Phase diagram of the frustrated gauge-invariant Ising model

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The phase diagram of the $Z(2)$ gauge-invariant Ising model is studied at negative gauge coupling in three dimensions. Exact procedures are applied to establish the ground states and find, at nonzero temperatures, the minima of the local mean-field free energy. The resulting phase diagram exhibits one unfrustrated and two frustrated phases, in good agreement with Monte Carlo results. The interpretation of the various phases in terms of a gas of random surfaces with free edges is also discussed.

I. INTRODUCTION

The $Z(2)$ gauge-invariant Ising model was introduced by Wegner[1] as an example of a system where a phase transition occurs without a spontaneous symmetry breaking. Later the model has become popular in the context of lattice gauge theories for strong interactions.[2]

In a (hyper-) cubic lattice, let σ_i and U_{ij} be Ising variables defined on sites and bonds (joining nearest-neighboring sites), respectively, with an interaction specified by the Hamiltonian

$$\beta H\{\sigma,U\}=-\beta_M\sum_{<ij>}\sigma_i U_{ij}\sigma_j-\beta_G\sum_{\|ijkl\|}U_{ij}U_{jk}U_{kl}U_{li}\ , \tag{1.1}$$

where the sums are over nearest-neighboring sites and elementary squares, or plaquettes. In Ref. 3 it was conjectured that the model (1.1) could have some interesting relation with spin-glass systems[4] at negative values of β_G along the trajectory $\beta_G=\beta_G(\beta_M)$ where the thermal average $\langle \mathcal{U}_p\rangle \equiv \langle U_{ij}U_{jk}U_{kl}U_{li}\rangle$ for each plaquette is zero. Since U_{ij} is equal to the sign of the coupling between the nearest neighboring spins σ_i and σ_j, at $\beta_G=\beta_G(\beta_M)$ the coupling configurations have the property that plaquettes are equally probable to be found frustrated, i.e., $\mathcal{U}_p<0$, and unfrustrated, i.e., $\mathcal{U}_p>0$. However, in spin-glass systems the average over coupling configurations is quenched,[4] while it is annealed in the model we are considering.[3]

The phase diagram of the gauge Ising model has been studied at negative coupling only by numerical simulations.[5] Here we study the same problem, applying a local mean-field approximation. We make use of methods which have been found convenient for studying frustrated configurations of spin models in a different context.[6] The main difference of the resulting phase diagram with the one of Ref. 5 is that the contour of zero frustration $\beta_G=\beta_G(\beta_M)$, as an effect of the mean-field approximation, intersects a transition line at finite temperatures.

In Sec. II the model is better specified and the phase diagram at zero temperature is exactly solved. It is also noticed that the model can describe open surfaces on the lattice, and the phase diagram is discussed in terms of surfaces. In Sec. III the whole mean-field phase diagram is presented and discussed. Some conclusions follow in Sec. IV.

II. GROUND STATES

The gauge Ising model on a cubic lattice is defined by the Hamiltonian (1.1) which is locally gauge invariant with respect to the set of transformations,

$$\sigma_i\to\gamma_i\sigma_i\ ,\qquad U_{ij}\to\gamma_i U_{ij}\gamma_j\ , \tag{2.1}$$

with the γ_i's equal to $+1$ or -1.

The ground states of the Hamiltonian (1.1) can be found by applying the plaquette method which is described in detail in Refs. 6 and 7. However, as a preliminary step, it is convenient to fix the gauge. This can be done in unitary gauge by inserting the identity

$$\sum_{\{\gamma\}}\prod_i\frac{1+\gamma_i\sigma_i}{2}=1 \tag{2.2}$$

in the evaluation of the partition function

$$\mathcal{Z}(\beta_M,\beta_G)=2^{-L^3}\sum_{\sigma,U}e^{-\beta H\{\sigma,U\}}\ , \tag{2.3}$$

where L^3 is the total number of sites. Performing the transformations (2.1) and using the gauge invariance of H, one easily gets

$$\mathcal{Z}(\beta_M,\beta_G)=\sum_U e^{-\beta H\{\sigma=1,U\}}\ . \tag{2.4}$$

The plaquette method is based on rewriting the Hamiltonian as a sum over elementary cubes, that is,

