Bosonization of Fermi liquids

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We bosonize a Fermi liquid in any number of dimensions in the limit of long wavelengths. From the bosons we construct a set of coherent states which are related to the displacement of the Fermi surface due to particle-hole excitations. We show that an interacting Hamiltonian in terms of the original fermions is quadratic in the bosons. We obtain a path-integral representation for the generating functional, which, in real time, in the semiclassical limit, gives the Landau equation for sound waves and in the imaginary time gives us the correct form of the specific heat for a Fermi liquid even with the corrections due to the interactions between the fermions. We also discuss the similarities between our results and the physics of quantum crystals.

I. INTRODUCTION

Attempts to describe fermionic systems by bosons date to the early days of second quantization. In the early 1950s, Tomonaga,[1] generalizing earlier work by Bloch[2] on sound waves in dense Fermi systems, gave an explicit construction of the Bloch waves for systems in one dimension. Three years after the seminal work of Tomonaga, Bohm, and Pines[3] showed that there was a natural connection between bosonization and the random-phase approximation (RPA). Subsequently, many authors[4] derived an explicit Fermi-Bose transmutation in one-dimensional systems. These works uncovered deep connections in relativistic field theories (both fermionic and bosonic) and with condensed matter systems.

The success of the bosonization approach in one dimension is related to phase space considerations. Even for noninteracting fermions, two excitations with arbitrarily low energies moving in the same direction move at the same speed (the Fermi velocity) and, hence, are almost a bound state. Consequently, even the weakest interactions can induce dramatic changes in the nature of the low-lying states. These phase space effects become manifest even in perturbation theory and result in the presence of both marginal and marginally relevant operators. Bosonization yields a simple and very general description of the possible fixed point structure of strongly correlated systems: marginally relevant operators are responsible for the gap generating instabilities while strictly marginal operators give rise to anomalous dimensions. Quite generally, the fixed points are determined by the marginal operators alone. In turn, the anomalous dimensions produce non-Fermi liquid behavior. Bosonization in one spacial dimension is due to the fact that the charge and current densities for states restricted to the vicinity of the two Fermi points, $\rho(x)$ and $j(x)$, of electrons obey the equal-time algebra (Kac-Moody)

$$[\rho(x), j(y)] = -\frac{i}{\pi}\frac{\partial}{\partial x}\delta(x-y). \tag{1.1}$$

This algebra implies that there exists of a free bosonic field $\phi(x)$ and its canonically conjugate momentum $\Pi(x)$, which obey the canonical equal-time commutation relation

$$[\phi(x), \Pi(y)] = i\delta(x-y), \tag{1.2}$$

provided that one makes the identifications

$$\rho(x) \equiv \frac{1}{\sqrt{\pi}}\frac{\partial}{\partial x}\phi(x), \qquad j(x) \equiv \frac{1}{\sqrt{\pi}}\Pi(x). \tag{1.3}$$

More complicated operators, such as various order parameters associated with $2k_F$ excitations, can also be identified with suitably chosen bosonic operators.[5] Typically, these operators involve nonlocal or nonlinear (or both) expressions in the bosonic field. In any event, the basic building blocks are the local density and current operators which describe electron-hole pairs at small momentum transfers.

In dimensions higher than one, phase space considerations change the physics of the low-lying states. The only marginal operators left [i.e., operators whose coupling constants (Landau parameters) do not change as the energy scale is lowered] are responsible only for changing the shape of the Fermi surface. However, the low temperature and low frequency behavior of physical quantities is insensitive to the presence of such marginal operators. In a sense, these operators are redundant. This observation is at the root of the stability of the Landau theory of the Fermi liquid.[6,7] It is, thus, hardly surprising that very few attempts have been made to generalize the bosonization approach to dimensions higher than one. In addition, phase space considerations also tell us that even for small momentum transfers, an electron-hole pair can decay into its electron and hole constituents. Electron-hole collective modes do exist (and are described by the random-phase approximation), but so does the electron-hole continuum. Hence, unlike what happens in one space dimension, kinematics alone does not require the generic existence of a bosonic bound state.

The first serious attempt at bosonization in higher dimensions was carried out by Luther,[8] who constructed

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a generalized bosonization formula in terms of the fluctuations of the Fermi sea along radial directions in momentum space. However, this approach was not pursued. Interest in the construction of bosonized versions of Fermi liquids has been revived recently in the context of strongly correlated systems[9] due to the possibility of nonconventional ground states that are not Fermi liquid like. Since in one dimension the bosonization approach is a powerful nonperturbative tool, the expectation is that if a suitably generalized analog does exist in higher dimensions, it may throw light on the nonperturbative aspects of Fermi liquids and, hopefully, it may help to define non-Fermi liquids. It has been well known since the early days of the Landau theory that particle-hole excitations have bosonic character, e.g., the sound waves (zero sound collective modes) of the Fermi surface of neutral liquids or plasmons in charged Fermi liquids.[3] Inspired by these identifications, and drawing from Luther's work, Haldane[10] has derived recently a bosonic algebra for density fluctuations of a Fermi liquid in the form of a generalized Kac-Moody algebra which generalizes Eq. (1.1). Haldane's construction has been examined recently by several authors.[11,12]

Recently, we developed a bosonization approach for Fermi fluids based on a coherent-state path integral approach.[13] In this paper, we generalize and apply the results of our earlier work.[13] Our approach has the same starting point as Haldane's. We also begin by constructing a bosonic algebra with creation and annihilation operators that act over a reference state describing a filled Fermi sea. This algebra is valid in a restricted Hilbert space of states generated by small deformation of the filled Fermi sea. Using these operators we construct a set of coherent states that can be regarded as a deformation of the Fermi surface. More precisely, we show that the eigenstate of the annihilation operators of the algebra, namely the coherent states, are related to the displacement of the Fermi surface in some direction. These displacements are a coherent superposition of particle-hole excitations close to the Fermi surface. This interpretation leads to the picture of the Fermi surface as a real dynamical object. We have two strong arguments in support of this picture. The first argument is based on the fact that, using these coherent states, we reproduce the phenomenological theory of Pomeranchuk[6] on the stability of a Fermi liquid and its modern reinterpretation (by Shankar[7]) in terms of the renormalization group method. Pomeranchuk's approach is based on the work of Landau on many-body systems. The second argument follows from the fact (on which we elaborate further below) that it is possible to define a surface tension associated with the Fermi surface itself.

Armed with these coherent states, we construct a coherent state path integral that can be viewed as a sum over the histories of the shape of the Fermi surface, a bosonic *shape field.* We show that there exists a class of simple interacting Hamiltonians (in terms of the original fermions of the theory), which are naturally related to the Landau theory of the Fermi liquid, which in the coherent-state bosonization leads to a simple quadratic Hamiltonian in terms of the bosons. These Hamiltonians are the natural generalization of the Luttinger-Thirring models of one-dimensional systems. According to the renormalization group analysis, these Hamiltonians represent the stable fixed points of Fermi liquids.

With the path integral we can study the thermodynamics and dynamics of interacting electronic systems in a very simple way. In order to prove the consistency of our method we calculate the low temperature specific heat and obtain the same results of the Fermi liquid theory (although we have only *real* bosons in the problem). We can go even further and calculate the corrections for the specific heat due to interactions and we show that our results also agree with the well known results of the Fermi liquid theory. This calculation reveals the new features of the bosonic field involved in the problem. They are not a usual free bosonic field but a *topologically constrained excitation.* The constraint comes from the need to sum over the bosonic excitations *tangent* to the Fermi surface and do not contribute to the energetics. This is most important since free nonrelativistic bosons in space dimensions larger than one do not have the linear specific heat that is characteristic of Fermi liquids.

In this paper we discuss the bosonization approach to the fixed point Hamiltonians. These Hamiltonians contain operators that are at most marginal. We deliberately leave out a number of relevant operators that lead to the well known instabilities of the Fermi liquid: superconductivity, magnetism, etc. We will discuss these operators elsewhere.

We also show that the Landau theory is the semiclassical approximation for the bosons, which is exact for the fixed point Hamiltonians, since the Hamiltonian is quadratic in bosons. This result means that the bosonic field is nothing but the field of sound waves that propagates on the Fermi surface. We applied these methods to study the physics of sound waves in two dimensions. We obtain a general equation for the sound modes and, in particular, we study the zero sound. In order to study the characteristics of these sound modes we go to the limit of large Fermi momentum and concentrate on the forward scattering direction. We introduce a scale in the problem, the range of interaction. We find a rather complex spectrum of sound modes that describe increasingly wrinkled Fermi surfaces. In the case of long range interactions, the shape of the Fermi surface can become unstable. The existence of strong local quantum fluctuations of the shape of the Fermi surface also suggests an analogy with the quantum mechanical fluctuations of the shape of a crystal at zero temperature. Thus it is natural to ask if it is possible that, for long range interactions, the Fermi surface could undergo a roughening transition. In the case of the shape of a three-dimensional crystal, it is known[14] that quantum fluctuations generally make the surface smooth, not rough. However, in the case of a "planar crystal," which has a one-dimensional surface, it is possible to have a quantum mechanical roughening transition. It turns out, however, that by an explicit calculation of the correlation function between different pieces of the Fermi surface, we can show that quantum fluctuations of the fixed point Hamiltonian wash out this interesting possibility. The reason is that the dynamics

described by the fixed point Hamiltonians enforce Luttinger's theorem as a local condition. It may be possible to find other Hamiltonians that enforce Luttinger's theorem only as a global constraint. The shape of the Fermi surface of the ground states of these Hamiltonians may exhibit a quantum roughening transition in two space dimensions.

The paper is organized as follows. In Secs. II and III we present two physical arguments which suggest that the Fermi surface should be regarded as a quantum mechanical object. In Sec. II we review the stability of a Fermi liquid and customize the discussion to the case of two dimensions. In Sec. III we refine the analogy with the theory of surfaces by defining the surface tension of the Fermi surface. In Sec. IV we describe the coherent-state bosonization construction for an interacting fermionic system. In Sec. V we show how a fermionic Hamiltonian can be rewritten in terms of the bosons and in Sec. VI we study the Fermi liquid properties, the thermodynamics, and classical dynamics. Section VII contains our conclusions. We also have included three appendixes. In Appendix A we show how the surface tension of the Fermi sea vanishes for non-Fermi-liquid behavior. In Appendix B we prove that our approach is consistent with the bosonization procedure in one spatial dimension. In Appendix C, in order to show that we can obtain all the important results of the Fermi liquid theory, we obtain the effective mass of the quasiparticles for an isotropic system.

