Correlation functions of the one-dimensional random-field Ising model at zero temperature

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We consider the one-dimensional random-field Ising model, where the spin-spin coupling J is ferromagnetic and the external field is chosen to be $+h$ with probability p and $-h$ with probability $1-p$. At zero temperature, we calculate an exact expression for the correlation length of the quenched average of the correlation function $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ in the case that $2J/h$ is not an integer. The result is a discontinuous function of $2J/h$. When $p = \frac{1}{2}$, we also place a bound on the correlation length of the quenched average of the correlation function $\langle s_0 s_n \rangle$.

I. INTRODUCTION

The one-dimensional random-field Ising model is an intriguing example of a system with nonzero entropy at zero temperature. The Hamiltonian for this system is

$$H = -J \sum_i s_i s_{i+1} - \sum_i h_i s_i , \qquad (1.1)$$

where the spin at each site, s_i, takes the values ± 1 and $J > 0$. The external field $\{h_i\}$ is frozen with the value of the field at each site chosen as an independent random variable with, say, 50% probability to be $+h$ and with 50% probability to be $-h$. For a fixed external field $\{h_i\}$ and $h \leq 2J$, there is more than one spin configuration $\{s_i\}$, which minimizes the energy. The zero-temperature entropy has been calculated,[1-3] and for $2J/h$ not equal to an integer it depends only on the integer q, defined by

$$q < \frac{2J}{h} < q+1 . \qquad (1.2)$$

For $2J/h$ equal to an integer, the entropy is larger than it is when $2J/h$ is slightly less than or greater than this integer. Thus, the entropy is a discontinuous function of $2J/h$. For $h > 2J$, i.e., $q = 0$, the entropy is zero, since the spins must follow the external field.

We are interested in the correlations between the spin at site j and the spin at site $j+n$. For a fixed external field configuration, even at zero temperature, we must average over the different degenerate spin configurations. We denote this thermal average by $\langle\ \rangle$. An object such as $\langle s_j s_{j+n} \rangle$ will depend on the particular external field configuration, most sensitively on the values of the external field near and between the sites j and $j+n$. What is typically measured in scattering experiments is the average over the sample

$$G(n) \equiv \lim_{N \to \infty} \frac{1}{N} \sum_j \langle s_j s_{j+n} \rangle , \qquad (1.3)$$

where the sum is over the N sites of the system. By the usual ergodic arguments we can replace the spatial average by an average over the various possible external field configurations. This allows us to write

$$G(n) = \overline{\langle s_0 s_n \rangle} , \qquad (1.4)$$

where the overbar means average over all external field configurations generated with the probabilistic rule introduced earlier for $\{h_i\}$.

The ordinary (nonrandom) one-dimensional Ising model has spontaneous magnetization at zero temperature. Any nonzero external random field, chosen with equal probabilities to be $\pm h$, destroys this magnetization. This means that for $h \neq 0$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_j \langle s_j \rangle = 0 \qquad (1.5)$$

or alternatively that $\overline{\langle s_j \rangle} = 0$. However, for a fixed external field configuration $\langle s_j \rangle$ will generally not be zero at the site j. In fact, $\langle s_j \rangle$ and $\langle s_{j+n} \rangle$ will be correlated so the calculation of

$$\chi(n) = \lim_{N \to \infty} \frac{1}{N} \sum_j [\langle s_j s_{j+n} \rangle - \langle s_j \rangle \langle s_{j+n} \rangle] , \qquad (1.6)$$

which is the same as

$$\chi(n) = \overline{\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle} , \qquad (1.7)$$

is rather different than the calculation of $G(n)$.[4]

In this paper we calculate the correlation length L of $\chi(n)$, which depends on its large-n behavior, i.e., $\chi(n) \sim e^{-n/L}$. If the external random field is chosen with probability p to be h and with probability $1-p$ to be $-h$, we find that

$$L = \frac{-1}{\ln\{2(p - p^2)^{1/2} \cos[\pi/(q+2)]\}} , \qquad (1.8)$$

where q is defined by (1.2) when $2J/h$ is not an integer.

We study the correlation function $G(n)$ in the case when the external field is chosen with equal probability to be $\pm h$. We are able to show that for any y such that $y > e^{-L}$ with L given by 1.8 for $p = \frac{1}{2}$, we have $|G(n)| \leq y^n$ for large enough n. We argue, but do not prove, that the bound is saturated, which would imply that the two correlation functions have the same correlation lengths. For both $\chi(n)$ and $G(n)$ we also discuss how our calculations are modified if $2J/h$ is an integer.

