Noether's theorem and the mechanics of nonlinear solitary waves

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Nonlinear solitary waves—or solitons in a loose sense—in $n+1$ dimensions driven by very general external fields that must only satisfy continuity conditions together with regularity conditions at the boundaries of the system obey a quite simple equation of motion. This equation is the exact generalization to this dynamical system of an infinite number of degrees of freedom—which may be conservative or not—of the second Newton's law, setting the basis of material point mechanics. Simple examples related to the three main classes of solitons in presence of a driving force (i.e., driven Korteweg–de Vries, nonlinear Schrödinger, and nonlinear Klein-Gordon scalar fields) are displayed in order to illustrate the physical meaning of this equation of motion.

I. INTRODUCTION

The dynamical problem consisting of a nonlinear solitary wave driven by an external force was considered by many authors from the point of view of building a classical mechanics of such waves.[1–12] Let us emphasize a recent exhaustive review of several aspects of the dynamics of solitons in nearly integrable systems where the most powerful techniques based upon the inverse scattering transform or (for more simple adiabatic effects) on the Lagrangian-Hamiltonian formalism are emphasized, together with "direct" methods adapted to particular cases.[13]

However, nonlinear solitary waves driven by external forces also display peculiar properties intrinsically related to their wave-particle duality. In some sense, although nonlinear, the are also "quantum objects."[6–11] In a recent paper[14] we emphasized that there is a continuous path leading from purely linear wave mechanics of the Schrödinger type described by the original Ehrenfest theorem[15] to the most general case of what we called the "generalized Ehrenfest theorem" concerning highly nonlinear solitary wave mechanics of the Klein-Gordon type. In particular we showed that the small wave-amplitude limit of such waves, described by the *nonlinear* Schrödinger system, obeys an equation of motion which has the *same* expression as the linear (quantum-mechanical) Ehrenfest theorem although the "wave-function" now obeys the *nonlinear* Schrödinger partial differential equation.

Therefore, investigating the principles of mechanics of nonlinear solitary waves when they are under the action of external forces—which may in special cases be derived from a potential—has a twofold interest. (i) It may concern those physicists who are taking part in the debate of the conceptual basis of quantum theory[16] as it provides them with exceptional examples of wave-particle duality facing external constraints. (ii) It ranges over many other aspects of theoretical physics a well as over applied physics (specifically, over condensed-matter physics). Indeed the extensively investigated fluxon dynamics in one-dimensional Josephson tunnel junctions (see, for instance, Refs. 17–27, and references therein) provides an excellent experimental field in order to study the behavior of nonlinear solitary waves of the Klein-Gordon type. Moreover, recent progresses were performed in order to extend such dynamics to two-dimensional spatial configurations.[28] On the other hand, the problem of the existence of solitons in the most general case of $n+1$ dimensions becomes an exciting challenge in theoretical research concerning nonlinear evolution equations related to the powerful techniques using the spectral transform.[29]

This present paper aims at performing an interlinking between these two domains of physics, as it provides a theorem which sets the equation of motion of *any* driven nonlinear solitary wave $u(x_1,x_2,\ldots,x_n,t)$ [labeled $u(x_i,t)$ for the sake of simplicity] propagating in a dynamical system of $n+1$ dimensions defined from the Lagrangian density $l[u,u_{x_i},u_t,x_i,t]$ by the usual Euler-Lagrange partial differential equation (the field equation):

$$\frac{\partial L}{\partial u}-\frac{d}{dt}\frac{\partial L}{\partial u_t}-\sum_{j=1}^{j=n}\frac{d}{dx_j}\frac{\partial L}{\partial u_{x_j}}=0\ . \tag{1}$$

We assume a dependence of L on the first spatial derivatives of the field u as usually done. There is actually no particular difficulty to extend the study to higher-order spatial derivatives, as the example treated below and concerning the Korteweg–de Vries field will show.