II. THE STABILITY OF THE FERMI LIQUID IN TWO DIMENSIONS

The route for the stability of a Fermi liquid was established by Pomeranchuk,[6] who studied the effect of the change of the Fermi surface on the free energy of the system using the Landau expansion for the change in the total energy,

$$\Delta E = \sum_{\vec{p}} \epsilon^0_{\vec{p}} \delta n_{\vec{p}} + \frac{1}{2V} \sum_{\vec{p},\vec{p'}} f_{\vec{p},\vec{p'}} \delta n_{\vec{p}} \delta n_{\vec{p'}}, \tag{2.1}$$

where V is the volume of the system, $\delta n_{\vec{p}}$ is the deviation of the occupation number where $\vec{p}$ is at the Fermi surface, $\epsilon^0_{\vec{p}}$ is the bare dispersion relation, and $f_{\vec{p},\vec{p'}}$ is the quasiparticle interaction.

In this section we essentially follow the work of Pomeranchuk[6,15] but we restrict ourselves to two dimensions since in three dimensions the results are well known and two-dimensional Fermi liquids are of current physical interest. Consider a change in the Fermi momentum given by

$$p_F(\theta) = p_F + \delta p_F(\theta), \tag{2.2}$$

where θ is the angle that parametrizes the position of the Fermi surface and p_F is the original Fermi momentum.

The new occupation number is simply given by

$$n_{\vec{p}} = \Theta[p_F(\theta) - p]. \tag{2.3}$$

Substituting (2.2) in (2.3) and expanding up to second order in the deviations we find

$$\delta n_{\vec{p}} = \delta p_F(\theta) \delta(p_F - p) - \frac{[\delta p_F(\theta)]^2}{2} \frac{d\delta(p_F - p)}{dp}. \tag{2.4}$$

Now, using (2.1) and (2.4) it is easy to show that the total change in the free energy, $F = E - \mu N$ (μ is the chemical potential and N is the total number of particles) is given by

$$\frac{\Delta F}{V} = \frac{p_F v_F}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} [\delta p_F(\theta)]^2 + \frac{p_F^2}{2\pi^2} \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\theta'}{2\pi} f(\vec{p},\vec{p'}) \delta p_F(\theta) \delta p_F(\theta'), \tag{2.5}$$

where $\vec{p}$ and $\vec{p'}$ are at the Fermi surface and v_F is the Fermi velocity.

We expand the interaction and the displacement of the Fermi surface in Fourier components (in two dimensions we can parametrize the points of the Fermi surface by one angle, θ) as

$$f(\vec{p},\vec{p'}) = \sum_{m=0}^{+\infty} f_m \cos[m(\theta - \theta')] \tag{2.6}$$

and

$$\delta p_F(\theta) = \sum_{m=0}^{+\infty} u_m \cos(m\theta). \tag{2.7}$$

Substituting these expressions in (2.5) and using the orthogonality between the Fourier components, one finds

$$\frac{\Delta F}{V} = \frac{p_F^2}{8\pi^2} \sum_{m=0}^{+\infty} \left(\frac{1}{N(0)} + f_m \right) u_m^2, \tag{2.8}$$

where $N(0) = \frac{p_F}{\pi v_F}$ is the density of states at the Fermi surface.

Observe that the Fermi liquid is stable if $\Delta F \geq 0$ or

$$N(0) f_n \geq -1 \tag{2.9}$$

for all n. This result implies that the Fermi liquid is a local minimum in the configuration space of the many-body system. Local stability, in terms of renormalization group, means that the effective Hamiltonian is at a fixed point.[7]

We notice that it is not a coincidence that the free energy (2.8) has the same form as the one for a drumhead. It means that the Fermi surface is a dynamical object with elastic properties. Furthermore, we see that the interaction term gives the same order of contribution as the free term of the energy. In terms of the renormalization group interpretation of the stability of a Fermi liquid,[7] it means that the interaction is a marginal operator while next order expansion in the functional (2.1) will give only irrelevant operators that do not contribute to the low energy physics of the problem.

Anderson[9] has argued that a singular interaction be-

tween the fermions could give rise to new features which could explain the anomalous behavior of cuprates. Haldane[10] has constructed an argument which is mainly based on the form of expression (2.8), which we reproduce here due to its simplicity and elegance. Suppose that instead of an overall density of states, $N(0)$, we introduce a *local* density of states at each point of the Fermi surface. In order to do so we have to define a cutoff on the Fermi surface, Λ, which is much smaller than the Fermi momentum yet large enough to count a macroscopic number of states. By naive dimensional analysis we conclude that this local density of states must scale like $\Lambda^{d-1}v_F^{-1}$. Therefore, from (2.8), the quantity $F_m = \Lambda^{d-1}v_F^{-1}f_m$ is the dimensionless coupling constant. Suppose that the interaction f_m and the Fermi velocity are well behaved (they do not diverge) in the limit where $\Lambda \to 0$ (the scaling limit). It is easy to see that F_m goes to zero for $d > 1$ and, therefore, only the noninteracting term is present. This result implies that the interaction is *marginally irrelevant*. However, for $d = 1$, F_m will always be finite, that is, the interaction is *marginally relevant* in one dimension. Therefore, even if the interaction is not singular in the scaling limit it has profound consequences in one dimension while it "scales away" in higher dimensions. However, we have the interesting possibility that the interaction term f_m or the Fermi velocity diverge in the scaling limit. If this is so, the dimensionless coupling constant can diverge and therefore the operator becomes *relevant* with respect to the noninteracting term. However, this singular behavior is never present in a Fermi liquid since the renormalization group flow is dominated by the noninteracting fixed point.

The approach in this section is essentially phenomenological because we have postulated (2.1) in order to get the stability condition (2.9). In the next section we study the elastic properties of the Fermi surface from the microscopic point of view. This point of view will bring a different perspective on the same subject, namely, the Fermi surface as a *real quantum object.*

III. THE TENSION OF THE FERMI SURFACE

In this section we explore in more detail the picture of the Fermi surface as a quantum object. In particular, we will establish that, for systems that obey the hypothesis of a Fermi liquid, it is possible to define a surface tension. Here we follow the work of Luttinger[16] on the properties of many-body systems. It can be shown that the total energy of a spinless interacting fermionic system can be written in the form[17] of an integral over the whole momentum space as

$$E = \sum_{\vec{k}} E_{\vec{k}}, \tag{3.1}$$

where

$$E_{\vec{k}} = \int_{-\infty}^{+\infty} \frac{dw}{2\pi i}\, e^{iw\eta}\, \frac{\left(w + \epsilon_{\vec{k}}^0\right)}{2}\, G(\vec{k}, w), \tag{3.2}$$

where $\eta \to 0^+$, $\epsilon_{\vec{k}}^0$ is the noninteracting dispersion relation and $G(\vec{k}, w)$ is the interacting Green's function in momentum space.

Equation (3.2) can be rewritten in terms of the spectral function of the system as[18]

$$E_{\vec{k}} = \int_{-\infty}^{+\mu} \frac{dw}{\pi}\, \frac{\left(w + \epsilon_{\vec{k}}^0\right)}{2}\, A(\vec{k}, w), \tag{3.3}$$

where the spectral function $A(\vec{k}, w)$ is given by

$$A(\vec{k}, w) = \frac{\Sigma_I(\vec{k}, w)}{\left[w - \epsilon_{\vec{k}}^0 - \Sigma_R(\vec{k}, w)\right]^2 + \Sigma_I(\vec{k}, w)^2}, \tag{3.4}$$

where $\Sigma_R(\vec{k}, w)$ and $\Sigma_I(\vec{k}, w)$ are the real and imaginary part of the self-energy and μ is the chemical potential of the problem.

The spectrum of the interacting system (the poles of the Green's function), $\epsilon_{\vec{k}}$, is given by the solution of the self-consistent equation

$$\epsilon_{\vec{k}} - \epsilon_{\vec{k}}^0 - \Sigma_R(\vec{k}, \epsilon_{\vec{k}}) = 0 \tag{3.5}$$

and, in particular, the Fermi surface is defined by the set of vectors $\{\vec{p}_F\}$ such that

$$\epsilon_{\vec{p}_F} = \mu. \tag{3.6}$$

It is clear from the above definitions that we expect that any singularity in this problem should appear near the Fermi surface. We, therefore, split the integral in (3.3) in two pieces, from $-\infty$ to $\mu - \zeta$ (free of singular terms) and from $\mu - \zeta$ to μ (the singular contribution), where ζ is a small energy scale that permits us to look only to the singular part of $E_{\vec{k}}$.

Moreover, consider the vectors $\vec{k}$ close to the Fermi surface,

$$\vec{k} = \vec{p}_F + \delta\vec{k}, \tag{3.7}$$

where the displacement $\delta\vec{k}$ is perpendicular to the Fermi surface, that is,

$$\delta\vec{k} \cdot \nabla\epsilon_{\vec{p}_F} = \delta k \mid \nabla\epsilon_{\vec{p}_F} \mid \tag{3.8}$$

and δk is positive if the vector points outside of the Fermi surface and negative if it points inside the Fermi surface.

Since we are interested in the integral close to the Fermi surface, we can expand the integrand using the above formulas. In particular, the denominator in (3.4) can be rewritten as

$$w - \epsilon_{\vec{k}}^0 - \Sigma_R(\vec{k}, w) = Z_{\vec{k}}^{-1}(w - \epsilon_{\vec{k}}), \tag{3.9}$$

where

$$Z_{\vec{k}}^{-1} = 1 - \left(\frac{\partial\Sigma_R(\vec{k}, w)}{\partial w}\right)_{w=\epsilon_{\vec{k}}} \tag{3.10}$$

is the quasiparticle residue.[18]

We can also write

$$\epsilon_{\vec{k}} = \mu + \delta k \mid \nabla \epsilon_{\vec{p}_F} \mid \tag{3.11}$$

where we used (3.6), (3.7), and (3.8).

Substituting these approximations in the singular part of the integral (3.3) and changing variables with respect to the chemical potential we get

$$E^s_{\vec{k}} = \int_0^{\zeta} \frac{dw}{\pi} \frac{\left(\mu + \epsilon^0_{\vec{k}} - w\right)}{2} \times \frac{\Sigma_I(\vec{k}, \mu - w)}{\left[Z^{-2}_{\vec{k}} \left(w + \delta k \mid \nabla \epsilon_{\vec{p}_F} \mid\right)^2 + \Sigma_I(\vec{k}, \mu - w)^2\right]}. \tag{3.12}$$

For a Fermi liquid we use the fact that the imaginary part of the self-energy can be written as[16–18]

$$\Sigma_I(\vec{k}, w) = C_{\vec{k}}(w - \mu)^2 \mathrm{sgn}(\mu - w), \tag{3.13}$$

where $C_{\vec{k}}$ depends only on $\vec{k}$ and $\mathrm{sgn}(x) = 1$ (-1) if $x > 0$ $(x < 0)$.