$$H\{\sigma=1,U\}=\sum_c h_c\{U\}\ , \tag{2.5}$$

where

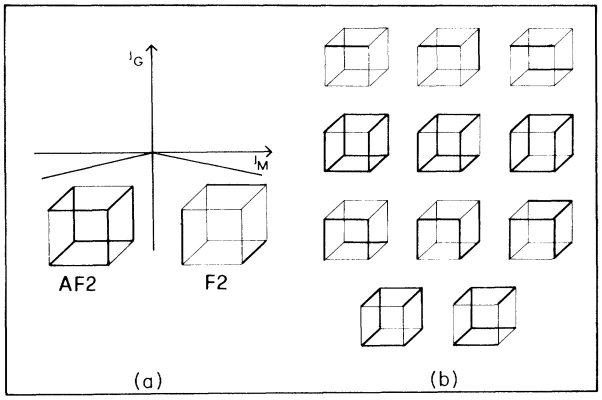

FIG. 1. The zero-temperature phase diagram of the $Z(2)$ gauge spin model in unitary gauge. The fully frustrated phases F2 and AF2, occuring at $J_G < -|J_M|/4$ for $J_M > 0$ and $J_M < 0$, respectively, are represented in terms of bond configurations over a single cube where the bold bonds are negative. At $J_G = -J_M/4$ and $J_M > 0$, the phase F2 and the phase with all the bonds positive are degenerate with the configurations in the first line of Fig. 1(b). The cubes of the second line can be used to build up ground states at $J_G = J_M/4$ and $J_M < 0$; they can be obtained by reversing the sign of the corresponding configurations at $J_M > 0$. The cubes of the last two lines are fully frustrated configurations, degenerate at $J_M = 0$.

$$h_c\{U\} = -\frac{J_M}{4} \sum_{<ij> \subset c} U_{ij} - \frac{J_G}{2} \times \sum_{||ijkl|| \subset c} U_{ij} U_{jk} U_{kl} U_{li} \ , \quad (2.6)$$

with $J_M = \beta_M/\beta$ and $J_G = \beta_G/\beta$. Since the whole lattice can be tiled by a given cube configuration of U's (two cubes with a common face have U configurations which are mirror images one of each other with respect to the plane containing the common face), one has

$$\min_U H\{\sigma = 1, U\} = L^3 \min_U h_c\{U\} \ . \quad (2.7)$$

Thus the bulk ground state is obtained tiling the whole lattice with the cube configuration of U's which minimizes $h_c\{U\}$. In some cases there are several cube configurations minimizing h_c and different tilings of the lattice are possible, which give rise to degenerate ground states.

Two gauge-invariant quantities are useful in characterizing the phases of the system: the thermal average of $\mathcal{U}_p \equiv \prod_{<ij> \in p} U_{ij}$ for each plaquette p, and of $\mathcal{L}_{ij} \equiv \sigma_i U_{ij} \sigma_j$ for each link $\langle ij \rangle$. We will also denote with $\mathcal{L}$ the space average of $\mathcal{L}_{ij}$.

In Fig. 1, the zero-temperature phase diagram is represented in terms of single cube configurations (in unitary gauge). At positive values of J_M and J_G, all the U_{ij} are equal to 1. For $J_G < -J_M/4$, the ground state can be obtained by replicating the cube shown in the picture over the whole lattice. The phase, denoted as F2, is fully frustrated in the sense that for each plaquette $\mathcal{U}_p = -1$; moreover $\mathcal{L}$ is equal to 0.5. The phase diagram at negative J_M can be simply obtained from the one at positive J_M by changing the sign of the link variables. The phase corresponding to F2 at $J_M < 0$ will be denoted as AF2. At the contact points between the homogeneous and the frustrated phases, other not fully frustrated configurations, shown in Fig. 1(b) (see the caption), are degenerate. At $J_M = 0$ and $J_G < 0$, all the fully frustrated phases with $\mathcal{L}$ ranging from -0.5 to 0.5 are degenerate.