Under the following trivial assumptions concerning the definition by use of its own Lagrangian density $L_0[u,u_{x_i},u_t]$ of a solitary wave driven by the "external" field $\Phi[u,x_i,t]$,

$$L[u,u_{x_i},u_t,x_i,t]=L_0[u,u_{x_i},u_t]+\Phi[u,x_i,t] \tag{2}$$

(where subscripts stand for partial derivations, as usual)

$$[L_0[u,u_{x_i},u_t]]_{\lim_{x_i\to-\infty}}^{\lim_{x_i\to+\infty}}=0 \quad (i=1,2,\ldots,n) , \tag{3}$$

which imply

$$\lim_{x_i\to\pm\infty}[u_{x_i},u_t]=0 \quad (i=1,2,\ldots,n) , \tag{4}$$

this theorem reads

$$\frac{d}{dt}\vec{\Pi}_{(i)}=\int_{\mathbf{R}^n}\left[\frac{\partial}{\partial x_i}-\frac{d}{dx_i}\right]L\,dx_1\cdots dx_n$$
$$=\int_{\mathbf{R}^n}\frac{\partial L}{\partial x_i}dx_1\cdots dx_n-[L[u,u_{x_i},u_t]]_{\lim_{x_i\to-\infty}}^{\lim x_i\to+\infty}$$
$$(i=1,2,\ldots,n) , \tag{5}$$

where the field momentum vector $\vec{\Pi}$ is defined as

$$\vec{\Pi}=\{\Pi_{(1)},\ldots,\Pi_{(n)}\}=-\int_{\mathbf{R}^n}\frac{\partial L}{\partial u_t}\nabla u\,dx_1\cdots dx_n . \tag{6}$$

Using definition (2), we obtain the following equivalent expression of the field equation of motion (5):

$$\frac{d}{dt}\vec{\Pi}=-\int_{\mathbf{R}^n}F\nabla u\,dx_1\cdots dx_n , \tag{7}$$

where the driving force F is defined as

$$F[u,x_i,t]=\frac{\partial\Phi[u,x_i,t]}{\partial u} . \tag{8}$$

II. THE NOETHER EQUATION OF MOTION OF THE FIELD

The demonstration of this theorem proceeds as follows. Equation (1) implies

$$\left[\frac{\partial L}{\partial u}-\frac{d}{dt}\frac{\partial L}{\partial u_t}-\sum_{j=1}^{j=n}\frac{d}{dx_j}\frac{\partial L}{\partial u_{x_j}}\right]u_{x_i}=0$$
$$(i=1,2,\ldots,n) . \tag{9}$$

Since we have

$$\frac{\partial L}{\partial u}u_{x_i}=\frac{dL}{dx_i}-\frac{\partial L}{\partial u_t}u_{tx_i}-\sum_{j=1}^{j=n}\frac{\partial L}{\partial u_{x_j}}u_{x_jx_i}-\frac{\partial L}{\partial x_i}$$
$$(i=1,2,\ldots,n) , \tag{10}$$

we obtain the following conservation equation of the field momentum density:

$$\frac{d}{dt}\left[u_{x_i}\frac{\partial L}{\partial u_t}\right]=\frac{dL}{dx_i}-\sum_{j=1}^{j=n}\frac{d}{dx_j}\left[u_{x_i}\frac{\partial L}{\partial u_{x_j}}\right]-\frac{\partial L}{\partial x_i}$$
$$(i=1,2,\ldots,n) . \tag{11}$$

The integration of this equation over the n-dimensional space yields Eq. (5) by use of conditions (4): QED. Therefore we obtain the following result: The equation of motion of any driven $(n+1)$-dimensional solitary wave is obtained by the mere projection of the original Euler-Lagrange partial differential equation which defines the total field onto the Goldstone translation mode ∇u which defines the driven wave. The following comment will emphasize this property.

The theorem [(5)–(8)]—and especially the definition of the field momentum (6)—should be considered in the frame of Noether's theorem which states that there is a conservation equation, the conservation equation of the momentum density, corresponding to the group of spatial translations for which the Lagrange variational principle leading to the field equation (1) is invariant.[30] Strictly speaking, the invariance of the system with respect to shifts in x_i obviously requires that x_i be an ignorable coordinate of the system:

$$\frac{\partial L}{\partial x_i}=0 . \tag{12}$$

The variational equation corresponding to this symmetry is simply Eq. (9). As a consequence, Noether's conservation equations of the field momentum density $[u_{x_i}(\partial L/\partial u_t)]$ are Eqs. (11) and (12). Therefore, Eq. (11) provides a physical explanation of the definition (6) of the field momentum $\vec{\Pi}$ in terms of the translational symmetry of the homogeneous system (2) and (12).

