Spin-wave bound-state energies from an Ising model

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We study the two-dimensional classical Ising model that is equivalent, via the Suzuki-Trotter mapping, to the *XXZ* Heisenberg quantum-spin chain. By imposing appropriate boundary conditions to the Ising model, the spin waves of the quantum model are studied. We reproduce the entire energy spectrum of the two-spin-wave states and derive bound-state energies of the three-spin-wave states. Thus, the continuum energetics of the elementary excitations of a d-dimensional quantum model are contained in the equivalent $(d+1)$-dimensional classical model, even though the latter is a discrete-spin model.

I. INTRODUCTION

Noncommuting quantum-mechanical operators bring an added degree of difficulty to the statistical-mechanical treatment of model systems. A general step along the direction of relieving this difficulty was taken by Suzuki,[1] who showed that the Trotter formula[2] can be employed to map, rigorously, d-dimensional quantum-mechanical systems onto $(d+1)$-dimensional classical systems with somewhat complicated, but local, interactions and constraints. This mapping for the partition function is similar to the Feynman path-integral formalism for particle propagators in many-body theory. The Suzuki-Trotter transformation has to date been exploited to enable Monte Carlo simulations, which are carried out on the equivalent $(d+1)$-dimensional classical system.[3-6] Unfortunately, it has not been much used within closed-form treatments of model systems.

The classical system that is the upshot of the Suzuki-Trotter mapping is composed of discrete, namely Ising-type, local degrees of freedom. Therefore, a question that arises is how the latter system incorporates elementary excitations of the initial quantum-mechanical system, such as spin waves, that have a continuously varying energy spectrum. We have investigated this question with *XXZ* Heisenberg magnetic chains. We find that its answer lies, quite generally, in the extreme spatial anisotropy of the $(d+1)$-dimensional classical system. In the process of this study, working with the equivalent classical Ising system, we have reproduced the entire energy spectrum of the two-spin-wave quantum states and we have derived bound-state energies of the three-spin-wave quantum states.

II. THE *XXZ* HEISENBERG MAGNETIC CHAIN AND ITS EQUIVALENT CLASSICAL ISING MODEL

The *XXZ* Heisenberg chain is defined by the Hamiltonian

$$-\beta\mathcal{H}_{XXZ}=K\sum_i(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y)+K_z\sum_i\sigma_i^z\sigma_{i+1}^z\ , \quad (1)$$

where $\beta=1/k_BT$, and σ_i^u are the Pauli spin matrices at site i. For $K=K_z$, $0<|K|<|K_z|$, $|K|>|K_z|>0$, $K=0\neq K_z$, and $K\neq 0=K_z$, the model respectively reduces to the Heisenberg, easy-axis Heisenberg, easy-plane Heisenberg, Ising, and XY models. This model has been treated by Bethe,[7] Dyson,[8] Orbach,[9] Wortis,[10] and others, within its quantum-mechanical formulation.

Suzuki has mapped[1] the *XXZ* chain onto a classical system as follows. The Hamiltonian is separated into two terms, each containing every other bond. The Trotter formula states that

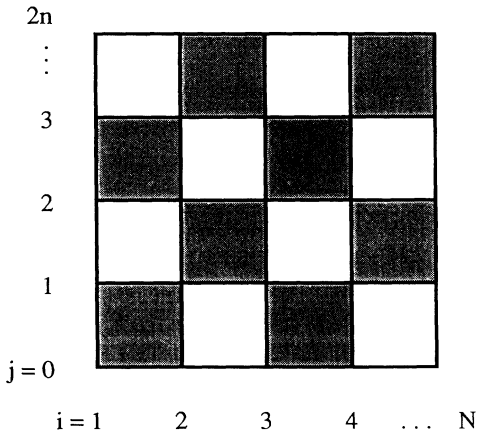

FIG. 1. The classical $d=2$ model that is equivalent to the *XXZ* Heisenberg quantum-spin chain (Ref. 1). There is a classical variable $m_{i,j}=\pm 1$ at each site (i,j). These are coupled only in the shaded squares, with the interaction of Eq. (3), which shows that the model is extremely anisotropic. There are N (the number of original *XXZ* Heisenberg spins) sites horizontally and $2n+1$ (the number of inserted sets of states plus 2) sites vertically. Various specifications of the horizontal boundary conditions determine the property of the original quantum system that is studied via the classical system (Sec. III).