The gauge Ising model can be also interpreted as a gas of open random surfaces made of plaquettes on the dual

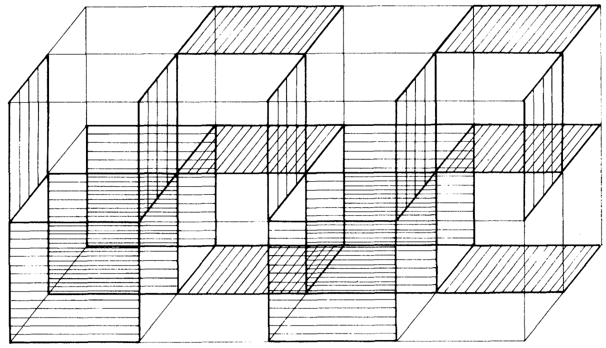

FIG. 2. The phase F2 in terms of surfaces made by plaquettes on the dual lattice.

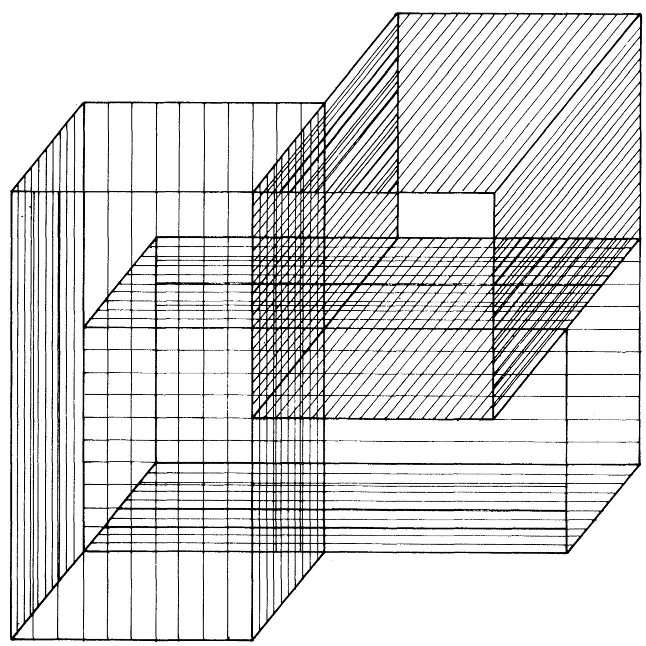

FIG. 3. The phase AF2 in terms of surfaces. The three tubes have to be intended of infinite extension. The structure is two-lattice-spacing periodic.

lattice.[6,8] A plaquette belongs to some surface if the link variable $\mathcal{L}_{ij}$ dual to that plaquette is negative. The Hamiltonian (1.1) can be mapped on a surface Hamiltonian where only the area and the edges are weighted. Intersections between edges and surfaces and between surfaces are allowed.[6] Examples of open surface configurations corresponding to the phases F2 and AF2 are given in Figs. 2 and 3.

III. MEAN-FIELD PHASE DIAGRAM

Mean-field theory for lattice gauge models has been introduced and discussed in Ref. 9. We will first study the model (1.1) without fixing the gauge.[10] In the following, the average of the variables σ_i and U_{ij} will be denoted by m_i and l_{ij}. The mean-field free-energy functional corresponding to Eq. (1.1) is

$$\beta\Gamma(\{m_i\},\{l_{ij}\}) = -\beta_M \sum_{<ij>} m_i l_{ij} m_j - \beta_G \sum_{\|ijkl\|} l_{ij} l_{jk} l_{kl} l_{li} + \sum_i \left[\frac{1+m_i}{2}\ln(1+m_i) + \frac{1-m_i}{2}\ln(1-m_i) \right] + \sum_{<ij>} \left[\frac{1+l_{ij}}{2}\ln(1+l_{ij}) + \frac{1-l_{ij}}{2}\ln(1-l_{ij}) \right] . \tag{3.1}$$

The translationally invariant minima of Eq. (3.1) have been discussed in Ref. 9. At negative gauge couplings, frustrated configurations are favored and one has to look for solutions which are not uniform.