Substituting (3.13) into (3.12), changing the variables of integration from w to $x = \frac{|\delta k||\nabla \epsilon_{\vec{p}_F}|}{w}$, we found

$$E^s_{\vec{k}} = \int_{\frac{|\delta k||\nabla \epsilon_{\vec{p}_F}|}{\zeta}}^{\infty} \frac{dx}{2\pi} \left(\mu + \epsilon^0_{\vec{k}} - \frac{1}{x} \mid \delta k \mid\mid \nabla \epsilon_{\vec{p}_F} \mid\right) \times \frac{C_{\vec{k}} \mid \delta k \mid\mid \nabla \epsilon_{\vec{p}_F} \mid}{\left\{Z^{-2}_{\vec{k}} x^2 [x + \mathrm{sgn}(\delta k)]^2 + C^2_{\vec{k}} \delta k^2 \mid \nabla \epsilon_{\vec{p}_F} \mid^2\right\}}. \tag{3.14}$$

Observe now that if we let $\delta k \to 0$ we can use the well known expression

$$\lim_{|\delta k| \to 0} \frac{C_{\vec{k}} \mid \delta k \mid\mid \nabla \epsilon_{\vec{p}_F} \mid}{\left\{Z^{-2}_{\vec{k}} x^2 [x + \mathrm{sgn}(\delta k)]^2 + C^2_{\vec{k}} \delta k^2 \mid \nabla \epsilon_{\vec{p}_F} \mid^2\right\}} = \pi \mid Z_{\vec{p}_F} \mid \{\delta[x + \mathrm{sgn}(\delta k)] - \delta(x)\}. \tag{3.15}$$

Substituting this result in (3.14), we find

$$E^s_{\vec{k}} = \mid Z_{\vec{p}_F} \mid \frac{\left(\epsilon^0_{\vec{p}_F} + \mu\right)}{2} \Theta(-\delta k), \tag{3.16}$$

where $\Theta(x) = 1(0)$ if $x > 0$ $(x < 0)$.

From (3.16) we conclude, therefore, that there is a discontinuity at the Fermi surface which is given by

$$\Delta E(\vec{p}_F) = \mid Z_{\vec{p}_F} \mid \frac{\left(\epsilon^0_{\vec{p}_F} + \mu\right)}{2}. \tag{3.17}$$

Therefore, there is no singularity when the quasiparticle residue vanishes and we have non-Fermi liquid behavior. In the noninteracting system it is easy to see that this discontinuity is simply given by μ, which is the energy needed to put another electron in the system.

We can also show (see Appendix A) that this result holds true if the imaginary part of the self-energy goes like $(w - \mu)^{1+\nu}$ with ν greater than zero. However, if ν is smaller than zero we can prove that the discontinuity in the $E(\vec{k})$ vanishes as $\delta k^{(1-\nu)^2}$ as $\delta k \to 0$. Therefore, $\nu = 0$ is a critical point (this is the case of a marginal Fermi liquid phenomenology[19]).

We can interpret this discontinuity in $E(\vec{k})$ as being due to a surface tension in momentum space, since $E(\vec{k})$ is the contribution of the mode $\vec{k}$ to the total energy, as we see by (3.1). The surface tension in three dimensions can be defined by the difference of energy density across the Fermi surface as follows:[20]

$$\sigma_F \left(\frac{1}{R^1_F} + \frac{1}{R^2_F}\right) = \frac{\Delta E(\vec{p}_F)}{V_F}, \tag{3.18}$$

where R^1_F and R^2_F are the principal radii of curvature at $\vec{p}_F$ and V_F is the volume of the Fermi sea.

In two dimensions the analog of (3.18) is

$$\frac{\sigma_F}{R_F} = \frac{\Delta E(\vec{p}_F)}{A_F}, \tag{3.19}$$

where A_F is the area of the Fermi sea.

Comparing (3.18) or (3.19) with (3.17) we conclude that the surface tension is proportional to the quasiparticle residue. Therefore, for a non-Fermi liquid behavior the surface tension vanishes with the quasiparticle residue. From this observation we can conclude that a transition from Fermi liquid to non-Fermi liquid behavior can be viewed as a phase transition in momentum space for the case where the quasiparticle residue can be tuned to zero adiabatically.

IV. BOSONIZATION AND COHERENT STATES

In this section we review and generalize the method of bosonization of Fermi fluids of Ref. 13. Our starting point to approach the bosonization resembles the microscopic approaches for the foundations of Fermi liquid theory.[18] However, instead of working with the dynamics of the response functions, we will work directly with the properties of the operators that generate the physical spectrum, in a restricted Hilbert space of states of the filled Fermi sea. This is the standard procedure of bosonization in one dimension and in a recent work of Haldane.[10] From this perspective, the algebra obeyed by the operators is not a property of the operators themselves but a property of the states.

For simplicity, we will consider a system of interacting spinless fermions. Generalizations to systems with an internal symmetry, such as spin SU(2), can be done with some minor but important modifications. The density of fermions at some point $\vec{r}$ at some time t is given by

$$\rho(\vec{r}, t) = \psi^\dagger(\vec{r}, t)\psi(\vec{r}, t) = \sum_{\vec{k}, \vec{q}} c^\dagger_{\vec{k} - \frac{\vec{q}}{2}}(t) c_{\vec{k} + \frac{\vec{q}}{2}}(t)\, e^{i\vec{q}\cdot\vec{r}}, \tag{4.1}$$

where $c^\dagger_{\vec{k}}$ and $c_{\vec{k}}$ are the creation and annihilation operators of an electron, at some momentum $\vec{k}$, which obey the fermionic algebra $\left\{c^\dagger_{\vec{k}}, c_{\vec{k}'}\right\} = \delta_{\vec{k},\vec{k}'}$, where $\{\cdots\}$ is the anticommutator and all other anticommutation relations are zero. In the Fermi liquid theory[15] the operator that

appears on the right-hand side (rhs) of (4.1) determines the behavior of the system. We will concentrate on this operator, which we will denote by

$$n_{\vec{q}}(\vec{k},t) = c^{\dagger}_{\vec{k}-\frac{\vec{q}}{2}}(t)c_{\vec{k}+\frac{\vec{q}}{2}}(t). \tag{4.2}$$

In particular, $n_0(\vec{k})$ is the number operator in momentum space. As in the standard approaches for the Fermi liquid theory, we will concentrate our interest in regions close to the Fermi surface ($\vec{k} \sim \vec{p}_F$) and consider long wavelength fluctuations around these regions ($\vec{q} \to 0$).

The equal-time commutation relation between the operators defined in (2) is easily obtained. In the long wavelength limit ($q \ll \Lambda$, where Λ is a cutoff) we get

$$\left[n_{\vec{q}}(\vec{k}), n_{-\vec{q}'}(\vec{k}')\right] = -\delta_{\vec{k},\vec{k}'}\delta_{\vec{q},\vec{q}'}\vec{q}\cdot\nabla n_0(\vec{k}) + 2n_0(\vec{k})\delta_{\vec{q},-\vec{q}'}\vec{q}\cdot\nabla\delta_{\vec{k},\vec{k}'}. \tag{4.3}$$

We are interested in the behavior of systems with a Hilbert space restricted to the vicinity of the Fermi surface. We define the Hilbert space to be the filled Fermi sea $|\mathrm{FS}\rangle$ and the tower of states obtained by acting finitely on it with local fermion operators. More specifically, we consider a shell of states of thickness D (measured in units of momentum) around the Fermi energy. Next we imagine dividing up the Fermi surface in patches[10,11] centered at points $\vec{k}$ on the Fermi surface of the filled Fermi sea. Each patch has thickness D and width Λ (in momentum units and tangent to the surface). Instead of the sharp operators $n_{\vec{q}}(\vec{k})$, we will consider operators *smeared* over each patch. We will replace Eq. (4.3) with a weaker identity valid in the restricted Hilbert space. Furthermore, we will make the explicit assumptions that we are in the thermodynamic limit (the momenta form a continuum) and that the Fermi surface is *macroscopically large* ($q \ll D \ll \Lambda \ll p_F$).

Since the vectors $\vec{k}$ are at the Fermi surface and $\vec{q}$ are very small, in the restricted Hilbert space it is possible to replace the rhs of (4.3) with its expectation value in the filled Fermi sea $|\mathrm{FS}\rangle$, namely,

$$\begin{aligned} n_0(\vec{k}) &\to \langle n_0(\vec{k})\rangle = \Theta(\mu - \epsilon_{\vec{k}}), \\ \nabla n_0(\vec{k}) &\to \nabla\langle n_0(\vec{k})\rangle = -\vec{v}_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}}), \end{aligned} \tag{4.4}$$

where μ is the chemical potential, $\epsilon_{\vec{k}}$ is the one-particle fermion spectrum (from which the Hilbert space is constructed), and $\vec{v}_{\vec{k}} = \nabla\epsilon_{\vec{k}}$ the velocity of the excitations. The terms ignored in (3) vanish as $\frac{\Lambda}{p_F^2}$ as the size of the Fermi surface diverges. More generally, all corrections vanish if the limit $q \ll D \ll \Lambda \ll p_F$ is satisfied. These are exactly the same assumptions that enter into the construction in one dimension (in Appendix B we prove the equivalence of our construction and the well known bosonization in one dimension). Notice that the state $|\mathrm{FS}\rangle$, which is used to normal order the operators, is not necessarily the ground state of the system of interest (as in the one-dimensional case). However, this approach will succeed only if the true ground state belongs to the restricted space.

Hence, within this approximation the commutators of the operators $n_{\vec{q}}(\vec{k})$ become c numbers, namely,

$$\left[n_{\vec{q}}(\vec{k}), n_{-\vec{q}'}(\vec{k}')\right] = \delta_{\vec{k},\vec{k}'}\delta_{\vec{q},\vec{q}'}\vec{q}\cdot\vec{v}_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}}), \tag{4.5}$$

where we drop the last term on the rhs of (4.3) because its matrix elements near the Fermi surface are down by powers of $\frac{\Lambda}{p_F}$. We now define the operator

$$a_{\vec{q}}(\vec{k}) = n_{\vec{q}}(\vec{k})\Theta[\mathrm{sgn}(q)] + n_{-\vec{q}}(\vec{k})\Theta[-\mathrm{sgn}(q)], \tag{4.6}$$

where $\mathrm{sgn}(q)$ is $+1$ if $\vec{q}$ points outside the Fermi surface and -1 if it points inside the Fermi surface at the point $\vec{k}$. The adjoint of (4.6) is simply

$$a^{\dagger}_{\vec{q}}(\vec{k}) = n_{-\vec{q}}(\vec{k})\Theta[\mathrm{sgn}(q)] + n_{\vec{q}}(\vec{k})\Theta[-\mathrm{sgn}(q)], \tag{4.7}$$

where we used, from the definition (4.2), that $n^{\dagger}_{\vec{q}}(\vec{k}) = n_{-\vec{q}}(\vec{k})$.

It is important to note that, by construction, the operator defined in (4.6) annihilates the filled Fermi sea

$$a_{\vec{q}}(\vec{k}) \mid \mathrm{FS}\rangle = 0. \tag{4.8}$$

Moreover, from the commutation relations (4.5) we easily obtain,

$$\left[a_{\vec{q}}(\vec{k}), a^{\dagger}_{\vec{q}'}(\vec{k}')\right] = \mid \vec{q}\cdot\vec{v}_{\vec{k}} \mid \delta(\mu-\epsilon_{\vec{k}})\delta_{\vec{k},\vec{k}'}\left(\delta_{\vec{q},\vec{q}'} + \delta_{\vec{q},-\vec{q}'}\right), \tag{4.9}$$

and all other commutators vanish.