Note that the field momentum component $\vec{\Pi}_{(i)}$ may *not* be conserved although the space coordinate x_i is assumed ignorable according to Eq. (12). This is directly seen in Eq. (5) when the surface term does not vanish. We recover the peculiar dynamical behavior of a *field* which, unlike a point particle, is not only sensitive to the local symmetry (12), but also to the boundary conditions of the system as a whole. A typical "historical" example—as it played a important role in the pioneering study of the sine-Gordon Newtonian dynamics[1–8]—is the one-dimensional sine-Gordon kink field accelerated by an uniform external field according to

$$L(u,u_{x_i},u_t)=\tfrac{1}{2}[u_t^2-u_x^2]-(1-\cos u)+\epsilon u(x,t) \tag{13}$$

(the application of a uniform electric field to a charge-density wave, for example, yields the potential of this form; in the Josephson physics, this is simply the potential corresponding to the bias current). Equation (5), or equivalently Eqs. (7) and (8), yields

$$\frac{d}{dt}\vec{\Pi}=-\epsilon[u(x=+\infty)-u(x=-\infty)]=-2\pi\epsilon \tag{14}$$

for a kink [and $(d/dt)\vec{\Pi}=2\pi\epsilon$ for an antikink]. This is a well-known result.[1–8]

III. FIELD MOMENTUM AND CENTER-OF-MASS IMPULSE

A time-independent (conservative) system defined by

$$\frac{\partial L}{\partial t}\equiv 0 \tag{15}$$

leads through Noether's theorem to the following conservation equation:

$$\frac{dH}{dt}+\sum_{j=1}^{j=n}\frac{d}{dx_j}\left[u_t\frac{\partial L}{\partial u_{x_j}}\right]=0 , \tag{16}$$

where the Hamiltonian density of the wave, H, is defined by the usual Legendre transformation:

$$H = u_t \frac{\partial L}{\partial u_t} - L \ . \tag{17}$$

As a consequence of conditions (4), the total energy of the system (the integral of H over the n-dimensional space) is conserved:

$$\int_{\mathbf{R}^n} H \, dx_1 \cdots dx_n = \text{const} = M \ . \tag{18}$$

Let

$$\vec{P} = \frac{d}{dt} \int_{\mathbf{R}^n} \mathbf{X} H \, dx_1 \cdots dx_n = M \frac{d}{dt} \langle \mathbf{X} \rangle \tag{19}$$

be the impulse of the center-of-mass of the nonlinear field u. We have by use of conditions (4)

$$\vec{P}_{(i)} = \int_{\mathbf{R}^n} u_t \frac{\partial L}{\partial u_{x_i}} dx_1 \cdots dx_n \ , \tag{20}$$

and therefore

$$[\vec{P} - \vec{\Pi}]_{(i)} = \int_{\mathbf{R}^n} \left[\frac{\partial Q}{\partial u_t} \frac{\partial L}{\partial u_{x_i}} - \frac{\partial Q}{\partial u_{x_i}} \frac{\partial L}{\partial u_t} \right] dx_1 \cdots dx_n \ , \tag{21}$$

where

$$Q = \frac{1}{2} \left[u_t^2 - \sum_{j=1}^{j=n} u_{x_j}^2 \right] . \tag{22}$$

In the case of conservative nonlinear Klein-Gordon systems (NKG) such that

$$L[u, u_{x_i}, u_t, x_i] = Q - U(u) + \Phi[u, x_i] \ , \tag{23}$$

we obviously have

$$\vec{\Pi}_{\text{NKG}} = \vec{P}_{\text{NKG}} = \frac{d}{dt} \int_{\mathbf{R}^n} \mathbf{X} H dx_1 \cdots dx_n \ . \tag{24}$$

This equation, or more generally Eqs. (19) and (21), justify the choice of the negative sign in definition (6).

When the system is not conservative, there appears the following additional term $-\partial L/\partial t = \partial H/\partial t$ at the right-hand side of Eq. (16). Then the definition of momentum $\vec{P}$ given by Eq. (19) may be generalized as follows:

$$\vec{P} = \int_{\mathbf{R}^n} \mathbf{X} \left[\frac{d}{dt} - \frac{\partial}{\partial t} \right] H dx_1 \cdots dx_n \ .$$

Equations (20) and (21) remain valid with this definition of momentum $\vec{P}$. Considering the Poisson bracket which is the integrant of this latter equation, we conclude that we recover the usual dual time-energy description of the space-impulse field dynamics described by Eqs. (5) and (6).