The random-field Ising model is closely related to the random-bond Ising model in a uniform field. To see this, replace each s_i in (1.1) with $h_i s_i / h$. The new Hamiltonian has a uniform field and random bonds $J_{i,i+1} = J h_i h_{i+1} / h^2$. If the h_i are selected with equal probabilities to be $\pm h$, then the bonds take the values $\pm J$ with equal probabilities, and additionally each bond is independent of the others. However, if $h_i = +h$ with $p \neq \frac{1}{2}$, then the associated random-bond model does not have the bonds chosen independently. The random-bond model with independently chosen bonds is what is usually considered in the literature. For example, in Refs. 1–3 the entropy is actually calculated in a random-bond model so these results only apply to the random field model if $p = \frac{1}{2}$.

The description[2] of the degenerate configurations, which contribute to the zero-temperature entropy, which we give in Sec. II, is needed before we can attempt to calculate the correlation functions (Secs. III and IV).

II. SPIN CONFIGURATIONS AT ZERO TEMPERATURE

For a given external field configuration $\{h_i\}$, the spin configurations $\{s_i\}$ are those that minimize the energy (1.1). The first term in the energy favors agreement between adjacent sites, whereas the second term prefers the spin at a site to agree with the random field at that site. A given $\{h_i\}$ will not uniquely specify the $\{s_i\}$. However, there can be stretches of the $\{h_i\}$ that force the associated spins to take unique values. For example, if we find a very long stretch where all of the $h_i = +1$, then clearly in that stretch all of the $s_i = +1$.

Consider a sequence of k sites at which $h_i = -1$ and imagine that this sequence is flanked on both sides by very long stretches where $h_i = +1$. In the flanking regions all $s_i = +1$. A simple calculation shows that if $k > 2J/h$, then the spins in the sequence all match the external field, i.e., $s_i = -1$, whereas if $k < 2J/h$ the spins are all $s_i = +1$. Note that if $2J/h$ is an integer and k is equal to $2J/h$, then these two spin configurations both minimize the energy. Here we see a source of entropy present only when $2J/h$ is an integer.

To understand how entropy arises for any noninteger value of $2J/h > 1$, consider a long stretch where $h_i = +1$ followed by a long stretch where $h_i = -1$. This is a deterministic situation where the spins follow the external field. Now at the break point imagine inserting two additional sites where the external field takes the values -1 and $+1$ so the $\{h_i\}$ configuration is

$$\{h_i\} = \cdots + + + + + - + - - - - - - \cdots \;.$$

The two spin configurations,

$$\{s_i\} = \cdots + + + + + + + - - - - - - \cdots$$

and

$$\{s_i\} = \cdots + + + + + - - - - - - - \cdots \;,$$

have the same energy (by symmetry), and by comparing with two other configurations (where the spins are $-+$ and $+-$ in the middle) we see that the two illustrated configurations minimize the energy. This is an example of zero-temperature entropy.

We have seen that regions of constant h_i longer than $2J/h$ force the spins to line up with the field and that there are configurations that do not determine the spins. We now state the general rules that dictate which regions of the $\{h_i\}$ configuration necessarily determine the spins.

We denote a region of sites as $[l,r]$ if the leftmost site is l and the rightmost site is r. We define

$$W[l,r] = \frac{1}{h} \sum_{i=l}^{r} h_i \;, \tag{2.1}$$

which measures the difference between the number of sites at which the random field is positive and the number at which it is negative in the region $[l,r]$. We further call $[l,r]$ an R^+ region if it meets the following three conditions:

$$R^+ \text{ conditions:} \begin{cases} \text{(i)} & W[l,r] > W[l,i], \quad l \leq i < r \\ \text{(ii)} & W[l,r] > W[i,r], \quad l < i \leq r \\ \text{(iii)} & W[l,r] \geq 2J/h \;. \end{cases}$$

$$\tag{2.2}$$

Condition (i) says that starting from l, the number of sites at which $h_i = +1$ minus the number at which $h_i = -1$ has a maximum in $[l,r]$ at r and condition (iii) tells us that this maximum exceeds (or equals) $2J/h$. Similarly, we call $[l,r]$ an R^- region if

$$R^- \text{ conditions:} \begin{cases} \text{(i)} & W[l,r] < W[l,i], \quad l \leq i < r \\ \text{(ii)} & W[l,r] < W[i,r], \quad l < i \leq r \\ \text{(iii)} & W[l,r] \leq -2J/h \;. \end{cases}$$

$$\tag{2.3}$$

Now an R^+ region favors having all of the spins in $[l,r]$ be $+1$ over having them all be -1, whereas an R^- region prefers all -1 spins over all $+1$. To guarantee that all spins in an R^+ region $[l,r]$ be $+1$, there should be no R^- subregions of $[l,r]$. We call the region $[l,r]$ a D^+ region if it is an R^+ region with no R^- subregion. Similarly, we call the region $[l,r]$ a D^- region if it is an R^- region with no R^+ subregion.