Equations (4.8) and (4.9) show that the elementary excitations have bosonic character, particle-hole pairs, and they are created and annihilated close to the Fermi surface by these operators; moreover, they span the Hilbert space of low energy. But we need an interpretation for these excitations. It is natural now to define the coherent state[21]

$$\mid u_{\vec{q}}(\vec{k})\rangle = U(\vec{k}) \mid \mathrm{FS}\rangle, \tag{4.10}$$

where

$$U(\vec{k}) = \exp\left(\sum_{\vec{q}} \frac{v_{\vec{k}}}{2 \mid \vec{q}\cdot\vec{v}_{\vec{k}} \mid} u_{\vec{q}}(\vec{k})\, a^{\dagger}_{\vec{q}}(\vec{k})\right), \tag{4.11}$$

where the sum (and all other sums that follow) are restricted to $q \ll \Lambda$. Observe that from the definition (4.7) we have $a^{\dagger}_{-\vec{q}}(\vec{k}) = a^{\dagger}_{\vec{q}}(\vec{k})$ and we choose $u_{\vec{q}}(\vec{k}) = u_{-\vec{q}}(\vec{k})$. Using this property and the commutation relation (4.9) we find

$$U^{-1}(\vec{k})\, a_{\vec{q}}(\vec{k})\, U(\vec{k}) = a_{\vec{q}}(\vec{k}) + \delta(\mu-\epsilon_{\vec{k}})v_{\vec{k}}u_{\vec{q}}(\vec{k}), \tag{4.12}$$

which, together with (4.8), leads us to the eigenvalue equation

$$a_{\vec{q}}(\vec{k}) \mid u_{\vec{q}}(\vec{k})\rangle = \delta(\mu-\epsilon_{\vec{k}})\, v_{\vec{k}}\, u_{\vec{q}}(\vec{k}) \mid u_{\vec{q}}(\vec{k})\rangle. \tag{4.13}$$

It is easy to see that $u_{\vec{q}}(\vec{k})$ is the displacement of the Fermi surface at the point $\vec{k}$ in the direction of $\vec{v}_{\vec{k}}$. Indeed,[22] suppose we change the shape of the Fermi surface at some point $\vec{k}$ by an amount $u(\vec{k})$. The occupation number changes up to leading order by $\delta\langle n_0(\vec{k})\rangle = -\frac{\partial\langle n_0(\vec{k})\rangle}{\partial\vec{k}}u(\vec{k}) = v_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}})u(\vec{k})$, which is precisely the quantity that appears in (4.13). Hence, the coherent states of (4.10) represent *deformed Fermi surfaces* parametrized by the bosonic field $u_{\vec{q}}(\vec{k})$.

Next we define a many-body state which is a direct product of the coherent states defined above,

$$\mid\{u\}\rangle = \prod_{\vec{k}}\otimes U(\vec{k})\mid \mathrm{FS}\rangle = \Xi[u]\mid \mathrm{FS}\rangle \tag{4.14}$$

where, due to the commutation relation at different $\vec{k}$'s,

$$\Xi[u] = \exp\left(\sum_{\vec{k},\vec{q}}\frac{v_{\vec{k}}}{2\mid\vec{q}\cdot\vec{v}_{\vec{k}}\mid}u_{\vec{q}}(\vec{k})\,a^{\dagger}_{\vec{q}}(\vec{k})\right). \tag{4.15}$$

The adjoint is simply

$$\Xi^{\dagger}[u] = \exp\left(\sum_{\vec{k},\vec{q}}\frac{v_{\vec{k}}}{2\mid\vec{q}\cdot\vec{v}_{\vec{k}}\mid}u^{*}_{\vec{q}}(\vec{k})\,a_{\vec{q}}(\vec{k})\right). \tag{4.16}$$

From the above equations we obtain the overlap of two of these coherent states,

$$\begin{aligned}\langle\{w\}\mid\{u\}\rangle &= \langle \mathrm{FS}\mid\Xi^{\dagger}[w]\Xi[u]\mid\mathrm{FS}\rangle \\ &= \exp\left(\sum_{\vec{k},\vec{q}}\frac{v^2_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}})}{2\mid\vec{q}\cdot\vec{v}_{\vec{k}}\mid}w^{*}_{\vec{q}}(\vec{k})\,u_{\vec{q}}(\vec{k})\right).\end{aligned} \tag{4.17}$$

It is also possible to find the resolution of the identity for this Hilbert space,

$$1 = \int\cdots\int\prod_{\vec{k},\vec{q}}\left(\frac{v^2_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}})}{2\pi\mid\vec{q}\cdot\vec{v}_{\vec{k}}\mid}du_{\vec{q}}(\vec{k})\,du^{*}_{\vec{q}}(\vec{k})\mid u_{\vec{q}}(\vec{k})\rangle\langle u_{\vec{q}}(\vec{k})\mid\right)\exp\left(-\sum_{\vec{k},\vec{q}}\frac{v^2_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}})}{2\mid\vec{q}\cdot\vec{v}_{\vec{k}}\mid}\mid u_{\vec{q}}(\vec{k})\mid^2\right), \tag{4.18}$$

and we conclude, as expected, that they are overcomplete.[23]

In order to study the dynamics of these modes, we can construct, from (4.17) and (4.18), a generating functional as a sum over the histories of the Fermi surface in terms of these coherent states in the form $Z = \int D^2u\,\exp\{i\,S[u]\}$, where S is the action whose Lagrangian density is given by ($\hbar = 1$),

$$\begin{aligned}L[u] = &\sum_{\vec{k},\vec{q}}\frac{v^2_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}})}{2\mid\vec{q}\cdot\vec{v}_{\vec{k}}\mid}\,i\,u^{*}_{\vec{q}}(\vec{k},t)\frac{\partial u_{\vec{q}}(\vec{k},t)}{\partial t} \\ &-\frac{\langle\{u\}\mid H\mid\{u\}\rangle}{\langle\{u\}\mid\{u\}\rangle},\end{aligned} \tag{4.19}$$

where H is the Hamiltonian written in the restricted Hilbert space which will be studied in the next section.

V. THE BOSONIC HAMILTONIAN

Consider a fermionic system described by a Hamiltonian of the form $H = K + U$ where,

$$K = \sum_{\vec{p}}\epsilon_{\vec{p}}\,c^{\dagger}_{\vec{p}}c_{\vec{p}} \tag{5.1}$$

is the noninteracting term and

$$U = \frac{1}{2V}\sum_{\vec{p},\vec{p}',\vec{q}}f_{\vec{p}-\vec{q},\vec{p}'+\vec{q}}\;c^{\dagger}_{\vec{p}+\frac{\vec{q}}{2}}c_{\vec{p}-\frac{\vec{q}}{2}}c^{\dagger}_{\vec{p}'-\frac{\vec{q}}{2}}c_{\vec{p}'+\frac{\vec{q}}{2}} \tag{5.2}$$

is the interaction between the fermions. We will assume that the interaction between the fermions is important only close to the Fermi surface. This means that the vectors $\vec{p}$ and $\vec{p}'$ are at the Fermi surface in what follows.

We see that the interaction is given in terms of the bosonic operators as

$$U = \frac{1}{2V}\sum_{\vec{p},\vec{p}',\vec{q}}f_{\vec{p}-\vec{q},\vec{p}'+\vec{q}}\;n_{-\vec{q}}(\vec{p})n_{\vec{q}}(\vec{p}'). \tag{5.3}$$

Of course, the free part K cannot be written directly in terms of the bosons, since it is quadratic from the beginning. However, since we are in a restricted Hilbert space, what really matters is the effective dynamics that it can generate. We know that the operators $n_{\vec{q}}(\vec{k})$ generate the restricted space; therefore, it is natural to look at commutation relation between K and $n_{\vec{q}}(\vec{k})$. Up to first order in q we find

$$\left[K, n_{\vec{q}}(\vec{k})\right] = -\vec{q}\cdot\vec{v}_{\vec{k}}n_{\vec{q}}(\vec{k}). \tag{5.4}$$

Observe that in order to keep the vector $\vec{k}$ at the Fermi surface we have to multiply the above expression by a distribution function of the form $\delta(\mu-\epsilon_{\vec{k}})/N(0)V$, where

$$N(0) = \frac{1}{V}\sum_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}}) \tag{5.5}$$

is the density of states at the Fermi surface.

Now we notice that there is another operator that gives the same commutation as in (5.4) in the restricted Hilbert space, namely, using (4.5) we find

$$\left[\sum_{\vec{p},\vec{q}} n_{-\vec{q}}(\vec{p}) n_{\vec{q}}(\vec{p}), n_{\vec{q}}(\vec{k})\right] = -2\,\vec{q}\cdot\vec{v}_{\vec{k}}\, n_{\vec{q}}(\vec{k})\,\delta(\mu-\epsilon_{\vec{k}}), \tag{5.6}$$

which means that in the restricted Hilbert space we can rewrite

$$K = \frac{1}{2N(0)V}\sum_{\vec{k},\vec{q}} n_{-\vec{q}}(\vec{k}) n_{\vec{q}}(\vec{k}). \tag{5.7}$$

Using the definition (4.6) and Eqs. (5.3) and (5.7), we conclude that in the restricted Hilbert space, the Hamiltonian for the interacting fermions in simply given by

$$H = \frac{1}{2V}\sum_{\vec{k},\vec{k}',\vec{q},\mathrm{sgn}(q)>0} G^{\vec{q}}_{\vec{k},\vec{k}'} a^{\dagger}_{\vec{q}}(\vec{k})\, a_{\vec{q}}(\vec{k}'), \tag{5.8}$$

where

$$G^{\vec{q}}_{\vec{k},\vec{k}'} = \frac{\delta_{\vec{k},\vec{k}'}}{N(0)} + f_{\vec{k}-\vec{q},\vec{k}'+\vec{q}}, \tag{5.9}$$

which describes a free, quadratic, theory.

It is worth noting the similarity between (5.8) and (5.9) and Pomeranchuk's expression (2.8). This resemblance is expected, since we are looking at the same kind of fluctuations. We would say that we have a quantum version of Pomeranchuk's construction. Moreover, all the scaling arguments of Sec. II can be used here in order to understand the stability of the Fermi liquid as a renormalization group fixed point.

VI. FERMI LIQUID PROPERTIES

This section is devoted to showing that we can get *all* the results of the Fermi liquid theory from the point of view of the bosons. In particular, the relationship between the bare mass of the fermions and the effective mass of the Landau theory is discussed in Appendix C. We start by studying the thermodynamics (imaginary time) of the system and then we study its dynamics (real time) from the point of view of our path integral.