IV. BY WAY OF ILLUSTRATION

A. (1+1)-dimensional Korteweg–de Vries system (KdV)

We consider a (1+1)-dimensional scalar field $u(x,t)$ defined by the following Lagrangian density:

$$L = L_0[u_t, u_x, u_{xx}] + \Phi = \tfrac{1}{2} u_{xx}^2 + u_x^3 - \tfrac{1}{2} u_t u_x + \Phi \ . \tag{25}$$

The corresponding partial differential equation (1) is

$$u_{xt} - 6 u_x u_{xx} + u_{xxxx} + F = 0 \ , \tag{26}$$

provided the driving field F is defined as usually by the variational derivative of Φ:

$$F = \frac{\delta \Phi}{\delta u} = \frac{\partial \Phi[u, u_x, \ldots]}{\partial u} - \frac{d}{dx} \frac{\partial \Phi[u, u_x, \ldots]}{\partial u_x} + \cdots \ . \tag{27}$$

The usual KdV equation defining the solitary-wave field $v(x,t)$ is obtained by assuming

$$v = u_x \ . \tag{28}$$

The Noether conservation equation (11) remains relevant provided its right-hand side now contains the additional term:

$$- \left[u_{xxx} \frac{\partial L}{\partial u_{xx}} - u_x \frac{d^2}{dx^2} \frac{\partial L}{\partial u_{xx}} \right] ,$$

which is due to the presence of higher-order space derivatives in the Lagrangian density (2). This term vanishes by integration over space for KdV solitons. Therefore, we recover the equation of motion (6) and (7) where the driving field F is defined according to Eq. (27). It yields [cf. definition (28)]

$$\frac{d\Pi}{dt} = - \int_{-\infty}^{+\infty} F u_x dx = - \int_{-\infty}^{+\infty} F v \, dx \ , \tag{29}$$

where

$$\Pi = - \int_{-\infty}^{+\infty} u_x \frac{\partial L_0}{\partial u_t} dx = \frac{1}{2} \int_{-\infty}^{+\infty} v^2 dx \ , \tag{30}$$

[see definitions (6) and (28)]. In particular, the simple choice $F = \text{const}$ (i.e., $\Phi = [\text{const}] u$) yields

$$\frac{d}{dt} \int_{-\infty}^{+\infty} v^2 dx = -2[\text{const}] \int_{-\infty}^{+\infty} v dx \ ,$$

a result which is otherwise trivial for a KdV soliton described by Eqs. (26)–(28).

B. $(n+1)$-dimensional nonlinear Klein-Gordon system

Consider the NKG field whose Lagrangian is given by Eq. (23) in the particular conservative case. As an example we may choose a parametric modulation of the nonlinear term according to

$$\Phi(u, x_1, \ldots, x_n) = -\mu V(x_1, \ldots, x_n) U(u) \ , \tag{31}$$

where $V(x_1, \ldots, x_n)$ is the external applied potential and $U(u)$ is the nonlinear solitary-wave potential, as usual. The corresponding equation of the field is

$$\sum_{j=1}^{j=n} \frac{\partial^2}{\partial x_j^2} u - \frac{\partial^2}{\partial t^2} u - [1 + \mu V(x_1, \ldots, x_n)] \frac{d}{du} U(u) = 0 \ . \tag{32}$$

It has been extensively studied in the one-dimensional case and was one of the early models proposed in order to build a Newtonian mechanics of sine-Gordon solitons.[10-11] Theorem (7) and (8) implies

$$\frac{d}{dt}\vec{\Pi}=\mu\int_{\mathbf{R}^n}V(x_1,\ldots,x_n)\nabla U\,dx_1,\ldots,dx_n\ ,\tag{33}$$

which yields by obvious integration by part [cf. assumption (3)],

$$\frac{d}{dt}\vec{\Pi}=-\mu\int_{\mathbf{R}^n}U\nabla V(x_1,\ldots,x_n)dx_1,\ldots,dx_n\ ,\tag{34}$$

where the field momentum $\vec{\Pi}$ is defined as the center-of-mass impulse according to Eq. (24). We recover the generalized Ehrenfest theorem.[14]