$$e^{-\beta\mathcal{H}_1-\beta\mathcal{H}_2}=\lim_{n\to\infty}(e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n})^n\ , \tag{2}$$

the corrections being of order n^{-1}. Suzuki uses this formula by inserting a complete set of states between each of the $2n$ factors in the right side. In each $-\beta\mathcal{H}_j/n$, the operators associated with a given bond $(i,i+1)$ commute with all other operators. Thus, the matrix elements, resulting from the insertion of the complete set of states, themselves factorize to local calculations of matrix elements of single-bond operators. The result of such a calculation amounts to a local coupling in a classical two-dimensional Hamiltonian, where the degrees of freedom are the quantum numbers of the single-bond operators in two adjoining inserted states.

The equivalent classical $d=2$ anisotropic model is finalized, after the insertion of states and the calculations just mentioned, by taking matrix elements of the operator in Eq. (2), in manners to be specified in Sec. III, which determines the classical boundary conditions. The resulting model is composed of classical spins $m_{i,j}=\pm1$ at each site (i,j) of a square lattice. These classical spins are coupled by local interactions that are grouped into every other square in a checkerboard pattern, as shown in Fig. 1. Thus, in this figure, each darkened square contributes to the exponentiated Hamiltonian a term

$$e^{-\beta\mathcal{H}_{\blacksquare}(m_{i,j},m_{i+1,j},m_{i,j+1},m_{i+1,j+1})}=\begin{bmatrix}A&0&0&0\\0&B&C&0\\0&C&B&0\\0&0&0&A\end{bmatrix}$$

$$\text{with } A=e^{K_z/n},\quad B=e^{-K_z/n}\cosh(2K/n)\ ,\quad \text{and } C=e^{-K_z/n}\sinh(2K/n)\ , \tag{3}$$

where the rows or columns are addressed by the states $(+1,+1)$, $(+1,-1)$, $(-1,+1)$, $(-1,-1)$ of $(m_{i,j},m_{i+1,j})$ or $(m_{i,j+1},m_{i+1,j+1})$, respectively. The index i ranges from 1 to N, the number of initial quantum spins, and the index j ranges from 0 to $2n$, the number of inserted sets of states plus the two states of the matrix element of Eq. (2). The expression in Eq. (3) is the direct result of the local calculation, described in the preceding paragraph, of

$$\langle m_{i,j},m_{i+1,j}|\exp[(K/n)(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y)+(K_z/n)\sigma_i^z\sigma_{i+1}^z]|m_{i,j+1},m_{i+1,j+1}\rangle\ , \tag{4}$$

where $m_{i,j}$ is the eigenvalue of σ_i^z in the inserted set j. The above clearly corresponds to a classical spin-$\frac{1}{2}$ Ising model, with local constraints, namely excluded nearest-neighbor quartets of states, due to the zeroes in the matrix in Eq. (3).

III. BOUNDARY CONDITIONS OF THE CLASSICAL MODEL

A. Corresponding to the partition function of the *XXZ* model

In Suzuki's original work, the trace of Eq. (2) is taken, in order to obtain the partition function of the *XXZ* Heisenberg model. This yields the partition function of the equivalent classical $d=2$ model with periodic boundary conditions along the j direction as defined in Eq. (3), $m_{i,0}\equiv m_{i,2n}$. The boundary condition along the i direction is always determined by that of the *XXZ* model, which we take as periodic in the entirety of this article. Furthermore, N is taken to be large (approaching the thermodynamic limit) and even.

