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Mass of a vortex line in superfluid ^{4}He: Effects of gauge-symmetry breaking

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In previous works the mass of a vortex line in superfluid ^{4}He was argued to arise from the tiny core of the vortex. We show that due to gauge-symmetry breaking, the condensate compressibility contributes to a vortex mass that far exceeds the core mass. This large vortex mass should play an important role in some problems of vortex dynamics.

A quantized vortex is a topological object that generally exists in a variety of materials called quantum liquids and solids, including the Bose superfluid ^{4}He,[1,2] type-II superconductors,[3] and the Fermi superfluid ^{3}He.[4] A vortex consists of a normal core region of the size of the coherence length ξ, and an outside region of circulating supercurrent. The rectilinear vortex lines form Abrikosov lattices[3] inside superfluid ^{4}He and superconductors. During the last two or three decades the vortex dynamics of this lattice was closely studied in rotating superfluid ^{4}He (Ref. 5) both theoretically and experimentally. Different kinds of collective modes of the vortex lattice have been studied, including the "inertial-wave" mode due to a finite vortex mass (e.g., Refs. 6 and 5), the interesting "Tkachenko mode,"[7] and so on. While in superconductors, vortex dynamics is of both theoretical interest and practical importance: the eventual industrial use of superconductors in energy transport is restricted by the critical currents of the superconductors, which is in turn determined by the interaction between vortex lines and thermal quasiparticles. Ever since the discovery of the layered high-transition-temperature cuprate superconductors, the study of vortex dynamics has become a rapidly expanding field. A short review can be found in Ref. 8.

The dynamics of an object is dictated by Newton's second law, $\mathbf{F} = m_{\rm dyn}\mathbf{a}$, namely, the total force $\mathbf{F}$ acted upon the object equals the dynamic mass $m_{\rm dyn}$ times the acceleration $\mathbf{a}$. One can see that $m_{\rm dyn}$ is the most important intrinsic property of an object in its dynamics. In the case of a vortex line $\mathbf{F}$ usually includes[5] interactions due to other vortex lines, viscous force, and the Magnus force. Regarding the *dynamic* mass, the conventional wisdom is that $m_{\rm dyn}$ is identical to the *inertial* mass $m_{\rm inert}$ determined from the kinetic energy $m_{\rm inert}\mathbf{v}^2/2$ of the object with velocity $\mathbf{v}$, which is usually correct. The inertial mass of a flux line in a type-II superconductor is well studied.[9–15] Corresponding to different origins of the kinetic energy, various inertial masses have been identified, such as the core mass,[9] the electromagnetic field mass[9–13] (see discussions in Ref. 11), and the strain field inertial mass.[14,15] However, no consideration on the dynamic mass has been reported so far.

In contrast, the discussion on the vortex mass in superfluid ^{4}He is particularly poor and did not go beyond a classical fluid model, so that the only obvious contribution is from the normal core[6,16,17] with $m_{\rm core} = \pi\rho\xi^2$ per unit length of the vortex line, where density $\rho = mN$ with m the mass of a ^{4}He atom and N the bulk number density of ^{4}He fluid. ξ is extremely small and only of the order of 1 Å. This small vortex mass is usually discarded in vortex dynamics except in the "inertial wave" mode.[5,6] In this paper, we show that due to gauge-symmetry breaking and the topology of a vortex, the condensate compressibility contributes to a vortex mass which is much larger than the core mass. We calculate both the inertial and the dynamic masses, they turn out to be the same (as expected), and diverge logarithmically with the system size. We start by noting that the superfluid ^{4}He is compressible,[18] as manifested by the existence of the hydrodynamic density sound (first sound). A superfluid system possesses an order parameter with its phase $S(\mathbf{r},t)$ describing the motion of the condensate. The superfluid velocity is defined as

$$\mathbf{v}_s(\mathbf{r},t) = \frac{\hbar}{m}\boldsymbol{\nabla} S(\mathbf{r},t). \tag{1}$$

For a slowly varying nonuniform condensate the order parameter $\phi(\mathbf{r},t)$ can be written as (see Ref. 18, Chap. 10),

$$\phi(\mathbf{r},t) = e^{iS(\mathbf{r},t)}[1+\lambda(\mathbf{r},t)]e^{-i\mu_0 t}\sqrt{n_0}. \tag{2}$$

Here n_0 is the uniform condensate number density which depends on temperature. In this paper we consider the zero temperature limit where $n_0 = N$. μ_0 is the chemical potential for the static uniform condensate. The square of the amplitude of ϕ equals the condensate density, so that λ describes the superfluid density change. For a small deviation from the equilibrium state, the changes of the chemical potential and the superfluid density are related to λ by $\delta\mu(\mathbf{r},t) = (ms^2/N)\delta\rho(\mathbf{r},t) \approx 2ms^2\lambda(\mathbf{r},t)$ with s the characteristic sound velocity. The dynamic equation for the condensate is[18]

$$\hbar\frac{\partial S}{\partial t} + \frac{1}{2}m\mathbf{v}_s^2 = -\delta\mu. \tag{3}$$

We note that if the first term ($\hbar\partial S/\partial t$) is absent, Eq. (3) is just the Bernoulli equation for classical fluids. The first term makes all the difference between a superfluid and a classical fluid, and there is a direct physical origin to this term: due to gauge-symmetry breaking, superfluid number density and the order parameter phase are a pair of conjugate variables. The rate of the phase change ($\hbar\partial S/\partial t$) corresponds to the superfluid density change.