The spin at a given site will be $+1$ if the site is in a D^+ region. Similarly, if a site is in a D^- region the spin at that site will be -1. Now a given site will be either in a D^+ region, a D^- region, or in a region where the spin is not forced, an E region. With our definitions a D^+ re-

gion can be contained in a larger D^+ region. We call a D^+ region D_m^+ (for maximal) if it is not contained in any other D^+ region. Similarly, a D_m^- region is a D^- region, which is not a subset of a larger D^- region. Every lattice site is either in a D_m^+, D_m^-, or E region. We cannot have consecutive D_m^+ regions, since together they would form a D^+ region that contained them both. Similarly, two consecutive E regions will be considered as one E region.

To understand what an E region looks like, consider three consecutive regions D_m^+, E, and D_m^-, and let l be the leftmost site of E and r be the rightmost site, i.e., $E = [l,r]$. Consider $W[l-1,i]$ as a function of i. Then $W[l-1,l-1] = +1$, since $l-1$ is the rightmost site of D_m^+, which must end in a $+1$ site. For any $k \in E$, $W[l-1,k] \leq 1$ because if this were not the case then the D_m^+ region could be extended. Similarly $W[r+1,r+1] = -1$ and $W[k,r+1] \geq -1$ for $k \in E$. These two inequalities imply that $W[l,r] = 0$, which means that the entropy region has the same number of $h_i = +1$ sites as $h_i = -1$ sites. The function $W[l-1,i]$ is equal to 1 at $i = l-1$ and at $i = r$. It can achieve the value 1 but not exceed it at other sites in E, and it also can never go below $1 - 2J/h$. If it did, subregions of the E region would meet the conditions for being D^+ or D^- regions. These properties of $W[l-1,i]$ will be used when we calculate the correlation functions.

We now describe the degenerate spin configurations associated with a $D_m^+ E D_m^-$ region. The spins are all $+1$ in D_m^+ and continue to be $+1$ until some point in E, where they switch to -1 and remain -1 through D_m^-. The last site at which s_i takes the value $+1$ must be at $i = l-1$ or r or any other possible site in E at which $W[l-1,i]$ happens to be $+1$. We will illustrate this with an example momentarily. First note that a mirror construction is used for a $D_m^- E D_m^+$ region. It is also possible to show that no $D_m^- E D_m^-$ or $D_m^+ E D_m^+$ regions can exist (when $2J/h$ is not an integer).

As an illustration, suppose $2 < 2J/h < 3$ and we have the $\{h_i\}$ configuration

$$\{h_i\} = \cdots + + + + - - + + - + - - + - - \cdots$$
$$\cdots \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ \cdots \ ,$$

where the numbers below are the site labels. It is useful to plot $W[1,i]$ from which we can infer the values of W on subregions (see Fig. 1). We can see that $[1,4]$ is a D^+ region, whereas $[11,15]$, which has $W[11,15] = -3$ is a D^- region. The region $[5,10]$ is an entropy region and if we look at $W[4,i]$ for $4 \leq i \leq 10$ we see that it is equal to

1 at $i = 4$, 8, and 10. The three degenerate spin configurations are

$$\{s_i\} = \cdots + + + + - - - - - - - - - - - \cdots \ ,$$
$$\{s_i\} = \cdots + + + + + + + + - - - - - - - \cdots \ ,$$
$$\{s_i\} = \cdots + + + + + + + + + + - - - - - - \cdots$$
$$\cdots \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ \cdots \ .$$

It is also interesting to study the same $\{h_i\}$ configuration if $1 < 2J/h < 2$. In this case the only E region is $[9,10]$ and the two possible spin configurations are

$$\{s_i\} = \cdots + + + + - - + + - - - - - - - \cdots \ ,$$
$$\{s_i\} = \cdots + + + + - - + + + + - - - - - \cdots$$
$$\cdots \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ \cdots \ ,$$

which do not coincide with any of the three possibilities for $2 < 2J/h < 3$. If $2J/h = 2$ then all five configurations are degenerate and the entropy is larger than it is on either side of $2J/h = 2$. In general, when $2J/h$ is an integer there are even more degenerate configurations than those one would discover by looking at $2J/h$ just above and just below its integer value. This is because there can be degenerate configurations, which within a single E region look in part like those for $2J/h$ just above its integer value and in part look like those for $2J/h$ just below.