B. Corresponding to the *z*-aligned state of the *XXZ* model

The diagonal matrix element of Eq. (2) with respect to the quantum state $|\{m_i=+1\}\rangle$ yields, in the classical $d=2$ model, the pinned "up" boundary conditions $m_{i,0}=+1=m_{i,2n}$. The constraints [Eq. (3)] do not allow the creation of a "down" spin $(m_{i,j}=-1)$, so that only one state, $\{m_{i,j}=+1\}$, occurs in the classical system. Thus,

$$\langle\{m_i=+1\}|e^{-\beta\mathcal{H}_{XXZ}}|\{m_i=+1\}\rangle=e^{-\beta\Sigma\mathcal{H}_{\blacksquare}(+1,+1,+1,+1)}$$
$$=A^{Nn}=e^{NK_z}\ . \tag{5}$$

The energy of the z-aligned state, NK_z, is the ground-state energy of the *XXZ* model for $K_z>|K|$ and $K_z=K$.

C. Corresponding to a single spin wave

Consider the matrix element of Eq. (2) between states such as $|\{m_{i\neq r}=+1\},m_{i=r}=-1\rangle\equiv|r\rangle$. This yields the partition function of the classical $d=2$ model with pinned up boundary conditions at rows $j=0$ and $j=2n$, except for the spins at $i=r_0$ and $i=r_{2n}$, respectively, which are pinned down. Since the constraints [Eq. (3)] do not allow the creation or destruction of a down spin, each row j has one and only one down spin, which, from row to row, may remain at the same position r, or move to $r\pm1$ within an interaction square, respectively, with Boltzmann weight B or C according to the interactions in Eq. (3). Consider the transfer matrix of the classical system, connecting every other row, with respect to the single spin-wave basis set

$$|k\rangle=\frac{1}{\sqrt{N}}\sum_r e^{ikr}|r\rangle,\quad k=2\pi p/N,\quad p=1,\ldots,N\ . \tag{6}$$

A little algebra starting from Eq. (3) yields, for this transfer matrix, to leading order in $1/n$,

$$\langle k'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|k\rangle=\delta(k',k)e^{[NK_z-4K_z+4K\cos k]/n}, \tag{7}$$

where $\delta(k',k)=1,0$ for $k'=k,k'\neq k$. Thus, the partition function of the entire system (of n pairs of rows) is

$$\langle k'|(e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n})^n|k\rangle=\delta(k',k)e^{NK_z-4K_z+4K\cos k}, \tag{8}$$

which yields the well-known single spin-wave energy

$$-\beta\varepsilon_1=-4K_z+4K\cos k . \tag{9}$$

D. Corresponding to two spin waves

Consider the matrix element of Eq. (2) between the states such as $|\{m_{i\neq r,r'}=+1\},m_{i=r,r'}=-1\rangle\equiv|r,r'\rangle$. This yields the partition function of the classical $d=2$ model with pinned up boundary conditions at rows $j=0$ and $j=2n$, except for the spins at $i=r_0,r_0'$ and $i=r_{2n},r_{2n}'$, respectively, which are pinned down. Similarly to the previous case, each row j has two and only two down spins, which, from row to row, may remain at the same positions r,r', or each or both move to neighboring positions within an interaction square, with Boltzmann weights dictated by Eq. (3). Again consider the transfer matrix of the classical system, connecting every other row,

$$\tfrac{1}{2}\{\langle r_1',r_2'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|r_1,r_2\rangle + \langle r_1',r_2'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|r_2,r_1\rangle\} . \tag{10}$$

There are two terms in Eq. (10) because, in going between the rows, either down spin can match to a given down spin in the other row. There is a leading factor of $\frac{1}{2}$ because the down spins are indistinguishable, so that a summation over (r_1,r_2) double counts the states. With respect to another two-spin-wave basis set,

$$|k,\rho\rangle=\frac{1}{\sqrt{N}}\sum_{r_1}e^{ik(r_1+r_2)/2}|r_1,r_2=r_1+\rho\rangle , \qquad k=2\pi p/N,\ p=1,\ldots,N , \tag{11}$$

this transfer matrix has the form

$$\tfrac{1}{2}\{\langle k',\rho'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|k,\rho\rangle + \langle k',\rho'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|k,-\rho\rangle\} . \tag{12}$$