To see the implications of Eq. (3) in vortex dynamics, we consider, for simplicity, a static rectilinear vortex line at the origin of an infinitely large superfluid system. Remember that a vortex is a topological object with a 2π phase change around it, determined by the single valuedness of the many-body wave function. We choose the vortex line pointing along $\hat{\mathbf{e}}_z$, with a supercurrent circulating counterclockwise. In polar coordinates (r,θ) at point $\mathbf{r}$ the supercurrent velocity is $\mathbf{v}_s(\mathbf{r}) = (\hbar/m)\boldsymbol{\nabla}S(\mathbf{r}) = (\hbar/mr)\hat{\mathbf{e}}_\theta$ with $\hat{\mathbf{e}}_\theta$ the unit vector along the θ angle. Now give the vortex an instant velocity $\mathbf{v} = v\hat{\mathbf{e}}_x$ with respect to the superfluid background, and the superfluid velocity is still zero at infinity. In common with previous works,[9–15] we use the *adiabatic phase assumption*, that is, the vortex motion is slow enough such that the phase will adjust and is only determined by the relative position of any point with respect to the center of the vortex. In other words, $S(\mathbf{r},t) \equiv S(\mathbf{r}-\mathbf{v}t)$, and we consider the $t \to 0$ limit. Note that an acceleration $\mathbf{a}$ of the vortex can also be included, but it does not play any role in the $t \to 0$ limit. By definition $\mathbf{v}_s(\mathbf{r}, t \to 0) = \mathbf{v}_s(\mathbf{r})$, and it is clear from Eq. (3) that the only change of the chemical potential is due to $\hbar\partial S/\partial t = -\hbar\mathbf{v}\cdot\boldsymbol{\nabla}S$. This leads to a global density change (for $r > \xi$) caused by the moving vortex

$$\delta\rho(\mathbf{r}) = -\frac{N\hbar\, v}{s^2}\frac{\sin\theta}{r}. \qquad (4)$$

We can estimate the relative change of the density by noting that the coherence length $\xi \approx (\hbar/ms)$ due to the Heisenberg indeterminism since ms is the characteristic momentum scale for the superfluid system. Taking $s \sim 2\times10^2$ m/s (Ref. 18) we estimate $\xi \sim 1$ Å, which is indeed the correct magnitude. The order of magnitude of the relative change outside the core region $|\delta\rho(\mathbf{r})/\rho| \sim (\xi/r)(v/s) \ll 1$ for slow motion $(v/s \ll 1)$. So the superfluid density of the whole system is perturbed by a moving vortex line. This density change is linearly proportional to the velocity v, and decays slowly as the inverse of the distance from the vortex center. The density change possesses a dipolelike distribution and satisfies the global superfluid continuity equation, where the maximum density change lies along the line perpendicular to the vortex velocity.[19] To the best of our knowledge the density change in Eq. (4) has not been discussed before in the context of superfluid ^{4}He. In the following we calculate the changes in energy, momentum, and angular momentum separately due to this density distribution.

First let us consider the energy change for a unit length of the vortex line. Computed to second order in the density fluctuation, the energy change is

$$E = \frac{1}{2}\int d^2\mathbf{r}\frac{\partial^2 u}{\partial\rho^2}[\delta\rho(\mathbf{r})]^2, \qquad (5)$$

where u is the energy density and $\partial^2u/\partial\rho^2 = \partial\mu/\partial\rho = ms^2/N$. One can see that E is proportional to v^2 since $\delta\rho(\mathbf{r})$ is linear in $|\mathbf{v}|$. This energy can be defined as the kinetic energy $m_{\text{inert}}\mathbf{v}^2/2$ of the vortex line. After the areal integration we get the inertial mass

$$m_{\text{inert}} = \pi N m\left(\frac{\hbar}{ms}\right)^2 \ln\frac{L}{\xi} = m_{\text{core}}\ln\frac{L}{\xi}, \qquad (6)$$

where L is the sample size. For a practical superfluid system $\ln(L/\xi) \sim 20$–30, so the m_{inert} is much larger than the core mass m_{core}. In terms of the vortex static energy ε_0 which is also logarithmically divergent as the sample size (e.g., Ref. 18), we have $m_{\text{inert}} \cong \varepsilon_0/s^2$, in agreement with the general dimensional arguments.[10,11] Note that the density fluctuation energy, which in our case is proportional to $\int(\partial S/\partial t)^2$, is actually the origin of the time variation term $\int|\partial\Delta/\partial t|^2$ (with Δ the order parameter) in the time-dependent Ginzburg-Landau theories for a Fermi superfluid near $T=0$.[10,11,20–22]