III. THE CORRELATION LENGTH OF $\chi(n)$

Recall that

$$\chi(n) = \overline{\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle} \ , \tag{3.1}$$

where $\langle \ \rangle$ is the average over different degenerate spin configurations for fixed $\{h_i\}$ and the overbar indicates the average over $\{h_i\}$. In this section we determine the dominant large-n behavior of $\chi(n)$ for $q < 2J/h < q+1$ with q an integer. Note that for a given $\{h_i\}$, s_0 and s_n are either determined by the external field or they are not. We can think of s_0 and s_n as random variables and $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ as their statistical covariance. If either s_0 or s_n is forced by the $\{h_i\}$ to take a particular value, then $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ vanishes. Thus for $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ to be nonvanishing, both s_0 and s_n must be in E regions. However, $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ also vanishes if s_0 and s_n are independent. Now if s_0 and s_n
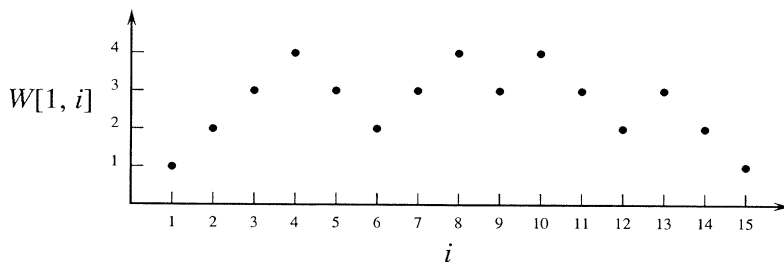

$W[1,i]$

FIG. 1. The function $W[1,i]$ for an external field configuration, which has associated entropy.

are in different E regions, that is, E regions separated by at least one D_m^+ or D_m^- region, then the value of s_0 is independent of the value of s_n and the correlation vanishes. For $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ to be nonzero for a given $\{h_i\}$, both s_0 and s_n must be in the same E region.

For a given value of n we will calculate the probability, i.e., the fraction of configurations $\{h_i\}$, which have 0 and n in the same E region. The distribution of $\{h_i\}$ configurations is given by assuming for simplicity that at site i, $h_i = +1$ or -1 each with probability one-half. (The calculation is easily carried through with $\frac{1}{2}$ replaced by p.) As a first step we calculate the fraction of E regions which have length R. Consider an E region, which, for example, begins at site 1 and ends at site R and has a D_m^+ region to the left and a D_m^- region to the right. (Note that the sites 1 and R have nothing to do with the sites 0 and n mentioned above.) The function $W[0,i]$ as

discussed in the preceding section has the following properties: (i) $W[0,0]=1$, $W[0,1]=0$, $W[0,R]=1$, $W[0,i]\leq 1$ for $i\in[1,R]$, and, since $W[0,i]>1-2J/h$ with $q<2J/h<q+1$ we also have (ii) $W[0,i]>-q$ for $i\in[0,R]$. The sites just to the right of R form the beginning of a D_m^- region. Therefore, the function $W[0,i]$ for $i>R$ must take the value $-q$ before it takes the value $+1$ or else the E region could have been extended beyond R. So (iii) $W[0,i]$ for $i>R$ goes through $-q$ before it goes through $+1$.

The randomly generated field $\{h_i\}$ can be thought of as determining (or as equivalent to) a random walk (RW), where position at time i changes by h_i/h. The function $W[0,i]$ with $W[0,0]=1$ is the position of the random walk at time i given that at $i=0$ the walk is at $+1$. If we call f_R the normalized probability that an E region has length R we see from (i), (ii), and (iii) above that

$$f_R = N \times \text{Prob (RW goes from 1 to 1 in } R \text{ steps without hitting } +2 \text{ or } -q)$$

$$\times \text{Prob (RW starting at 1 goes to } -q \text{ before returning to 1) ,} \tag{3.2}$$

where N is the normalization factor. Actually, we will calculate the transform

$$\bar{f}(\lambda) = \sum_{R=0}^{\infty} f_R \lambda^R , \tag{3.3}$$

which is more useful for our purposes and from which we can infer f_R. (The $R=0$ term in the sum corresponds to an E region of zero size, which occurs when there are no sites between a D_m^+ region and a D_m^- region.) Note that the transform (3.3) is intimately connected to the transform of the correlation function $\bar{\chi}(\lambda) = \sum_{R=0}^{\infty} \chi(R) \lambda^R$; however we only need to calculate (3.3) to infer the large n behavior of $\chi(n)$.

Turning to the first term in (3.2), let

$$Z_1(R) = \text{Prob (RW goes from 1 to 1 in } R \text{ steps without hitting } +2 \text{ or } -q) . \tag{3.4}$$

To find this, we solve for the more general function

$$Z_j(R) = \text{Prob (RW goes from } j \text{ to 1 in } R \text{ steps without hitting } +2 \text{ or } -q) \tag{3.5}$$

and then set $j=1$. Since a walk starts at j and immediately goes to $j+1$ or $j-1$ we have for $-q+1\leq j\leq 1$,

$$Z_j(R) = \tfrac{1}{2} Z_{j-1}(R-1) + \tfrac{1}{2} Z_{j+1}(R-1) . \tag{3.6}$$

By (3.5), $Z_{-q}(R) = Z_2(R) \equiv 0$ for $R\geq 0$ and $Z_1(0)=1$. If we define $Z_2(-1)\equiv 2$ and $Z_j(-1)\equiv 0$ for $-q\leq j\leq 1$, then (3.6) holds for $R=0$ as well as for $R>0$.