It reduces to leading order in $1/n$ to

$$\tfrac{1}{2}\delta(k',k)A^{N-4}[M(\rho',\rho)+e^{ikN/2}M(\rho',N-\rho)] , \tag{13}$$

where

$$M(\rho',\rho)=\begin{bmatrix} 1 & \gamma & & & & \\ \gamma & 1-\alpha & \gamma & & & \\ & \gamma & 1-\alpha & \gamma & & \\ & & & \cdots & & \\ & & & & \gamma & 1 \end{bmatrix},$$

with

$$\alpha=4K_z/n,\quad \gamma=(4K/n)\cos(k/2) ,$$

and where ρ', ρ are between 1 and $N-1$ inclusive. The eigenvectors of this transfer matrix have the form

$$Y(\rho)=x^\rho+bx^{N-\rho},\ \text{with } b=e^{-ikN/2}=\pm1 . \tag{14}$$

One set of eigensolutions are extended (unbound) spin-wave pair states with $x=e^{iq}$ in Eq. (14). Their eigenvalues are

$$e^{[(N-8)K_z+8K\cos(k/2)\cos q]/n}, \tag{15}$$

where, as specified above, $k=2\pi p/N$, $p=1,\ldots,N$, and q is determined for $b=+1$ by

$$(K/K_z)\cos(k/2)-\cos q=\sin q\tan(Nq/2) , \tag{16}$$

and for $b=-1$ by

$$(K/K_z)\cos(k/2)-\cos q=-\sin q\cot(Nq/2) . \tag{17}$$

For $|(K/K_z)\cos(k/2)|\geq1$, graphical analysis shows that Eq. (16) accounts for $(N/2)(N/2)$ solutions and Eq. (17) accounts for $(N/2)[(N/2)-1]$ solutions. For $|(K/K_z)\cos(k/2)|<1$, again graphical analysis shows that Eq. (16) accounts for $(N/2)[(N/2)-1]$ solutions and Eq. (17) accounts for $(N/2)[(N/2)-2]$ solutions. In the latter case, a set of N eigensolutions of bound spin-wave pair states occurs, with $x=|(K/K_z)\cos(k/2)|$ in Eq. (14), and eigenvalue

$$e^{\{(N-4)K_z+4[K\cos(k/2)]^2/K_z\}/n} . \tag{18}$$

The expressions in (15) and (18), with their exponents multiplied by n, yield the corresponding Boltzmann weights of the entire system. Thus, the extended spin-wave pair energies are

$$-\beta\varepsilon_2=-8K_z+8K\cos(k/2)\cos q =-8K_z+4K[\cos(k/2+q)+\cos(k/2-q)] , \tag{19}$$

which in fact equals the sum of the energies [Eq. (9)] of two single spin waves with wave numbers $(k/2)\pm q$. The bound spin-wave pair states, occurring for $|(K/K_z)\cos(k/2)|<1$ as above, have energy

$$-\beta\varepsilon_2=-4K_z+4[K\cos(k/2)]^2/K_z . \tag{20}$$

The above account for all of the spin-wave pair states, and agree with the previous works.[7-10]

A particulate analogy to two interacting spin waves, from the diagonalization of the matrix in Eq. (13), is given in Appendix A. A renormalization-group transformation is derived in Appendix B, with asymptotic flow behavior that is fixed point or chaotic, distinguishing the bound or extended spin-wave pairs, respectively.