Second we consider the total momentum $\mathbf{P}$ of the system outside the vortex core. Newton's second law $\mathbf{F} = d\mathbf{P}/dt$ implies that the dynamic mass can be obtained from $\mathbf{P}$. Note that if there were no density change the net momentum of the superfluid would be zero, since the superfluid velocity $\mathbf{v}_s(\mathbf{r}) = (\hbar/mr)\hat{\mathbf{e}}_\theta$ has perfect rotational symmetry. We look at the system from the z axis. With a counterclockwise circulating supercurrent, the density decrease in the half space $(\sin\theta > 0)$ contributes to a momentum along the x axis (i.e., parallel to $\mathbf{v}$), while the density increase in another half-space $(\sin\theta < 0)$ also contributes a momentum along $\mathbf{v}$. Hence the total momentum $\mathbf{P}$ is nonzero. A direct simple integration of

$$\mathbf{P} = \int d^2\mathbf{r}\,\delta\rho(\mathbf{r})\mathbf{v}_s(\mathbf{r}) \qquad (7)$$

leads to $\mathbf{P} = m_{\text{dyn}}\mathbf{v}$, with the dynamic mass given by

$$m_{\text{dyn}} = m_{\text{core}}\ln\frac{L}{\xi} = m_{\text{inert}}, \qquad (8)$$

which is expected.

Third we consider the change of the angular momentum of the superfluid. This can serve as a self-consistency check, since, by definition, a quantized vortex forbids any continuous change in the angular momentum. The dipolar distribution in Eq. (4) does not change the total angular momentum and $\int d^2\mathbf{r}\,\delta\rho(\mathbf{r})\mathbf{r}\times\mathbf{v}_s(\mathbf{r}) = 0$. So a quantized vortex remains well defined.

With the above results on vortex mass, let us discuss some theoretical as well as experimental consequences. The most direct manifestation of the vortex mass is the so-called "inertial wave" mode, whose existence is purely due to a nonzero mass of a vortex.[6,5] A direct calculation by Baym and Chandler shows that the mode frequency is approximately *inversely* proportional to the vortex mass [see Eqs. (83)–(85) in Ref. 6]. The core mass m_{core} was used, and the result of the inertial mode frequency is sufficiently high that Baym and Chandler began to question the validity of their calculation on the basis of hydrodynamics. We point out that they missed the more important contribution to the mass from the outside of the vortex core. Using our result [Eq. (6)] one can easily reduce their calculated inertial mode frequency by one order of magnitude.

Previous theoretical works on the vortex dynamics in superfluid ^{4}He are based on the equation $\mathbf{F} = m\mathbf{a}$, and the right-hand side of the equation is set to zero due to the smallness of the core mass.[5] So the dynamic equation for a vortex is that the summation of the forces equals zero. A large mass in Eq. (6) should not be arbitrarily ignored. Based on dimensional grounds, the right-hand side of the equation for vortex oscillations (i.e., the $m\mathbf{a}$ term) should contribute a term of $(\omega/s)^2$ to the dispersion of the collective oscillations, with ω the corresponding frequency.

In an influential paper,[16] Muirhead, Vinen, and Donnelly used the core mass $m_{\rm core}$ as the vortex mass to calculate the quantum nucleation rate of a vortex ring. Since the vortex mass may influence the quantum nucleation rate exponentially, one must be careful to use the correct vortex mass. We estimate that in addition to the $m_{\rm core}$, one should include a dynamic mass of the order $m_{\rm dyn} \approx m_{\rm core} \ln(R/\xi)$, where R is the radius of the vortex ring. For any reasonable size of R ($R \gg \xi$), $m_{\rm dyn}$ still dominates over $m_{\rm core}$. Note that the radius R replaces the sample size L in Eq. (6). This is an example showing that the total mass of an assembly of vortices is not the simple addition of the masses belonging to each individual vortex—one should take into account the effect due to interference of the phase of the order parameter.

In a recent paper,[17] Niu, Ao, and Thouless strongly argued that the vortex mass cannot be greater than the core mass unless, e.g., a heavy ion is trapped inside the vortex core. In view of our equations (6) and (8), their argument is incorrect. A large vortex mass makes their suggestions of quantized Landau Levels for a vortex less appealing.

Finally, without giving any details, we mention our calculations for the *dynamic* mass of a vortex inside a bulk superconductor or a thin superconducting film. In a superconductor the screening for a charge density and a current density are described by the Debye shielding length and the London penetration depth separately. The charge density distribution corresponding to Eq. (4) is greatly reduced due to the Debye screening. The inertial mass based on energy considerations of the charge density deviations is the previously called "electromagnetic mass" $m_{\rm EM}$ (see results and discussions in Ref. 11). Now we calculate the total momentum of the system due to the density change. Using the expressions for the charge density and the formulations in Ref. 11, we find that the results for *dynamic* masses are identical to the corresponding inertial masses given in Ref. 11.

I recently became aware of the work by Popov.[23] By mapping the vortices and phonons into charged particles and photons in relativistic electrodynamics, he reached the conclusion that the vortex inertial mass equals the vortex static energy divided by the square of sound velocity.

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