First we will rewrite the Lagrangian in (4.19) using the Hamiltonian we have just derived in the preceding section. Namely, we substitute (5.8) in (4.19) and use the eigenvalue equation (4.13) in order to get

$$L[u] = \sum_{\vec{k},\vec{k}',\vec{q},\mathrm{sgn}(q)>0} \frac{v_{\vec{k}}^2\,\delta(\mu-\epsilon_{\vec{k}})}{2\,\vec{q}\cdot\vec{v}_{\vec{k}}}\, u^{*}_{\vec{q}}(\vec{k},t) \times\left(i\delta_{\vec{k},\vec{k}'}\frac{\partial}{\partial t} - \frac{1}{V}W^{\vec{q}}_{\vec{k},\vec{k}'}\delta(\mu-\epsilon_{\vec{k}'})\right) u_{\vec{q}}(\vec{k}',t), \tag{6.1}$$

where

$$W^{\vec{q}}_{\vec{k},\vec{k}'} = \frac{\vec{q}\cdot\vec{v}_{\vec{k}}\; v_{\vec{k}'}}{v_{\vec{k}}} G^{\vec{q}}_{\vec{k},\vec{k}'}. \tag{6.2}$$

Observe that the vectors $\vec{k}$ and $\vec{k}'$ in the above expressions are restricted to the Fermi surface due to Dirac's delta function. We therefore parametrize these vectors by the solid angles Ω and Ω', respectively. We will assume throughout this paper that the density of states is a smooth function across the Fermi surface (which means that there is no Van Hove singularity present). In our language we make the following substitution:

$$\sum_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}}) f_k = \frac{N(0)V}{S_d}\int d\Omega f(\Omega), \tag{6.3}$$

where S_d is the integral over the whole solid angle, $S_d = \int d\Omega$. Furthermore, using the fact that the density of states is smooth we can replace the velocity with

$$\vec{v}_{\vec{k}} = v_F\vec{n}_{\vec{k}} \tag{6.4}$$

where $\vec{n}_{\vec{k}}$ is an unit vector perpendicular to the Fermi surface at the point $\vec{k}$.

With these substitutions we can rewrite the Lagrangian (6.1) as

$$L = \frac{N(0)V}{S_d}\int d\Omega\int d\Omega' \sum_{\vec{q},\mathrm{sgn}(q)>0}\frac{v_F}{2\vec{q}\cdot\vec{n}(\Omega)} u^{*}_{\vec{q}}(\Omega) \times\left(i\delta_{\Omega,\Omega'}\frac{\partial}{\partial t} - W_{\vec{q}}(\Omega,\Omega')\right) u_{\vec{q}}(\Omega'), \tag{6.5}$$

where

$$W_{\vec{q}}(\Omega,\Omega') = v_F\vec{q}\cdot\vec{n}(\Omega)\left(\delta_{\Omega,\Omega'} + \frac{F_{\vec{q}}(\Omega,\Omega')}{S_d}\right) \tag{6.6}$$

and $F_{\vec{k},\vec{k}'} = N(0) f_{\vec{k},\vec{k}'}$.

Observe that the field $u_{\vec{q}}(\Omega)$ has dimensions of momentum, as required by our interpretation; however, we can rewrite the action in terms of a new dimensionless bosonic field $\phi_{\vec{q}}(\Omega)$ by a simple scaling of the original field as

$$\phi_{\vec{q}}(\Omega) = \left(\frac{v_F N(0)V}{2S_d|\vec{q}\cdot\vec{n}(\Omega)|}\right)^{1/2} u_{\vec{q}}(\Omega) \tag{6.7}$$

and rewrite the generating functional as

$$Z = \int\cdots\int\prod_{\vec{q},\Omega,t,\mathrm{sgn}(q)>0}\left(\frac{d\phi_{\vec{q}}(\Omega,t)d\phi^{*}_{\vec{q}}(\Omega,t)}{2\pi}\right) e^{iS[\phi,\phi^{*}]}, \tag{6.8}$$

where

$$S = \int dt\left(\int d\Omega\sum_{\vec{q},\mathrm{sgn}(q)>0} i\phi^{*}_{\vec{q}}(\Omega)\frac{\partial\phi_{\vec{q}}(\Omega)}{\partial t} - \int d\Omega\int d\Omega'\sum_{\vec{q},\mathrm{sgn}(q)>0}\phi^{*}_{\vec{q}}(\Omega) M^{\vec{q}}(\Omega,\Omega')\phi_{\vec{q}}(\Omega')\right), \tag{6.9}$$

where

$$M^{\vec{q}}(\Omega,\Omega') = \frac{N(0)v_F}{S_d}\sqrt{|\vec{q}\cdot\vec{n}(\Omega)||\vec{q}\cdot\vec{n}(\Omega')|} \times\left(\frac{S_d\delta(\Omega-\Omega')}{N(0)} + f(\Omega,\Omega')\right). \tag{6.10}$$

We can split the field $\phi_{\vec{q}}(\Omega)$ into real and imaginary parts as

$$\phi_{\vec{q}}(\Omega) = \varphi_{\vec{q}}(\Omega) + i\eta_{\vec{q}}(\Omega) \tag{6.11}$$

and we trace over the imaginary part in order to get

$$Z = (\det M)^{-1/2} \int \cdots \int \prod_{\vec{q},\Omega,t,\mathrm{sgn}(q)>0} \times \left[\left(\frac{i}{\pi} \right)^{1/2} d\varphi_{\vec{q}}(\Omega,t) \right] e^{iS_{\mathrm{eff}}[\varphi]} \tag{6.12}$$

where,

$$S_{\mathrm{eff}} = \int dt \int d\Omega \int d\Omega' \sum_{\vec{q},\mathrm{sgn}(q)>0} \times \left(\frac{\partial \varphi_{\vec{q}}(\Omega)}{\partial t} \left[M^{\vec{q}}(\Omega,\Omega') \right]^{-1} \frac{\partial \varphi_{\vec{q}}(\Omega')}{\partial t} - \varphi_{\vec{q}}(\Omega) M^{\vec{q}}(\Omega,\Omega') \varphi_{\vec{q}}(\Omega') \right). \tag{6.13}$$

We proceed further and expand the action in its eigenmodes by rewriting

$$\varphi_{\vec{q}}(\Omega,t) = \sum_{\lambda} v_{\vec{q}}^{\lambda}(\Omega) \Phi_{\vec{q}}^{\lambda}(t), \tag{6.14}$$

where

$$\int d\Omega' M^{\vec{q}}(\Omega,\Omega') v_{\vec{q}}^{\lambda}(\Omega') = \omega_{\vec{q}}^{\lambda} v_{\vec{q}}^{\lambda}(\Omega). \tag{6.15}$$

Equation (6.15) can also be rewritten in terms of the original fields if we use (6.7) and (6.10),

$$v_F \, \vec{q} \cdot \vec{n}(\Omega) \left(u_{\vec{q}}^{\lambda}(\Omega) + \int \frac{d\Omega'}{S_d} F(\Omega,\Omega') u_{\vec{q}}^{\lambda}(\Omega') \right) = \omega_{\vec{q}}^{\lambda} u_{\vec{q}}^{\lambda}(\Omega). \tag{6.16}$$

Using the completeness of the states defined in (6.15) we rewrite the generating functional as

$$Z = (\det M)^{-1/2} \int \cdots \int \prod_{\vec{q},\lambda,t} \left[\left(\frac{i}{\pi} \right)^{1/2} d\Phi_{\vec{q}}^{\lambda}(t) \right] e^{iS_{\mathrm{eff}}[\Phi]}, \tag{6.17}$$

where

$$S_{\mathrm{eff}} = \int dt \sum_{\lambda,\vec{q},\mathrm{sgn}(q)>0} \frac{1}{\omega_{\vec{q}}^{\lambda}} \left[\left(\frac{\partial \Phi_{\vec{q}}^{\lambda}(t)}{\partial t} \right)^2 - \left(\omega_{\vec{q}}^{\lambda} \right)^2 \left[\Phi_{\vec{q}}^{\lambda}(t) \right]^2 \right], \tag{6.18}$$

which, as expected, is the action for a free massive bosonic field.

A. Thermodynamic properties

The extension of the Lagrangian (6.5) to imaginary time (i.e., finite temperature) is trivial. We will consider first the case in which only the diagonal term in Eq. (6.6) is present, namely,

$$W_{\vec{q}}(\Omega,\Omega') = E_{\vec{q}}(\Omega) \delta_{\Omega,\Omega'}, \tag{6.19}$$

and therefore it is straightforward to show that the partition function is given by

$$Z = \exp(-\beta F) = \prod_{\Omega,\vec{q},\mathrm{sgn}(q)>0} \left[\sinh \left(\frac{E_{\vec{q}}(\Omega)\beta}{2} \right) \right]^{-1}, \tag{6.20}$$

where $\beta = 1/T$ is the inverse of temperature (our units are such that $k_B = \hbar = 1$).

Using the expression for the free energy F given in (6.20), we easily calculate the specific heat,

$$C_V = \frac{\beta^2}{4} \sum_{\Omega,\vec{q},\mathrm{sgn}(q)>0} \frac{E_{\vec{q}}^2(\Omega)}{\sinh^2 \left(\frac{E_{\vec{q}}(\Omega)\beta}{2} \right)}. \tag{6.21}$$

We have to be careful in order to understand the integrals in (6.21). We built a reference frame that runs over the surface (a Fresnel frame as in differential geometry). The Fermi velocity (normal to the Fermi surface at $\vec{p}_F$) and gradient of the Fermi velocity (which spans the vectors in the plane tangent to the Fermi surface at $\vec{p}_F$) form a *local* orthogonal basis for the vectors $\vec{q}$. We therefore split the integral in (6.21) into one part which is normal to the surface, q_N, and another which is in the plane tangent to the surface, q_P. In the absence of interactions the dispersion is given by (6.6) as

$$E(q_N) = v_F \, q_N. \tag{6.22}$$

First we observe that the density of states is given as an integral over the tangent component. Indeed, from (5.5) and the definition of the local frame, we have

$$N(0) = \frac{1}{S_d} \int \frac{dS}{v_F} = \frac{1}{S_d} \int d\Omega \int dq_P(\Omega) \frac{q_P^{d-2}(\Omega)}{v_F}. \tag{6.23}$$

We also have assumed that the normal component is not affected by the curvature [see (6.4)]. Thus we can integrate out the tangent component in (6.21) using (6.23). The final result is,

$$\frac{C_V}{V} = \frac{\beta^2}{4} N(0) v_F \int_0^{\infty} dq_N \frac{E^2(q_N)}{\sinh^2 \left(\frac{E(q_N)\beta}{2} \right)}. \tag{6.24}$$

Using (6.22) we finally find

$$\frac{C_V}{V} = \frac{\pi^2}{3} N(0) T, \tag{6.25}$$

which is the expected result for a Fermi liquid.[15] Observe that the dimensionality plays no role here except in the calculation of the density of states.