E. Corresponding to three spin waves

The matrix element of Eq. (2) between the states such as $|\{m_{i\neq r,r',r''}=+1\},m_{i=r,r',r''}=-1\rangle$ is considered. The

classical $d=2$ model has pinned up boundary conditions at rows $j=0$ and $j=2n$, except for the spins at $i=r_0,r_0',r_0''$ and $i=r_{2n},r_{2n}',r_{2n}''$, respectively, which are pinned down. Each row j has three and only three down spins, which, from row to row, may remain at the same positions r,r',r'', or move to neighboring positions within an interaction square, with Boltzmann weights dictated by Eq. (3). The transfer matrix of the classical system, connecting every other row, with respect to the three-spin-wave basis set

$$|k,\rho_1,\rho_3\rangle=\frac{1}{\sqrt{N}}\sum_{r_2}e^{ik(r_1+r_2+r_3)/3}|r_1=r_2-\rho_1,r_2,r_3=r_2+\rho_3\rangle\ ,$$

$$\text{with } k=2\pi p/N,\quad p=1,\ldots,N,\quad \rho_1=1,\ldots,N-2,\text{ and } \rho_3=1,\ldots,N-1-\rho_1\ , \tag{21}$$

has the form

$$\tfrac{1}{3}\{\langle k',\rho_1',\rho_3'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|k,\rho_1,\rho_3\rangle+\langle k',\rho_1',\rho_3'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|k,-\rho_1-\rho_3,\rho_1\rangle$$

$$+\langle k',\rho_1',\rho_3'|e^{-\beta\mathcal{H}_1/n}e^{-\beta\mathcal{H}_2/n}|k,\rho_3,-\rho_1-\rho_3\rangle\}\ . \tag{22}$$

Again, there are three terms in Eq. (22) because, in going between the rows, any one of the three down spins can match to a given down spin in the other row (which fixes the other two matches). There is a leading factor of $\frac{1}{3}$ because the down spins are indistinguishable, so that a summation over (ρ_1,r_2,ρ_3) triple counts the states. The first term in the parentheses of Eq. (22), for example, reduces to leading order in $1/n$ to

$$\delta(k',k)A^{N-6}B^6(\delta(\rho_1',\rho_1)\delta(\rho_3',\rho_3)\{1+\alpha[\delta(\rho_1,1)+\delta(\rho_3,1)+\delta(\rho_1+\rho_3,N-1)]\}$$

$$+C\{[e^{ik/3}\delta(\rho_1',\rho_1-1)+e^{-ik/3}\delta(\rho_1',\rho_1+1)]\delta(\rho_3',\rho_3)+\delta(\rho_1',\rho_1)[e^{-ik/3}\delta(\rho_3',\rho_3-1)+e^{ik/3}\delta(\rho_3',\rho_3+1)]$$

$$+e^{ik/3}\delta(\rho_1',\rho_1+1)\delta(\rho_3',\rho_3-1)+e^{-ik/3}\delta(\rho_1',\rho_1-1)\delta(\rho_3',\rho_3+1)\})\ . \tag{23}$$

The transfer matrix of Eq. (22) has a set of bound-state eigenvectors of the form

$$Y(\rho_1,\rho_3)=y(\rho_1,\rho_3)+by(N-\rho_1-\rho_3,\rho_1)+b^2y(\rho_3,N-\rho_1-\rho_3)\ , \tag{24}$$

with

$$b=e^{-iNk/3},\quad y(\rho_1,\rho_3)=e^{-i\phi(\rho_1-\rho_3)}x^{\rho_1+\rho_3},\quad \sin(k/3+\phi)=\sin k/\sqrt{1+4(K_z/K)\cos(k)+4(K_z/K)^2}\ ,$$

where the spatial decay is determined by

$$x=\sin(k/3+\phi)/\sin(k/3-2\phi),\quad |x|<1\ , \tag{25}$$

where the bound-state restriction $|x|<1$ is satisfied for $(K/K_z)\cos k<[4(K_z/K)^2-3]$. The corresponding eigenvalues are

$$\exp\{[(N-4)K_z+4Kx\cos(k/3+\phi)]/n\}=\exp\{[(N-4)K_z+4K^2(2K_z+K\cos k)/(4K_z^2-K^2)]/n\}\ . \tag{26}$$