We define the transform

$$\bar{Z}_j(\lambda) = \sum_{R=-1}^{\infty} Z_j(R) \lambda^R , \tag{3.7}$$

which from (3.6) gives for $-q+1\leq j\leq 1$,

$$\bar{Z}_j(\lambda) = \tfrac{1}{2} \lambda \bar{Z}_{j-1}(\lambda) + \tfrac{1}{2} \lambda \bar{Z}_{j+1}(\lambda) . \tag{3.8}$$

We can solve (3.8) by making the ansatz that

$$\bar{Z}_j(\lambda) = \alpha u^j + \beta v^j \tag{3.9}$$

with the boundary condition that $\bar{Z}_{-q}(\lambda)=0$ and $\bar{Z}_2(\lambda)=2/\lambda$ as explained above. The solution is

$$\bar{Z}_j(\lambda) = \frac{2}{\lambda} \left[\frac{u^{q+j} - v^{q+j}}{u^{q+2} - v^{q+2}} \right] \tag{3.10}$$

with

$$u = \frac{1}{\lambda} + \left[\frac{1}{\lambda^2} - 1 \right]^{1/2} \text{ and } v = \frac{1}{\lambda} - \left[\frac{1}{\lambda^2} - 1 \right]^{1/2} . \tag{3.11}$$

Thus we obtain

$$\bar{Z}_1(\lambda) = \frac{2}{\lambda} \left[\frac{u^{q+1} - v^{q+1}}{u^{q+2} - v^{q+2}} \right] . \tag{3.12}$$

We now return to (3.2) and we see that the second

probability factor is independent of R and can therefore be absorbed in the normalization factor N. Thus we have for the transform $\bar{f}(\lambda)$ defined by (3.3),

$$\bar{f}(\lambda) = N\bar{Z}_1(\lambda) \tag{3.13}$$

which by (3.12) gives

$$\bar{f}(\lambda) = \frac{2N}{\lambda}\left[\frac{u^{q+1}-v^{q+1}}{u^{q+2}-v^{q+2}}\right]. \tag{3.14}$$

Now (3.14) can be expanded in only non-negative powers of λ as in (3.3). The normalization condition $\sum_{R=0}^{\infty} f_R = 1$ is equivalent to $\bar{f}(1) = 1$, which allows us to solve for N and we obtain

$$\bar{f}(\lambda) = \frac{q+2}{q+1}\frac{1}{\lambda}\left[\frac{u^{q+1}-v^{q+1}}{u^{q+2}-v^{q+2}}\right]. \tag{3.15}$$

Again, if we expand $\bar{f}(\lambda)$ as a power series in λ, the coefficient f_R is the probability that an E region has length R.

We now turn to finding

$$Q_n = \text{Prob } (0 \text{ and } n \text{ are in the same } E \text{ region}) . \tag{3.16}$$

The answer is

$$Q_n = N' \sum_{R>n} Rf_R\left[\frac{R-n}{R}\right], \tag{3.17}$$

where we explain each factor in turn. The factor N' contains the probability that 0 is in an E region and other n-independent factors. For the E region to contain 0 and n it must have length $R>n$. The factor Rf_R is proportional to the probability that an E region has length R given that 0 is in it. The factor $(R-n)/R$ is the probability that n is in an E region of length R given that 0 is in it.

We can also define the transform

$$\bar{Q}(\lambda) = \sum_{n=0}^{\infty} Q_n\lambda^n , \tag{3.18}$$

which by (3.17) is

$$\bar{Q}(\lambda) = N' \sum_{R=1}^{\infty} \sum_{n=0}^{R-1} f_R(R-n)\lambda^n . \tag{3.19}$$

It is straightforward to do the sum on n and then on R to obtain

$$\bar{Q}(\lambda) = N'\frac{1}{(1-\lambda)}\left[\frac{d\bar{f}}{d\lambda}\bigg|_{\lambda=1} - \frac{\lambda}{1-\lambda}\bar{f}(1)\right.$$
$$\left. + \frac{\lambda}{1-\lambda}\bar{f}(\lambda)\right]. \tag{3.20}$$

We have the explicit form of $\bar{f}(\lambda)$ through (3.15) so the coefficients Q_n in (3.18) can be determined for all n. Thus we have computed the (unnormalized) probability that 0 and n are in the same E region.