We can clearly see the differences between the bosonic excitations described in this paper, which are due to coherent superposition of particle-hole pairs, and the usual free bosonic excitations. First of all, naively we would expect that the specific heat should behave as T^3 as in usual free bosonic theories, since the bosonic Hamiltonian (5.8) is quadratic in the bosons. However, the bosonic field in the Fermi liquid *lives* on the Fermi surface, that is, it is topologically constrained. The Fermi velocity, through the density of states, defines the metric of the manifold where the fields propagate. In the case without interactions the bosonic fields oscillate without coherence (like decoupled harmonic oscillators with phases distributed at random). These oscillations are responsible for the contribution of the particle-hole continuum to the specific heat as shown in (6.25).

We can proceed further and evaluate the effect of the interactions in this problem. The general approach would be to calculate the eigenvalues of Eq. (6.15) and evaluate the bosonic determinant in the partition function. However, we use an approach inspired by the calculations of the Fermi liquid theory.[24] First, we expand the interaction term for small angles by defining the vector $\vec{p} = \vec{k}' - \vec{k}$ such that

$$f_{\Omega,\Omega'} = a + b\left(\frac{\vec{k}}{k}\cdot\frac{\vec{p}}{p}\right), \tag{6.26}$$

where a and b are parameters that depend on the specific form of the interaction. Due to the geometry of the interaction we can rewrite (6.26) as[11]

$$f_{\Omega,\Omega'} = a + b\frac{q_N^2}{q_N^2 + p^2/4}. \tag{6.27}$$

Since we are working with a local expansion, we will replace the nonlocal term in (6.6) with a local one which is an average over the surface,

$$f_{\Omega,\Omega'} \to \left(\int d\Omega' f_{\Omega,\Omega'}\right)\delta_{\Omega,\Omega'} \to b\sum_{\vec{p}}\frac{q_N^2}{q_N^2 + p^2/4}\delta_{\Omega,\Omega'}. \tag{6.28}$$

Changing the sum to an integral and taking into account that this integral must be done on the surface, we found that the change in the energy due to interactions can be written as

$$\Delta E(q_N) = 16 v_F b q_N^3 \frac{N(0) S_{d-1}}{p_F^{d-1} S_d}\int_0^\Lambda dp \frac{p^{d-2}}{p^2 + 4q_N^2}, \tag{6.29}$$

where Λ is the cutoff. In three dimensions, we have

$$E(q_N) = v_F q_N - \frac{8 b v_F N(0)}{p_F^2} q_N^3 \ln\left(\frac{q_N}{\Lambda}\right) \tag{6.30}$$

and in two dimensions one finds

$$E(q_N) = v_F q_N + \pi b \frac{v_F N(0)}{p_F} q_N^2. \tag{6.31}$$

Observe that these expressions depend only on the normal component and we can use Eq. (6.24) to calculate the corrections for the specific heat due to interactions. Up to first order in b we have

$$\frac{\delta C_V}{V} = -\alpha T^3 \ln\left(\frac{T}{\Lambda v_F}\right) \tag{6.32}$$

in three dimensions and

$$\frac{\delta C_V}{V} = \eta T^2 \tag{6.33}$$

in two dimensions, where

$$\alpha = \frac{16 p_F^2 b}{15\pi^2 v_F^4}, \tag{6.34}$$

and

$$\eta = \frac{3\zeta(3) p_F b}{\pi v_F^3}, \tag{6.35}$$

where $\zeta(n)$ is the Riemann ζ function of n.

The result (6.32) is well known[15,24] [apart from the prefactor, which is different due to the average over the Fermi surface we have carried out in (6.28)] and the result (6.33) was obtained recently using the RPA approximation.[25]

We conclude, therefore, that in our approximation we are counting the correct number of states at low temperatures. This result proves the consistency of our method.

B. The semiclassical dynamics

Notice that the Lagrangian (6.5) is quadratic in the fields and, therefore, the semiclassical approximation is exact. The semiclassical equations of motion for these Lagrangians are given by the saddle point equation (the Euler-Lagrange equations) derived from L, namely,

$$i\frac{\partial u_{\vec{q}}(\Omega,t)}{\partial t} = q v_F \cos\theta\, u_{\vec{q}}(\Omega,t) + q v_F \cos\theta \int \frac{d\Omega'}{S_d} F(\Omega,\Omega')\, u_{\vec{q}}(\Omega',t), \tag{6.36}$$

where we have defined the angle θ by $\vec{q}\cdot\vec{n}(\Omega) = \cos\theta$. Observe that if

$$u_{\vec{q}}(\Omega,t) = e^{-i\omega_{\vec{q}}^{\lambda} t} u_{\vec{q}}^{\lambda}(\Omega), \tag{6.37}$$

we recover Eq. (6.16).

Equation (6.36) is the Landau equation of motion for sound waves (the collective modes) of a neutral Fermi liquid where $f_{\vec{k},\vec{k}'}$ is the scattering amplitude for particle-hole pairs.[15] Observe that the Landau equation gives the eigenmodes of the problem [see (6.15)].

We conclude, therefore, that our bosons are the sound waves that propagate around the Fermi surface at zero temperature. The solution of (6.16) [or (6.36)] will give the possible values for the frequencies of oscillation for these modes and they will depend essentially on the Landau parameters of the theory. The Landau equation, (6.16), yields solutions that represents both stable collective modes ("sounds") and solutions with imaginary frequency that represent the particle-hole continuum. This behavior is a direct consequence of the phase space. Note that in one dimension these unstable solutions are absent and only the collective mode is left.

In order to illustrate the behavior of these sound modes we present here an explicit calculation in two dimensions. This calculation will enable us to understand many interesting features of these sound modes.

As in Sec. II, we expand the interaction and the displacement in Fourier components,

$$F(\theta,\theta') = \sum_{n=-\infty}^{\infty} F_n e^{i(\theta-\theta')n}, \tag{6.38}$$

where $F_{-n} = F_n^* = F_n$ due to the symmetry of the problem, and

$$u(\theta) = \sum_{n=-\infty}^{\infty} u_n e^{i\theta n}, \tag{6.39}$$

where $u_{-n} = u_n^*$.

From (6.16) we have

$$(s-\cos\theta)u(\theta) = \cos\theta \int_0^{2\pi} \frac{d\theta'}{2\pi} F(\theta,\theta')u(\theta'), \tag{6.40}$$

where $v_F s = \omega/q$ is the velocity of the sound waves.

Using (6.38), (6.39), and the orthogonality relation of the Fourier components it is easy to show that (6.40) can be written as a matrix equation,

$$(1+F_n)u_n = \sum_{m=-\infty}^{\infty} F_m K(m-n)u_m, \tag{6.41}$$

where

$$K(n) = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{s}{s-\cos\theta} e^{in\theta}. \tag{6.42}$$

We allow s to have an infinitesimal imaginary part and we integrate (6.42) into the complex plane in a contour in the unit circle around the origin. We found

$$K(n) = \frac{|s|}{\sqrt{s^2-1}} (\mathrm{sgn}(s))^n \left(s-\sqrt{s^2-1}\right)^{|n|}, \quad |s|>1;$$

$$= \frac{s}{i\sqrt{1-s^2}} \left(s-i\sqrt{1-s^2}\right)^{|n|}, \quad |s|<1. \tag{6.43}$$

It is easy to see from (6.43) that all the modes are damped if the velocity is smaller than the Fermi velocity $|s|<1$. This result is clearly expected; it is nothing but the Landau damping of the collective modes.[15]

Here we are only interested in the stable solutions of (6.41). Therefore, for $s>1$ we can replace (6.41) with

$$(1+F_n)u_n = \sum_{m=-\infty}^{\infty} F_m K(m-n)u_m, \tag{6.44}$$

where we define a kernel,

$$K(n) = A(s)e^{-|n|\alpha(s)}, \tag{6.45}$$

with

$$A(s) = \frac{s}{\sqrt{s^2-1}} \tag{6.46}$$

and

$$\alpha(s) = \ln\left(s+\sqrt{s^2-1}\right). \tag{6.47}$$

For a purely local interaction in the real space, the interaction is angle independent (long range over the surface in momentum space) and we define

$$F_n = F_0\delta_{n,0}. \tag{6.48}$$

Substituting (6.48) into (6.44) for the $n=0$ we obtain the allowed value of s,

$$s = \frac{1+\frac{1}{F_0}}{\sqrt{\left(1+\frac{1}{F_0}\right)^2-1}} \tag{6.49}$$

and for $n\neq 0$ we obtain the value of the Fourier components

$$u_n = F_0 K(n) u_0. \tag{6.50}$$

Substituting (6.50) into (6.39) and summing up an ordinary geometric series with a little help from (6.49), we easily get

$$u(\theta) = u_0 F_0 \frac{\cos\theta}{s-\cos\theta}, \tag{6.51}$$

which is the expected result if we have just solved (6.40) for a constant interaction. The result (6.51) is the well known zero sound mode, which has the same shape as the zero sound in the three-dimensional case.[15]

We can explore even further the matrix equation (6.44) due to the simplicity of the kernel in (6.45). We are mainly interested in the behavior of the Fermi surface in the forward scattering direction since it has been argued[9] that it is in this direction that pathological effects can happen.

Since we are not interested in the behavior over the whole Fermi surface, we will transform the matrix equation (6.44) into a simple second order differential equation by expanding the kernel in terms of the Fermi wavelength.

Consider the arc length defined on the Fermi surface by

$$\nu = p_F\theta. \tag{6.52}$$

The expansion (6.39) is rewritten as

$$u(\nu) = \sum_{\lambda} e^{i\lambda\nu} u(\lambda) \tag{6.53}$$

and, due to the periodic boundary conditions $u(\nu + 2\pi p_F) = u(\nu)$, we must have

$$\lambda_n = \frac{n}{p_F}. \tag{6.54}$$

Observe from (6.54) that in the limit of $p_F \to \infty$, the matrix equation (6.44) can be replaced by an integral equation,

$$[1 + F(\lambda)]u(\lambda) = \int_{-\infty}^{+\infty} d\lambda' G(\lambda - \lambda') F(\lambda') u(\lambda'), \tag{6.55}$$

where

$$G(\lambda) = 2\frac{A}{\alpha} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \frac{e^{ix\lambda}}{1 + \left(\frac{x}{p_F \alpha}\right)^2}, \tag{6.56}$$

where we have used (6.45) and an integral representation for the exponential function. Expanding (6.56) up to second order in p_F^{-1} one finds

$$G(\lambda) = 2\frac{A}{\alpha} \left(\delta(\lambda) + \frac{1}{(p_F \alpha)^2} \frac{d^2 \delta(\lambda)}{d\lambda^2} \right). \tag{6.57}$$

If we substitute (6.57) into the original equation (6.55) we find a differential equation which should be valid for small arc lengths compared to $2\pi p_F$. We proceed further and define the following quantity,

$$z(\lambda) = F(\lambda) u(\lambda), \tag{6.58}$$

which, from the arguments given above, obeys the differential equation

$$\left(-\frac{d^2}{d\lambda^2} + U(s, \lambda) \right) z(\lambda) = \epsilon(s) z(\lambda), \tag{6.59}$$

where

$$U(s, \lambda) = \frac{[p_F \alpha(s)]^2}{F(\lambda)} \tag{6.60}$$

and

$$\epsilon(s) = [p_F \alpha(s)]^2 \left(2\frac{A(s)}{\alpha(s)} - 1 \right). \tag{6.61}$$

Equation (6.59) has the form of a time-independent Schrödinger equation for a particle in a potential U with energy ϵ. Once we know the boundary conditions for the problem we can determine its eigenfunctions [and therefore we determine u using (6.58)] and its eigenenergies [and we determine s using (6.61)].