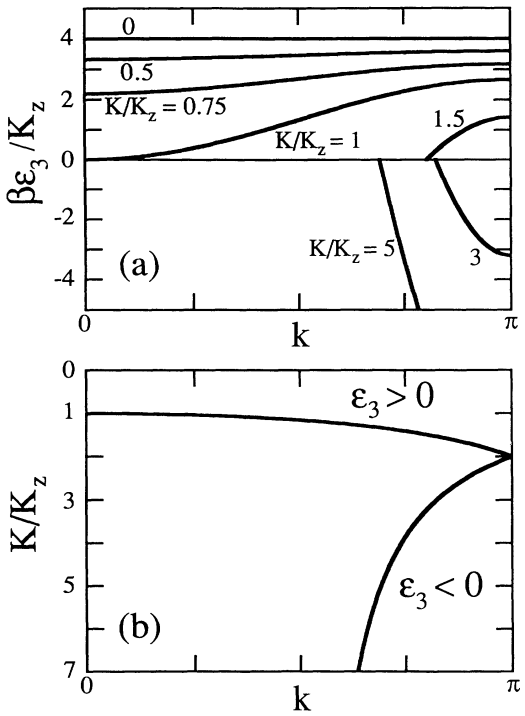

FIG. 2. (a) Bound-state energy spectra for three spin waves, as derived in Sec. III E and given in Eq. (27). Bound-state energy spectra for $K/K_z<0$ or $k<0$ are related to these curves by $\varepsilon_3(K/K_z,k)=\varepsilon_3(-K/K_z,k+\pi)=\varepsilon_3(K/K_z,-k)$. (b) Regions in which the three-spin-wave bound states with energies given in Eq. (27) occur. The lower boundary reaches $k=\pi/2$ as K/K_z goes to infinity.

Thus, the corresponding three-spin-wave bound-state energies are

$$-\beta\varepsilon_3=-4K_z+4Kx\cos(k/3+\phi)$$

$$=-4K_z+4K^2(2K_z+K\cos k)/(4K_z^2-K^2)\ . \qquad (27)$$

These bound-state energies are depicted in Fig. 2. The particulate analogy to three interacting spin waves, from the diagonalization of the matrix in Eq. (23), is noted in Appendix A.

IV. CONCLUSION

As seen above, the continuum energetics of the elementary excitations of a d-dimensional quantum model are contained in the equivalent $(d+1)$-dimensional classical model, even though the latter is a discrete-spin model. This is due to the fact that the extreme anisotropy of the classical model reduces the problem to a diagonalization of the Hamiltonian, as $n\to\infty$. In this process, we have derived three-spin-wave bound-state energies for the XXZ Heisenberg chain.

The implementation of our method is rather different from Bethe ansatz[7] studies of quantum systems. The method can also be generalized to, for example, spin-s systems. Our method is also much simpler, and therefore much more transparent, than the "quantum inverse scattering method."[11]

The procedure introduced here of using the Suzuki-Trotter formula with restricted boundary conditions may be useful for obtaining "renormalized" or "dressed" energy spectra for elementary excitations in more difficult problems, such as ones in which the transfer matrix does not conserve the number of fluctuations. More generally, it is likely that diverse effective studies of quantum systems can be built around the Suzuki-Trotter mapping.

ACKNOWLEDGMENTS

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APPENDIX A: PARTICULATE ANALOGY TO INTERACTING SPIN WAVES

The matrix in Eq. (13) can be written as

$$M(\rho',\rho)=\{\gamma\Delta^2+\alpha[\delta(\rho,1)+\delta(\rho,N-1)]$$
$$+(1-\alpha+2\gamma)\}\delta(\rho',\rho)\ , \qquad (A1)$$

where Δ^2 is the discrete Laplacian operator,

$$\Delta^2Y(\rho')=Y(\rho'+1)-2Y(\rho')+Y(\rho'-1)\ . \qquad (A2)$$

Accordingly, diagonalizing M is equivalent to solving the one-dimensional discrete Schrödinger equation for a particle of mass m in an infinite well from $\rho=1$ to $\rho=N-1$, subject to a potential at the edge sites, namely

$$V(\rho)=-\hbar^2/|2m(K/K_z)\cos(k/2)|$$
$$\times[\delta(\rho,1)+\delta(\rho,N-1)]\ . \qquad (A3)$$