The large n behavior of Q_n can be extracted if we know the smallest value of $\lambda > 0$, say λ_*, at which (3.20) blows up. This is because the expansion (3.18) will blow up first

at $\lambda = \lambda_*$ if $Q_n \sim \lambda_*^{-n}$. Now (3.20) does not blow up at $\lambda = 1$ as can be seen by expanding $\bar{f}(\lambda)$ about $\lambda = 1$. The only way for (3.20) to blow up at $\lambda \neq 1$ is for $\bar{f}(\lambda)$ to blow up. From (3.15) we see that this can occur only if $u^{q+2} = v^{q+2}$, where again u and v are given by (3.11). A simple calculation gives

$$\lambda_*^{-1} = \cos\left[\frac{\pi}{q+2}\right], \tag{3.21}$$

from which we conclude that, for large n, the probability that sites 0 and n are in an E region goes as

$$Q_n \sim \cos^n\left[\frac{\pi}{q+2}\right]. \tag{3.22}$$

Given that 0 and n are in the same E region we need to calculate $\overline{\langle s_0 s_n \rangle - \langle s_0 \rangle\langle s_n \rangle}^E$ where the E on the overbar denotes average only over those $\{h_i\}$ for which 0 and n are in the same E region. By examining the degenerate spin configurations in an E region one can see that $\langle s_0 s_n \rangle - \langle s_0 \rangle\langle s_n \rangle \geq 0$ for all $\{h_i\}$, so no cancellations take place in this average. We are interested in n large and the most probable configurations contributing to this average are those for which the E region is just slightly longer than n so the site 0 and the site n are near the edges of E. The number of degenerate spin configurations associated with an E region is proportional to its length from which we can estimate that $\overline{\langle s_0 s_n \rangle - \langle s_0 \rangle\langle s_n \rangle}^E \sim 1/n^2$.

Combining (3.22) with the estimate of the last paragraph gives, for large n,

$$\chi(n) = \overline{\langle s_0 s_n \rangle - \langle s_0 \rangle\langle s_n \rangle} \sim \cos^n\left[\frac{\pi}{q+2}\right], \tag{3.23}$$

from which we infer that the correlation length is

$$L = \frac{-1}{\ln\cos[\pi/(q+2)]} \tag{3.24}$$

It is worth noting that if we add a constant external field, no matter how small, then we destroy the zero-temperature entropy, since the degeneracy is lifted. In this case $\chi(n) = 0$ and there is also a nonzero magnetization, i.e., $\langle s_0 \rangle \neq 0$. Alternatively we can pick the random field at each site to be $+h$ with probability p and to be $-h$ with probability $1-p$, and then $\langle s_0 \rangle \neq 0$ unless $p = \frac{1}{2}$. In this case it is straightforward to redo the calculation of the correlation length, and we get (1.8).

If $2J/h$ is an integer, say k, then the calculation of the large n behavior of $\chi(n)$ changes in two ways. First, the probability that sites 0 and n are in the same E region is larger when $2J/h$ is equal to k than when $2J/h$ is slightly greater than k. For example, when $2J/h = 1$, site 0 and site n are in the same E region in the following configuration:

$$\{h_i\} = \cdots + + - + - + - + - + + \cdots ,$$
$$\phantom{\{h_i\} = \cdots + + - +}0n$$

whereas the entire pictured region is D^+ if $1 < 2J/h < 2$. But this only approximately doubles the chance that 0 and n are in the same E region and has no effect on the

correlation length.

However the typical value of $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ also changes if $2J/h = k$ as opposed to $k < 2J/h < k+1$. In an E region of length n when $2J/h$ is not an integer there are of order n configurations and $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle^E$ is of order n^{-2}. When $2J/h$ is an integer there are more configurations. For example, when $2J/h = 1$ an E region of length L has F_{L+2} configurations where F_L is the Lth Fibonacci number[5] ($F_1 = F_2 = 1$, $F_{L+2} = F_{L+1} + F_L$). Now for each $\{h_i\}$, $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle$ is still non-negative, but $\langle s_0 s_n \rangle - \langle s_0 \rangle \langle s_n \rangle^E$ is of order F_n^{-2}. For $1 < 2J/h < 2$ we have that $\chi(n) \sim (\frac{1}{2})^n$ as can be seen from (3.22). For $2J/h = 1$ we have $\chi(n) \sim (\frac{1}{2})^n F_n^{-2}$, and, since $F_n \sim [(1+\sqrt{5})/2]^n$, we infer that $L^{-1} = \ln[(1+\sqrt{5})^2/2]$.

IV. A BOUND ON THE CORRELATION FUNCTION $G(n)$

We are interested in the correlation length of $\overline{\langle s_0 s_n \rangle}$ in the case when the external field is chosen with equal probabilities to be $\pm h$ at each site. In this case $\overline{\langle s_0 \rangle} = 0$. We begin by using the symmetry of the problem to identify a class of the $\{h_i\}$, which has the property that $\langle s_0 s_n \rangle$ averaged over this class is zero. Roughly, this is the class of $\{h_i\}$, where sites 0 and n are separated by at least two disjoint D regions. We then will estimate the probability that the sites 0 and n are not in this class. This turns out to have the same large n behavior as the probability that sites 0 and n are in the same E region, which was relevant in calculating $\chi(n)$.