Let us now introduce an additional canonical damping $+\alpha u_t$ on the right-hand side of the partial differential equation (32). Multiplying both sides of the resulting field equation by ∇u and integrating by parts over the n-dimensional space leads [when taking account of the solitary-wave assumptions (3) and (4)] to the additional contribution

$$\left[\frac{d}{dt}\vec{\Pi}\right]_{\text{damping}}=-\alpha\vec{\Pi}\ .\tag{35}$$

Therefore, we obtain an additional exponential decrease of the field momentum $\vec{\Pi}$. We summarize the above results: In the case of a conservative nth-dimensional nonlinear Klein-Gordon system, the equation of motion of the center of mass of any driven solitary wave is Newtonian.

C. $(n+1)$-dimensional nonlinear Schrödinger system (NLS)

Instead of returning to the original Lagrangian formulation of the field equation of motion according to Eqs. (5)–(8), it is more elegant to deduce the parametrically perturbed NLS equation of motion from the above NKG case (32) by considering the following multiscale change of variables (stretched coordinates related to the small parameter ϵ):

$$\begin{aligned}&X_i=\epsilon x_i\ ,\\&T=\tfrac{1}{2}\epsilon^2 t\ ,\\&u(x_i,t)=\epsilon e^{i\Omega t}F(X_i,T)+\text{c.c.}\ ,\quad \epsilon\ll 1\ .\end{aligned}\tag{36}$$

The "stroboscopic frequency" Ω, which defines the scanning of the NKG wave, is of the order unity in the present reduced units; it yields[31] $\Omega^2=2p$, where

$$\lim_{u\to 0}U(u)=pu^2+qu^4+\cdots\ .\tag{37}$$

The corresponding NLS field equation reads

$$i\Omega F_T+\sum_{j=1}^{j=n}\frac{\partial^2}{\partial X_j^2}F+\Omega^2\frac{\mu V}{\epsilon^2}F+12q|F|^2F=0\ .\tag{38}$$

The potential μV should obviously be scaled in units of the NKG squared wave-amplitude ϵ^2: $\lim_{\epsilon\to 0}(\mu/\epsilon^2)<\infty$. The NKG equation of motion of the field (34) becomes, in terms of the stretched coordinates (36)

$$\frac{d^2}{dT^2}\int_{R^n}|F|^2\mathbf{X}dX_1\cdots dX_n=-\frac{2\mu}{\epsilon^2}\int_{R^n}|F|^2\nabla V dX_1\cdots dX_n\ .\tag{39}$$

As we already emphasized it in the Introduction, the quantum-mechanical (linear) formulation of the Ehrenfest theorem remains relevant even for *nonlinear* waves described by NLS equation. One may catch a feeling of it by taking into account the highest-order generalization (34) of this theorem. Clearly, this generalization is "scaled" according to NKG nonlinearities which are strong. Then the above "reduction"—but not cancellation—of the nonlinearity performed by Eqs. (36) and (37), which leads to the NLS system, has the property to restore the exact original expression of Ehrenfest's theorem.

Finally, concluding, let us mention how the theorem (5)–(8) may be applied in other concrete cases. We have two main approaches: (i) It is applied in a perturbative approach $u\sim u_0+\epsilon u_1$, where u in definition (6) should be replaced by u_0. A similar procedure based upon the assumption of the superimposition of two fields in order to define the scalar field u was used in Ref. 14 for the description of the collision of two two-dimensional pulsons defined by a logarithmic nonlinearity in the potential $U(u)$. (ii) We consider the description of the wave dynamics in terms of the so-called "collective coordinates" defined as

$$u(x_i,t)\equiv u(x_i,\{\Delta_j(t)\})\ ,\tag{40}$$

where the family of coordinates $\{\Delta_j(t)\}$ includes all the time dependence of the field u. Then theorem (5)–(8), which is equivalent to the integral over space of Eq. (9) as shown by Eqs. (10) and (11), is simply the Lagrange equation for the coordinate $\Delta_j(t)$:

$$\frac{\partial L}{\partial\Delta_j}-\frac{d}{dt}\frac{\partial L}{\partial\Delta_{j_t}}=0\ ,\tag{41}$$

where the Lagrangian L of the system is the integral over space of the Lagrangian density (2).[32] The use of this Lagrange equation provides the equation of motion of the degree of freedom $\Delta_j(t)$.

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