[4 - \ln \frac{1 + \xi_m^2 p_2^2}{1 + \xi_m^2 p_1^2} \right] . \qquad (20)$$

Here

$$4 > \ln \frac{1 + \xi_m^2 p_2^2}{1 + \xi_m^2 p_1^2}$$

and $N(0)$ is the 2D electronic density of states:

$$N(0) = m/2\pi . \qquad (21)$$

In the low-frequency domain $\omega_{SF} \ll E_F$ and Eq. (20) shows that the Migdal theorem is valid. If we take the susceptibility χ_Q as (see Ref. 2)

$$\chi_Q \simeq \frac{\Gamma}{\pi\omega_{SF}} \chi_0 \qquad (22)$$

we get from (20)

$$\Gamma_1^{(1)} \sim \frac{g^3 N(0)}{\pi^2} \chi_0 \frac{\Gamma}{E_F} \left[4 - \ln \frac{1+\xi_m^2 p_2^2}{1+\xi_m^2 p_1^2} \right] . \quad (23)$$

Usual values for Γ are of order of 0.4 eV and $E_F \pi^2$ is of order of few eV. Then $\Gamma / E_F \pi^2 \ll 1$ and in the high-T_c materials it is claimed that $\chi_0 \Gamma / \pi^2 \simeq 0.1 \ll 1$ (see Ref. 4). With these observations and with (23) we see that the Migdal theorem is also valid.

The second term from Eq. (7) corresponds to the high-frequency limit and is given by

$$\Gamma_2^{(1)} \sim \frac{g^3 \chi_Q \omega_{SF}}{(2\pi)^2 \pi} \int_0^{2k_F} p\, dp \int_0^{2\pi} d\theta \left[\int_{\tilde{\omega}}^{\infty} d\omega \frac{1}{\omega(\omega - \varepsilon_{\mathbf{k}_F+\mathbf{p}})^2} + \int_{\tilde{\omega}}^{\infty} d\omega \frac{1}{\omega(\omega + \varepsilon_{\mathbf{k}_F+\mathbf{p}})^2} \right] . \quad (24)$$

After performing the ω integration in (24) we get

$$\Gamma_2^{(1)} \sim \frac{g^3 \chi_Q \omega_{SF}}{(2\pi)^2 \pi} \int_0^{2k_F} p\, dp \int_0^{2\pi} d\theta \frac{1}{\varepsilon_{\mathbf{k}_F+\mathbf{p}}^2} \left[\ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}} - \tilde{\omega}| + \ln|\varepsilon_{\mathbf{k}_F+\mathbf{p}} + \tilde{\omega}| - 2\ln|\tilde{\omega}| - \frac{\varepsilon_{\mathbf{k}_F+\mathbf{p}}}{\varepsilon_{\mathbf{k}_F+\mathbf{p}} - \tilde{\omega}} - \frac{\varepsilon_{\mathbf{k}_F+\mathbf{p}}}{\varepsilon_{\mathbf{k}_F+\mathbf{p}} + \tilde{\omega}} \right] . \quad (25)$$

Here the angular integral and, as a consequence, the momentum integral are more complicated than in the case of the low frequencies. This is so because of the presence of $\varepsilon_{\mathbf{k}_F+\mathbf{p}}^2$ in the denominator of (25). However, the order of magnitude can be estimated and it is not very difficult to see that the first-order vertex correction is now given by

$$\Gamma_2^{(1)} \sim \frac{2}{\pi} \frac{g^3 \chi_Q N(0)}{\pi} \frac{\omega_{SF}}{E_F} . \quad (26)$$

There is a small difference between (26) and (20) but both formulas have the same order of magnitude. Because $\omega_{SF} \ll E_F$, $\Gamma_2^{(1)} \ll 1$ and the Migdal theorem is valid.

B. Diffusive paramagnons

The importance of the diffusive paramagnons in HTS has been pointed out by Brenig and Monien [3] in connection with the Raman scattering.

For the diffusive paramagnons, $\chi(\mathbf{p},\omega)$ has the form

$$\chi(\mathbf{p},\omega) = \chi_0 \frac{Dp^2}{Dp^2 - i\omega} , \quad (27)$$

D being the diffusion constant. Using (2) we find (roughly)

$$\Gamma_d^{(1)} \sim \frac{1}{2} g^3 \frac{\chi_0}{(2\pi)^3} \int_0^{2p_c} p\, dp\, Dp^2 \ln \left[1 + \left[\frac{\omega_c}{Dp^2} \right]^2 \right] \left[\int_0^{2\pi} d\theta \frac{1}{\left[\frac{k_F^2}{2m} + \frac{p^2}{2m} - \frac{1}{2}\omega_c + \frac{pk_F}{m}\cos\theta \right]^2} + \int_0^{2\pi} d\theta \frac{1}{\left[\frac{k_F^2}{2m} + \frac{p^2}{2m} + \frac{1}{2}\omega_c + \frac{pk_F}{m}\cos\theta \right]^2} \right] . \quad (28)$$

Here ω_c and p_c are the frequency and wave vector cutoffs, respectively. Taking $p_c = k_F$ we get

$$\Gamma_d^{(1)} \sim \frac{g^3 \chi_0}{(2\pi)^2 D} \left[\frac{\omega_c}{E_F} \right]^2 \quad (29)$$

and if $\omega_c \sim \omega_{SF}$, $\Gamma_d^{(1)} \ll 1$ and the Migdal theorem is valid.

III. DISCUSSIONS

We showed the validity of the Migdal theorem for the phenomenological MMP model of the electron-antiferromagnetic paramagnons interaction. This result, in agreement with the Millis[4] analytic Eliashberg approach, is different from the standard statement concerning the absence of a Migdal theorem in the electron-paramagnon system because the MMP model implies paramagnons with relaxational dynamics which is not the case of the Berk-Schrieffer[8] paramagnons. The Migdal theorem can be invalidated if we use for the staggered susceptibility χ_Q the Moriya-Tekeheshi-Ueda[9] self-consistent spin fluctuation theory. In this case we get $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ proportional to g and the Migdal theorem is no longer valid. However, the MMP model has as an essential characteristic of the form of $\chi''(\mathbf{p},\omega)$, which is not calculated as in Ref. 9, by taking the form (3) and, in this case, χ_Q is given by (22). Our results can also have a restricted validity because of many phenomenological parameters, but at the present stage of the model it is the only way to discuss the problem.

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