In this paper we will assume a particularly simple form for the interaction term which has been used in the literature in order to investigate the problem of interactions of finite range[26]

$$F(\nu) = \pi \gamma p_F \lambda_c e^{-\lambda_c |\nu|}. \tag{6.62}$$

There are two parameters here: γ gives the strength of the interaction and λ_c plays the role of the range of the interaction. When $\lambda_c \to 0$ the interaction is angle independent (is short range in the real space) and we obtain the zero sound solution already discussed; when $\lambda_c \to \infty$ the interaction is long range in real space and strongly angle dependent in momentum space. This parameter will enable us to study the crossover from short range to long range interaction in terms of the dynamics of the sound waves.

The Fourier transform of (6.62) is given by

$$F(\lambda) = \gamma \frac{\lambda_c^2}{\lambda^2 + \lambda_c^2}, \tag{6.63}$$

which is a Lorentzian.

Using (6.63) in (6.59) we obtain the equation for the one-dimensional harmonic oscillator,

$$\left(-\frac{d^2}{d\lambda^2} + \beta^4(s) \lambda^2 \right) z(\lambda) = E(s) z(\lambda), \tag{6.64}$$

where

$$\beta(s) = \left(\frac{p_F \alpha(s)}{\gamma^{1/2} \lambda_c} \right)^{1/2} \tag{6.65}$$

and

$$E(s) = \epsilon(s) - \left(\frac{p_F \alpha(s)}{\gamma^{1/2}} \right)^2. \tag{6.66}$$

Of course, the natural boundary condition is that function z vanishes at infinity.

The solution of (6.64) is standard. The eigenfunctions are

$$z_n(\lambda) = e^{-\beta^4 \lambda^2} H_n(\beta^2 \lambda), \tag{6.67}$$

where H_n is the Hermite polynomial of order n and the eigenenergies are

$$E(s_n) = 2\beta^2(s_n) \left(n + \frac{1}{2} \right). \tag{6.68}$$

Equation (6.68) is self-consistent and it gives the allowed values of s. If we substitute (6.46) and (6.47) into (6.68) we get the following transcendental equation,

$$\left(1 + \frac{1}{\gamma} \right) \alpha(s_n) + 2 \left[\frac{1}{\gamma^{1/2} p_F \lambda_c} \left(n + \frac{1}{2} \right) - A(s_n) \right] = 0. \tag{6.69}$$

It is very easy to check that the value of s_n decreases monotonically with n and, in particular, $\lim_{n \to \infty} s_n = 1$. This result means that the modes approach asymptotically the particle-hole continuum.

Furthermore, substituting (6.67) and (6.63) into (6.58) we find

$$u_n(\lambda) = (\lambda^2 + \lambda_c^2)e^{-\beta^4\lambda^2} H_n(\beta^2\lambda). \tag{6.70}$$

We now Fourier transform (6.70) back to the arc length in order to get

$$u_n(\nu) = \left[2n + 1 + \beta^2(s_n)\lambda_c - \left(\frac{\nu}{\beta(s_n)}\right)^2\right] \times e^{-\frac{1}{2}\left(\frac{\nu}{\beta(s_n)}\right)^2} H_n\left(\frac{\nu}{\beta(s_n)}\right). \tag{6.71}$$

Equations (6.69) and (6.71) are the solutions of (6.44) in the limit when the Fermi momentum is much larger than the momentum transfer between the particle-hole pairs (which means, as we said, that we are looking close to the forward scattering direction).

We want to change the range of the interaction λ_c but we keep the density of the system constant. In this case the product $p_F\lambda_c$ is finite ($\lambda_c p_F \sim \frac{a_0}{r_0}$ where r_0 is the mean distance between the particles and a_0 is the Bohr radius). In other words, it means that $\beta\lambda_c$ is finite in what follows. Therefore, keeping the above product constant and sending $\lambda_c \to 0$ we find from (6.65)

$$\beta \to \infty \tag{6.72}$$

and from (6.71),

$$u_n(\nu) \to \text{const}, \tag{6.73}$$

if n is even and $u_n(\nu) = 0$ if n is odd. Observe that this result is in agreement with the zero sound result, (6.51), which is constant in the forward direction ($\theta \sim 0$).

However, for $\lambda_c \to \infty$ it is easy to see that

$$\beta \to 0 \tag{6.74}$$

and

$$u_n(\nu) \sim \left(\frac{\nu}{\beta}\right)^2 e^{-\frac{1}{2}\left(\frac{\nu}{\beta}\right)^2} H_n\left(\frac{\nu}{\beta}\right), \tag{6.75}$$

which oscillates strongly near $\nu = 0$. We would argue, therefore, that while for short range interactions ($\lambda_c \to 0$) the field $u(\nu)$ is smooth around the forward scattering direction, for long range interactions ($\lambda_c \to \infty$) the field becomes rough near the same direction. This could represent a signal that there is something particularly nonconventional happening there. Notice, on one hand, that this result has similarities to the problem of long-range order in low dimensional systems,[27] since it is well known that one-dimensional systems at zero temperature with local interactions do not have long range order. However, on the other hand, quantum one-dimensional interfaces at zero temperature also cannot have long range order while two-dimensional interfaces always have long range order.[14] Since the Fermi surface has the properties of a real interface in momentum space, it could be argued that the Fermi surface could undergo a roughening transition of the Kosterlitz-Thouless type[28] for singular interactions.

However, the calculations we have carried out are purely classical and up to now we do not know if quantum fluctuations can wash out this behavior. It is worth mentioning that in quantum crystals the quantum fluctuations can wash out the classical ones.[14]

C. Quantum fluctuations of the Fermi surface

It was shown some time ago[27] that quantum one-dimensional crystals can lose long range order when the interactions are sufficiently local. Quantitatively it means that the equal-time correlation function for the deviations of the equilibrium position of the atoms, $u(R)$, diverges logarithmically for long distances, that is, $\langle u(R)u(R')\rangle \sim \ln|R-R'|$ as $|R-R'| \to \infty$. We would argue, due to the classical calculations we have carried out that the same could happen for the fields $u_{\vec{q}}(\Omega)$ for long range interactions in real space. Therefore, in order to understand this problem we calculate the quantum correlation function for different pieces of the Fermi surface.

From (6.18) is straightforward to calculate the correlation function for different pieces of the Fermi surface, namely,

$$\langle T\varphi_{\vec{q}}(\Omega,t)\varphi_{\vec{q}'}(\Omega',t')\rangle = \sum_{\lambda,\lambda'} v_{\vec{q}}^{\lambda}(\Omega)v_{\vec{q}'}^{\lambda'}(\Omega') \times \langle T\Phi_{\vec{q}}^{\lambda}(t)\Phi_{\vec{q}'}^{\lambda'}(t')\rangle \tag{6.76}$$

where T is the chronological operator and $v_{\vec{q}}^{\lambda}(\Omega)$ are the solutions of (6.15). The correlation function in the rhs of (6.76) is easily calculated from (6.18) (we assume that the eigenfrequencies $\omega_{\vec{q}}^{\lambda}$ are well behaved functions of $\vec{q}$ and λ) and it reads

$$\langle T\Phi_{\vec{q}}^{\lambda}(t)\Phi_{\vec{q}'}^{\lambda'}(t')\rangle = \delta_{\lambda,\lambda'}\delta_{\vec{q},\vec{q}'}e^{i\omega_{\vec{q}}^{\lambda}|t-t'|}. \tag{6.77}$$

Thus, from (6.76) and (6.77),

$$\langle T\varphi_{\vec{q}}(\Omega,t)\varphi_{\vec{q}'}(\Omega',t')\rangle = \delta_{\vec{q},\vec{q}'}\sum_{\lambda} v_{\vec{q}}^{\lambda}(\Omega)v_{\vec{q}}^{\lambda}(\Omega')\ e^{i\omega_{\vec{q}}^{\lambda}|t-t'|} \tag{6.78}$$

and therefore the equal-time correlation function is simply [using the completeness of the states in (6.15)],

$$\langle T\varphi_{\vec{q}}(\Omega,t)\varphi_{\vec{q}'}(\Omega',t)\rangle = \delta_{\vec{q},\vec{q}'}\delta(\Omega-\Omega'). \tag{6.79}$$

Equation (6.79) shows that the correlation function is always finite and therefore there is no roughening (there is long range order), that is, the Fermi surface is always smooth.

Through comparison between this result and the result of quantum crystals,[14] we know that this result is due to a conservation law, namely, the local conservation of the volume of the Fermi surface (the Luttinger's theorem[16]). From the other point of view this is expected, since the sound waves obey a diffusionlike equation, (6.36), and,

therefore, a bump on the Fermi surface diffuses leaving nothing behind it.

VII. CONCLUSIONS

We construct a bosonization of a Fermi liquid in any number of dimensions in the limit of long wavelengths. The bosonization is valid for a restricted set of states close to the Fermi surface. We generate a set of creation and annihilation operators that span the restricted Hilbert space. We showed that it is possible to construct a set of coherent states (bosonic shape fields) which are coherent superpositions of particle-hole excitation close to the Fermi surface and which are the displacements of the Fermi surface in some direction. The physical interpretation of the existence of these fields comes from the observation that the Fermi surface is a real quantum object which is responsible for the whole physics of Fermi liquid systems at low energies. From the coherent states we generate all the thermodynamics and semiclassical dynamics of a Fermi liquid.

We have shown that an interacting Hamiltonian of the original fermions is simply quadratic in terms of the bosons, which means that the bosons are free and move on the Fermi surface like sound waves. They are not free fields in the usual sense, since they are topologically constrained to live in the Fermi surface. This feature produces very interesting results. In particular, it reproduces *all* the results of the Fermi liquid theory from a bosonic point of view.

We obtain the correct thermodynamics of a Fermi liquid even with the corrections due to the scattering between the fermions. This observation leads us to conclude that our approach is consistent and that we are counting the correct number of states in the Hilbert space.

Moreover, we have shown that the semiclassical dynamics of these sound waves is described by the Landau theory. In two dimensions we calculate the form of these sound waves explicitly for short range interactions (zero sound) and long range interactions. We have shown that, at classical level, we could conclude that it is possible to have a rougheninglike transition of the Kosterlitz-Thouless type, which is closely related to the absence of long range order in one dimensional crystals at zero temperature. However, we show that quantum fluctuations destroy this effect due to the local conservation of the volume of the Fermi surface (Luttinger's theorem).