The absolute value is obtained by considering the vectors $(-1)^\rho Y(\rho)$ when $(K/K_z)\cos(k/2)<0$. The particle of this Schrödinger equation may have eigenstates bound to the edges. The condition for this turns out to be $|(K/K_z)\cos(k/2)|<1$. The combination of two matrices in Eq. (13) selects half of the even and odd eigenfunctions of M, for both bound and extended states.

Similarly, the three-spin-wave problem of Sec. III E is equivalent to the two-dimensional discrete Schrödinger equation for a particle in an equilateral-triangle infinite well, with a potential along the sides that doubles at the corners.

APPENDIX B: RENORMALIZATION-GROUP ANALYSIS OF THE TWO-SPIN-WAVE EIGENVALUE PROBLEM

The eigenvalue Λ problem for the matrix M in Eq. (13) reduces to the $N-1$ equations

$$Fx_1+x_2=0,\quad x_{\rho-1}+Gx_\rho+x_{\rho+1}=0 \text{ for } \rho=2 \text{ to } N-2,$$
$$x_{N-2}+Fx_{N-1}=0\ , \qquad (B1)$$

where

$$F=(1-\Lambda)/\gamma \text{ and } G=(1-\alpha-\Lambda)/\gamma\ . \qquad (B2)$$

A renormalization-group recursion[12] can be constructed

by substituting every other equation into the remaining ones, resulting in $N/2$ equations of the same form, but with renormalized coefficients:

$$F'=1-FG \text{ and } G'=2-G^2 . \tag{B3}$$

The procedure is repeated, but with one of the edge coefficients having its recursion modified as

$$F'=1-G^2+G/F , \tag{B4}$$

if the starting number of equations is even, in which case this number is halved. In any case, the recursion of the inner coefficient G depends only on itself. It has unstable fixed points at $G^*=1$ and -2, with a preimage of the latter at $G=2$. This recursion remains chaotic in the interval $|G|<2$, and runs away to the fixed point $G^*=-\infty$ and $F^*=-\infty$ for $|G|>2$. The extended and bound states found in Sec. III D, in fact, respectively fall into the chaotic and runaway fixed-point renormalization-group behaviors. In previous works on electronic[13] and harmonic[14] chains, it was similarly found that extended and localized states, respectively, have chaotic and fixed-point renormalization-group behaviors.[15]

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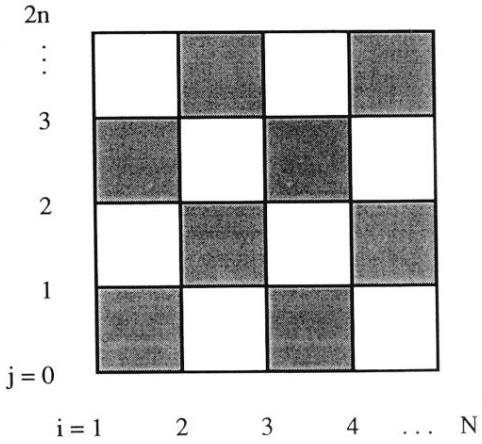

FIG. 1. The classical $d=2$ model that is equivalent to the *XXZ* Heisenberg quantum-spin chain (Ref. 1). There is a classical variable $m_{i,j}=\pm 1$ at each site (i,j). These are coupled only in the shaded squares, with the interaction of Eq. (3), which shows that the model is extremely anisotropic. There are N (the number of original *XXZ* Heisenberg spins) sites horizontally and $2n+1$ (the number of inserted sets of states plus 2) sites vertically. Various specifications of the horizontal boundary conditions determine the property of the original quantum system that is studied via the classical system (Sec. III).