We will study $\Gamma(\{m_i\},\{l_{ij}\})$ for all values of the couplings, applying the method described in Ref. 6. The method consists in writing $\Gamma(\{m_i\},\{l_{ij}\})$ as a sum of free energies relative to single cubes. A minimum configuration over a single cube gives a global minimum if the whole lattice can be tiled by that cube configuration. This can be generally done due to the invariance of the free energy (3.1) with respect to space reflections; therefore only minimum configurations over a single cube have to be found. This problem can be solved numerically and, in the following, we describe the results obtained by this procedure.

In Fig. 4 the phase diagram is shown for $\beta_M \geq 0$. At positive gauge coupling β_G, the standard results are reproduced.[11] The almost horizontal line at $\beta_G \approx 0.68$ separates the disordered high-temperature phase from a region where the link average $\langle \mathcal{L} \rangle$ is still zero and the plaquette average $\langle \mathcal{U}_p \rangle$ is positive; the line at $\beta_M \approx 0.16$, which ends in the triple point $T_1 \equiv (\beta_M = 0.16, \beta_G = 0.68)$, is a second-order transition line through which $\langle \mathcal{L} \rangle$ becomes positive too.

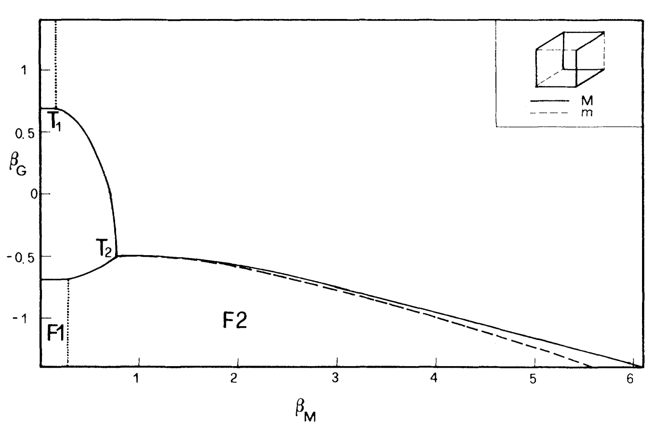

FIG. 4. The phase diagram of the $Z(2)$ gauge-invariant spin model. The full lines and the dotted lines represent first-order and second-order transition lines, respectively; the regions F1 and F2 correspond to fully frustrated phases. The dashed line is the contour $\beta_G = \beta_G(\beta_M)$ where the frustration is zero. In the narrow region between this contour and the transition line all the bond variables are positive, but assume two different values m and M, as shown in the inset.

For negative β_G, the line at $\beta_M \approx 0.29$ separates two fully frustrated phases with $\langle \mathcal{L} \rangle = 0$ (phase F1) and $\langle \mathcal{L} \rangle \approx 0.5$ (phase F2), at low and high β_M values, respectively. This line can be shown to be second order, since it is the locus where the determinant of the second derivatives of the free energy with respect to the site magnetizations vanishes. The line ends at the triple point $T_2 \equiv (\beta_M = 0.29, \beta_G = -0.68)$. In Ref. 5 this line is at $\beta_M = 1.25 \pm 0.5$ and terminates at the triple point $(\beta_M = 1.3 + 0.05, \beta_G = -0.79 \pm 0.01)$. In Fig. 4 the disordered phase, everywhere limited by a first-order transition line, does not extend itself towards zero temperature, as it does in the Monte Carlo phase diagram. Indeed, the mean-field approximation (3.1) predicts the existence of a transition line joining the triple points T_1 and T_2. This line should be interrupted at least around the $\beta_G = 0$ axis where the model becomes trivial.[9] A reminiscence of this discrepancy could be that here, at sufficiently large values of β_M, the transition along the direction $\beta_G = -\beta_M/4$ is from an ordered homogeneous phase to an inhomogeneous still not frustrated configuration where, in unitary gauge, the bond variables are all positive, but with two different values as in the inset of Fig. 4. This mean-field peculiarity is confirmed by a low-temperature expansion of the state equations which predict the above transition at $\beta_G \approx -\beta_M/4 + 0.125 \ln\beta_M$. Moving from the transition line towards more negative values of β_G, this inhomogeneous phase changes continuously without transition into the fully frustrated configuration with $\langle \mathcal{U}_p \rangle \approx -1$ and $\langle \mathcal{L} \rangle \approx 0.5$. The transition at $\beta_G \approx -\beta_M/4 + 0.125 \ln\beta_M$ can be checked to be first order.