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APPENDIX A: FERMI LIQUID AND NON-FERMI LIQUID BEHAVIOR

In this appendix we discuss the possibility of behaviors for the imaginary part of the self-energy other than the conventional Fermi liquid one, (3.13). We will still assume that the real part of the self-energy is linear in the frequency close to the Fermi surface, but for the imaginary part we will assume that

$$\Sigma_I(\vec{k},w) = C_{\vec{k}}\,|w-\mu|^{1+\nu}\,\mathrm{sgn}(\mu-w). \tag{A1}$$

If we substitute (A1) into (3.12) and make the same change of variables that lead to (3.14) we find

$$E^s_{\vec{k}} = \int_{\zeta}^{\infty} \frac{dx}{2\pi}\left(\mu+\epsilon^0_{\vec{k}}-\frac{1}{x}\mid \delta k \mid\mid \nabla\epsilon_{\vec{p}_F}\mid\right) \times \frac{C_{\vec{k}}x^{\nu-1}\mid \delta k\mid^{\nu}\mid \nabla\epsilon_{\vec{p}_F}\mid^{\nu}}{\left\{Z^{-2}_{\vec{k}}x^{2\nu}[x+\mathrm{sgn}(\delta k)]^2 + C^2_{\vec{k}}\delta k^{2\nu}\mid\nabla\epsilon_{\vec{p}_F}\mid^{2\nu}\right\}}, \tag{A2}$$

where $\zeta = |\delta k||\nabla\epsilon_{\vec{p}_F}|$.

When $\nu > 0$ it is easy to calculate the limit analogous to (3.15),

$$\lim_{\delta k\to 0}\frac{C_{\vec{k}}\mid\delta k\mid^{\nu}\mid\nabla\epsilon_{\vec{p}_F}\mid^{\nu}}{\left\{Z^{-2}_{\vec{k}}x^{2\nu}[x+\mathrm{sgn}(\delta k)]^2 + C^2_{\vec{k}}\delta k^{2\nu}\mid\nabla\epsilon_{\vec{p}_F}\mid^{2\nu}\right\}} = \pi\mid Z_{\vec{p}_F}\mid\delta\{x^{\nu}[x+\mathrm{sgn}(\delta k)]\}, \tag{A3}$$

which gives the same result as in (3.16) (Fermi liquid behavior). However, if $\nu < 0$ the above limit takes a different form, namely,

$$\lim_{\delta k\to 0}\frac{C_{\vec{k}}\mid\delta k\mid^{\nu}\mid\nabla\epsilon_{\vec{p}_F}\mid^{\nu}}{\left\{Z^{-2}_{\vec{k}}x^{2\nu}[x+\mathrm{sgn}(\delta k)]^2 + C^2_{\vec{k}}\delta k^{2\nu}\mid\nabla\epsilon_{\vec{p}_F}\mid^{2}\right\}} = \frac{1}{C_{\vec{k}}|\delta k|^{\nu}\,|\nabla\epsilon^0_{\vec{p}_F}|^{\nu}}, \tag{A4}$$

and, therefore, substituting in (A2) we find

$$E^s_{\vec{k}} = \left(\frac{1}{2(1-\nu)}C^{\nu-2}_{\vec{k}}|\delta k|\nabla\epsilon^0_{\vec{p}_F}||^{(\nu-1)^2}\right)\Theta(-\delta k). \tag{A5}$$

The discontinuity is easily calculated in the limit as $\delta k \to 0$,

$$\Delta E(\vec{p}_F) = \left(\frac{C^{\nu-2}_{\vec{p}_F}|\nabla\epsilon^0_{\vec{p}_F}|^{(\nu-1)^2}}{2(1-\nu)}\right)|\delta k|^{(\nu-1)^2}\to 0, \tag{A6}$$

and therefore there is no singularity left (non-Fermi liquid behavior).

APPENDIX B: BOSONIZATION IN ONE SPATIAL DIMENSION

In this appendix we show that the formulas obtained in Sec. IV of this paper are consistent with the bosonization procedure in one dimension. In order to do so we prove that we can obtain the same commutation relations for the densities as is obtained in the standard procedures of bosonization. Therefore, all other results relative to the calculation of the fermionic operators and the one-particle Green's function follows from this result.

We will concentrate particularly on the Tomonaga model,[1] but the extension of the arguments to the Luttinger model[4] is absolutely straightforward.

The Tomonaga model for spinless electrons is described by the following Hamiltonian:

$$H = \sum_p \epsilon_p c_p^\dagger c_p + \frac{1}{2L}\sum_q V_q \rho(q)\rho(-q), \tag{B1}$$

where the dispersion relation is given by

$$\epsilon_p = v_F \mid p \mid . \tag{B2}$$

V_q is the Fourier transform of the electron-electron interaction, L is the length of the system, and

$$\rho(q) = \sum_k n_q(k) \tag{B3}$$

is the density operator.

Now we split the density operator in two terms, one for right movers and other for left movers,

$$\begin{aligned}\rho_1(q) &= \sum_{k>0} n_q(k),\\ \rho_2(q) &= \sum_{k<0} n_q(k);\end{aligned} \tag{B4}$$

thus, $\rho(q) = \rho_1(q) + \rho_2(q)$.

The one-dimensional version of the commutation relation (4.5) is

$$[n_q(k), n_{-q'}(k')] = \mathrm{sgn}(k)\,\delta_{k,k'}\delta_{q,q'}\,qv_F\,\delta(\mu - v_F \mid k \mid). \tag{B5}$$

Therefore, from (B4) we easily find

$$[\rho_i(q), \rho_j(-q')] = (-1)^{j+1}\delta_{i,j}\,qv_f \times \sum_{k(-1)^j<0} \delta(\mu - v_F \mid k \mid), \tag{B6}$$

where the chemical potential is written as $\mu = v_F p_F$.

In the thermodynamic limit we transform the sum into an integral ($\sum_k \to \frac{L}{2\pi}\int dk$) and we finally get

$$[\rho_i(q), \rho_j(-q')] = \delta_{i,j}(-1)^{j+1}\frac{qL}{2\pi}, \quad i,j = 1,2\ , \tag{B7}$$

as expected for spinless fermions.[1]

The bosonic operators are defined in terms of the densities for $q > 0$ as [compared with (6.7)]

$$\begin{aligned} b_q &= \sqrt{\frac{2\pi}{qL}}\rho_1(q),\\ b_{-q} &= \sqrt{\frac{2\pi}{qL}}\rho_2(-q);\end{aligned} \tag{B8}$$

with their adjoint they obey the usual bosonic algebra,

$$\left[b_k, b_{k'}^\dagger\right] = \delta_{k,k'}. \tag{B9}$$

Following the same kind of argument as presented in Sec. V, we can prove that the Tomonaga model is purely quadratic in terms of the bosons and can be easily diagonalized by a Bogoliubov transformation. In particular, the calculation of the correlation functions follows exactly as in the work of Mattis and Lieb[4] and Luther and Peschel[29] for the Luttinger's model.

APPENDIX C: THE EFFECTIVE MASS

In this appendix we show the complete consistency of our approach and the Landau theory of the Fermi liquid via the calculation of the effective mass of the Landau theory in terms of the bare electronic mass.

Suppose we displace all pieces of the Fermi surface in momentum space by the same infinitesimal amount $\vec{q}$. This is equivalent to looking at the system from a reference point that moves relative to it with constant velocity. By simple geometric arguments it is easy to see that the displacements of the Fermi surface will change by the quantity

$$u_{\vec{q}}(\vec{k}) \to u_{\vec{q}}(\vec{k}) + \frac{\vec{q}\cdot\vec{v}_{\vec{k}}}{v_{\vec{k}}}. \tag{C1}$$

The total change in the action can be obtained from (6.1). Up to first order in q one gets

$$\delta L = -\sum_{\vec{k},\vec{k}',\vec{q}} \frac{v_{\vec{k}}^2\delta(\mu-\epsilon_{\vec{k}})}{V\vec{q}\cdot\vec{v}_{\vec{k}}} u_{\vec{q}}^*(\vec{k})\left(W_{\vec{k},\vec{k}'}^{\vec{q}}\delta(\mu-\epsilon_{\vec{k}'})\frac{\vec{q}\cdot\vec{v}_{\vec{k}'}}{v_{\vec{k}'}}\right). \tag{C2}$$

Using (6.2) and (5.9) we can rewrite the above expression as

$$\delta L = -\sum_{\vec{k},\vec{q}} v_{\vec{k}}\delta(\mu-\epsilon_{\vec{k}})u_{\vec{q}}^*(\vec{k})\left(\vec{q}\cdot\vec{J}_{\vec{k}}\right), \tag{C3}$$

where

$$\vec{J}_{\vec{k}} = \vec{v}_{\vec{k}} + \frac{1}{V}\sum_{\vec{k}'} f_{\vec{k},\vec{k}'}\,\delta(\mu-\epsilon_{\vec{k}'})\,\vec{v}_{\vec{k}'}. \tag{C4}$$

Notice that since we are transporting the whole Fermi sea by a constant vector the change in the Lagrangian must be minus the total change in the energy of the system. Therefore, $\vec{J}_{\vec{k}}$ is the contribution for a current

carried by a fermion with momentum $\vec{k}$.[15] Moreover, if the system is homogenous and isotropic the current and the velocity must point in the direction of $\vec{k}$. Naturally, the current transports a mass equal to the mass of the fermion m, namely,

$$\vec{J}_{\vec{k}} = \frac{\vec{k}}{m}. \tag{C5}$$

In the Landau theory the effective mass m^* is defined as the coefficient of the velocity,

$$\vec{v}_{\vec{k}} = \frac{\vec{k}}{m^*}. \tag{C6}$$

Substituting (C5) and (C6) into (C4) and assuming $\vec{k}$ and $\vec{k}'$ at the Fermi surface and $\vec{k}\cdot\vec{k}' = p_F^2 \cos\theta$, we easily get

$$\frac{m^*}{m} = 1 + \frac{1}{V}\sum_{\vec{k}'} f_{\vec{k},\vec{k}'}\, \delta(\mu - \epsilon_{\vec{k}'})\, \cos\theta, \tag{C7}$$

which is the usual relation between the effective mass and the bare mass in the Landau theory of the Fermi liquid.[15]

In three dimensions we can expand the interactions in Legendre polynomials,

$$F(\theta) = \sum_L F_L P_L(\cos\theta), \tag{C8}$$

where $F = N(0)f$. Use the density of states as defined in (5.5) and the orthogonality between the Legendre polynomials in order to get

$$\frac{m^*}{m} = 1 + \frac{F_1}{3}. \tag{C9}$$

In two dimensions we use (2.6) and the above definitions and we find

$$\frac{m^*}{m} = 1 + \frac{F_1}{2}. \tag{C10}$$

The Galilean invariance present in the argument of this appendix shows that the mass that appears in the Fermi velocity in the case of a homogenous system is not the bare mass but the effective mass which is calculated in the above expressions.

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