Suppose we are given a particular external field configuration $\{h_i\}$. Let $[l_0, r_0]$ be a minimal D^+ or D^- region with $r_0 \geq 0$ and r_0 as small as possible. (A minimal D^+ or D^- region has no subregion that is a D^+ or D^- region.) Let $[l_n, r_n]$ be a minimal D^+ or D^- region with $l_n \leq n$ and l_n as large as possible. Suppose $[l_0, r_0]$ and $[l_n, r_n]$ are disjoint, that is $r_0 < l_n$ (which is likely if n is large). Switching all the random field signs at $l_n, l_{n+1}, l_{n+2}, \ldots$ has the effect of leaving $\langle s_0 \rangle$ unchanged but reversing the sign of $\langle s_n \rangle$. Note that $\langle s_0 \rangle \langle s_n \rangle = \langle s_0 s_n \rangle$ in these cases and also that performing the switch twice returns us to the original configuration. Thus $\langle s_0 s_n \rangle$ averaged over all of these configurations is zero.

Next we estimate the probability of obtaining a configuration of external fields $\{h_i\}$ with $r_0 \geq l_n$. It is these configurations that produce a nonzero correlation function. The reader who wishes to skip the details of this estimate should proceed to Eq. (4.11), which gives the result.

We begin by calculating the distribution of r_0, which is the smallest non-negative site which is the rightmost site of a D region. Fix $h_{-1}, h_{-2}, h_{-3}, \ldots$. We actually calculate the distribution of r_0 conditional on these values. We will see, however, that the probability of r_0 being large will be essentially the same for any choice $h_{-1}, h_{-2}, h_{-3}, \ldots$. Given the random field at the negative sites, find the largest value of $a < 0$ so that $[b, a]$ is a D region for some $b < a$. Without loss of generality, assume it is a D^+ region. (Note that the external field at the non-negative sites may make $[b, a]$ part of an even larger D region. However, a is defined only using the values of the fields at the negative sites.)

Consider $W[a+1, r]$ as a function of $r \geq a+1$. If $W[a+1, r]$ reaches the value 1 before it goes through $-q-1$, then the D^+ region $[b, a]$ can be extended. This cannot happen for $r < 0$, for if it did a would not be the largest negative site ending a D region on the right. If $W[a+1, r]$ reaches the value $-q-1$ at some r before reaching the value 1, then there is a D^- region with r as its right end. Again, but assumption, this cannot happen for $r < 0$. We can see now that r_0 is the smallest value of $r \geq 0$ such that $W[a+1, r] = 1$ or $W[a+1, r] = -q-1$. Because $W[a+1, r] = W[a+1, -1] + W[0, r]$ for $r \geq 0$, we can say that r_0 is the first r such that $W[0, r] = A$ or $W[0, r] = B$ with

$$
\begin{aligned}
A &= 1 - W[a+1, -1] , \\
B &= -q - 1 - W[a+1, -1]
\end{aligned}
\tag{4.1}
$$

(where for $a = -1$ we define $W[0, -1] \equiv 0$). Thus for a fixed configuration at the negative sites, the distribution of $r_0 \geq 0$ depends only on the single number $W[a+1, -1]$, which obeys $-q \leq W[a+1, -1] \leq 0$.

As in the preceding section, we view $W[0, r]$ as equivalent to a random walk. Let

$$
Y_j(R) = \text{Prob (RW starting at } j \text{ first hits } A > 0 \text{ or } B < 0 \text{ at step } R) .
\tag{4.2}
$$

Note that

$$
\text{Prob}(r_0 = R \mid h_{-1}, h_{-2}, \ldots) = Y_0(R+1) .
\tag{4.3}
$$

Now

$$
Y_j(R) = \tfrac{1}{2} Y_{j-1}(R-1) + \tfrac{1}{2} Y_{j+1}(R-1)
\tag{4.4}
$$

with the boundary conditions that $Y_A(0) = Y_B(0) = 1$ and $Y_A(R) = Y_B(R) = 0$ for $R > 0$. We can solve for the transform of $Y_j(R)$,

$$
\overline{Y}_j(\lambda) = \sum_{R=0}^{\infty} Y_j(R) \lambda^R ,
\tag{4.5}
$$

as we did in the preceding section to obtain

$$
\overline{Y}_0(\lambda) = \frac{u^A - v^A + u^{-B} - v^{-B}}{u^{q+2} - v^{q+2}} ,
\tag{4.6}
$$

where again u and v are given by (3.11) and we have used the fact that $A - B = q+2$. Note that $\overline{Y}_0(\lambda)$ blows up for the first time at λ_* given by (3.21), which is independent of A and B.