The contour of zero frustration, which is inside the in-

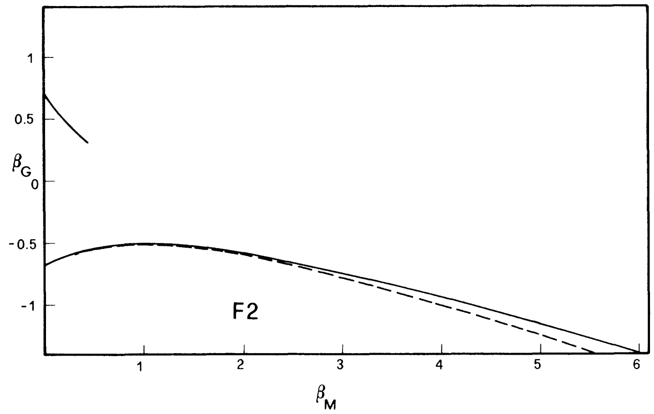

FIG. 5. The phase diagram of the $Z(2)$ gauge spin model in unitary gauge. The symbols are the same as in Fig. 4.

homogeneous phase, becomes undistinguishable from the transition line at small values of β_M. The contour crosses the transition line joining T_1 to T_2 at the point ($\beta_M=0.775$, $\beta_G=-0.505$). Finally, the horizontal line at $\beta_G \approx -0.68$, which separates the frustrated phase F1 from the disordered phase, is symmetric with respect to the corresponding line at positive β_G. Since at $\beta_M=0$ the Hamiltonian is an even function of the bond variables, and there exists a transformation changing the sign to all the $\mathcal{U}_p$'s [e.g., the one making three links of each cube negative as in Fig. 1(a)], one has $\mathcal{Z}(\beta_M=0,\beta_G)=\mathcal{Z}(\beta_M=0,-\beta_G)$. Thus the phase diagram has to be symmetric for $\beta_M=0$.

The phase diagram in unitary gauge is given for completeness in Fig. 5. At negative B_G, under the transition line, there is the F2 phase. Also here the zero frustration contour is inside the F2 phase and becomes undistinguishable from the transition line at small values of β_M. As expected, all the transition lines corresponding to a jump in the value of the m_i's have disappeared.

IV. CONCLUSIONS

We have presented an exact analysis at $T=0$ and a local mean-field calculation of the coupled $Z(2)$ gauge spin system. The results have been described in relation with the phase diagram of Ref. 5 which has been obtained by numerical simulations. The main difference is the existence here of the contour suggested in Ref. 3 as an approximate model of spin glasses. This contour, with zero frustration or zero unit Wilson loop, intersects the disorder-F2 transition line at finite values of the parameters. As an effect of the mean-field approximation, the critical dimension is lower than the one suggested in Ref. 5. It has to be noticed that the procedures used in this work can also be applied to the same model in higher dimensions.

Moreover, we have already observed that the gauge Ising model can also describe a system of open random surfaces.[6,8] The mean-field phase diagram can be interpreted in terms of open surfaces by saying that the value $(1-\langle\mathcal{L}_{ij}\rangle)/2$ represents the probability that the plaquette dual to the link ij is present. When $\langle\mathcal{L}_{ij}\rangle=0$ the probability for a plaquette to belong or not to some surface is the same. The phase F2 at very low temperatures has been represented in terms of surfaces in Fig. 2. The phase F1 with $\langle\mathcal{U}_p\rangle=-1$ and $\langle\mathcal{L}_{ij}\rangle=0$ corresponds to a configuration with plaquettes in the same local order as in F2 but with convoluted disordered configurations on large scales. Therefore, in conclusion, we observe that the $Z(2)$ gauge-invariant Ising model can also describe less conventional physical systems.

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