The coefficient of λ^{R+1} in (4.6) gives the probability that r_0 has the value of R given a fixed $h_{-1}, h_{-2}, \ldots$, which determine A and B. Similarly we could obtain an identical expression for the probability that $n - l_n$ has a given value for a fixed $h_{n+1}, h_{n+2}, \ldots$. We are interested in calculating the probability that $r_0 \geq l_n$ with $h_{-1}, h_{-2}, \ldots \equiv \{h_<\}$ and $h_{n+1}, h_{n+2}, \ldots \equiv \{h_>\}$ both fixed. Now

$$\text{Prob}(r_0 \geq l_n | \{h_<\}, \{h_>\}) = 1 - \text{Prob}(r_0 < l_n | \{h_<\}, \{h_>\})$$

$$= 1 - \sum_{\substack{i,j \geq 0 \\ i+j < n}} \text{Prob}(r_0 = i, n - l_n = j | \{h_<\}, \{h_>\}) . \tag{4.7}$$

The probability that $r_0 = i$ depends on the random field at sites $\leq i$, while the probability that $n - l_n = j$ depends on the sites $\geq n - j$, which do not overlap in the sum in (4.7), so the distribution can be taken as independent. It then follows that

$$\text{Prob}(r_0 \geq l_n | \{h_<\}, \{h_>\}) = \sum_{\substack{i,j \geq 0 \\ i+j \geq n}} \text{Prob}(r_0 = i | \{h_<\}) \, \text{Prob}(n - l_n = j | \{h_>\}) , \tag{4.8}$$

so for the purposes of our calculation we can treat the full distributions of r_0 and $n - l_n$ as independent. Consider the transform of the probability that $r_0 + n - l_n$ has the value k:

$$\sum_{k=0}^{\infty} \text{Prob}(r_0 + n - l_n = k | \{h_>\}, \{h_<\}) \lambda^k = \left[\sum_{i=0}^{\infty} \text{Prob}(r_0 = i | \{h_<\}) \lambda^i \right] \left[\sum_{j=0}^{\infty} \text{Prob}(n - l_n = j | \{h_>\}) \lambda^j \right] . \tag{4.9}$$

Both transforms on the right-hand side are of the form (4.6) and the product blows up at λ_* given by (3.21) so we can say that for k large

$$\text{Prob}(r_0 + n - l_n = k | \{h_<\}, \{h_>\}) \sim \cos^k \left[\frac{\pi}{q+2} \right] \tag{4.10}$$

from which we infer that

$$\text{Prob}(r_0 \geq l_n) \sim \cos^n \left[\frac{\pi}{q+2} \right] \tag{4.11}$$

for large n.

The only configurations of the external field which contribute to $\langle s_0 s_n \rangle$ are those for which $r_0 \geq l_n$. Given a configuration with $r_0 \geq l_n$ we expect $\langle s_0 s_n \rangle \sim +1$ or -1. We have not shown that cancellations do not conspire to make the average of $\langle s_0 s_n \rangle$ over those configurations with $r_0 \geq l_n$ of order x^n with $|x| < 1$. Hence, we can only assert that

$$\text{if } y > \cos \left[\frac{\pi}{q+2} \right], \text{ then } |\overline{\langle s_0 s_n \rangle}| \leq y^n ,$$

$$\text{for } n \text{ large enough} . \tag{4.12}$$

We can write $\overline{\langle s_0 s_n \rangle}$ as

$$\overline{\langle s_0 s_n \rangle} = \chi(n) + \overline{\langle s_0 \rangle \langle s_n \rangle} . \tag{4.13}$$

We know that $\chi(n) \geq 0$, and from (3.23) we see that it decays as $\cos^n[\pi/(q+2)]$. Therefore, if $\overline{\langle s_0 \rangle \langle s_n \rangle}$ is nonnegative, we can conclude that $\overline{\langle s_0 s_n \rangle}$ has as its correlation length L, given by (3.23). However, we have not been able to prove that $\overline{\langle s_0 \rangle \langle s_n \rangle} \geq 0$, although the following argument makes us believe that it is. Consider setting the spin-spin coupling, J, equal to zero, which gives $\overline{\langle s_0 \rangle \langle s_n \rangle} = 0$. For $J > 0$ we expect the ferromagnetic coupling to induce a positive correlation between s_0 and s_n, even in the quenched average. For this reason we believe, but have not proven, that the correlation length of $\overline{\langle s_0 s_n \rangle}$ is L given by (3.24).

Finally we remark that if $2J/h = k$, an integer, the argument leading to (4.11) again gives (4.12) with $q = k$.

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