Stability of skyrmions in thin films of superfluid ^{3}He-A

Gareth Williams
Department of Physics, Faculty of Liberal Arts, Shizuoka University, Shizuoka 422, Japan
(Received 2 April 1993; revised manuscript received 2 June 1993)

Topological considerations show that thin films of ^{3}He-A might support skyrmions. These are topological excitations in the spin part of the order parameter. By deriving an approximation to the solitonic free energy that is exact for both small and large sizes we show that the skyrmions are unstable against collapse.

I. INTRODUCTION

There are three known phases of superfluid ^{3}He, all described within the BSC formalism, through a spin-triplet p-wave pairing. The order parameter is the vacuum expectation value of

$$\frac{3g}{4k_F}\sum_{\sigma\tau}\psi_\sigma\partial_i(\sigma_2\sigma_a)_{\sigma\tau}\psi_\tau\ ,$$

where $\psi=(\psi_\uparrow,\psi_\downarrow)$ is the ^{3}He fermion operator and g is a coupling constant measuring the strength of the BSC binding interaction.[1] In what follows, we will be interested in the A phase, for which the order parameter is given by

$$A_{ai}=(\Delta/2k_F)d_aR_{ij}(\hat{\mathbf{x}}_j+i\hat{\mathbf{y}}_j)\ , \tag{1}$$

where Δ is the zero-temperature gap, d_a is the unit spin antiferromagnetism vector, and R_{ij} is a rotation matrix in orbital space.

When the superfluid is confined to a thin film, the orbital ferromagnetism axis $\hat{l}$ is forced to lie perpendicular to the film, that is $\hat{l}=\hat{\mathbf{x}}\times\hat{\mathbf{y}}=\pm\hat{\mathbf{z}}$. The weak spin-orbit interaction leaves the spin vector free to rotate, $d_a=d_a(x,y,t)$, leading to the possibility of solitonic configurations. This follows because the spin vector must approach a constant at spatial infinity for finiteness of the free energy. Hence, at each time t, the spin vector defines a map from S^2 to S^2. Such maps are characterized by integers, corresponding to the winding number of the map, referred to as the solitonic charge. These solitons are called $(2+1)$-dimensional skyrmions, after the solitons of the $(3+1)$-dimensional nonlinear σ model discovered by Skyrme.[2]

The spin statistics of the skyrmions are determined by a Chern-Simons term in the effective action.[3,4] That is, in the limit of an adiabatic process the effective action reduces to

$$S_{\text{eff}}=\frac{i}{12\pi^2}\theta\int dx\ \epsilon^{\mu\nu\rho}\text{tr}(A_\mu A_\nu A_\rho)\ . \tag{2}$$

Here the non-Abelian gauge field is $A_\mu=iU\partial_\mu U^\dagger$ and the SU(2) group element U is related to the spin vector through $\mathbf{d}\cdot\boldsymbol{\sigma}=U^\dagger\sigma_3U$. The statistical parameter θ is a functional of momentum space Green's functions in the absence of the gauge field.

We will not derive Eq. (2), but note that θ can take on values other than those corresponding to fermionic or bosonic statistics. Hence, there is a physical scenario for the emergence of anyons. However, for the physical picture to be complete we need to show that such objects have a finite size.

The stability analysis is greatly simplified if we neglect excitations in the momentum perpendicular to the film. This amounts to considering a purely two-dimensional system and enables us to linearize the spectrum pertinent to the problem. Here the statistical parameter is $\pi/2$ and the solitons have statistics halfway between fermions and bosons. We do not expect a fundamental alteration in the shape of the free energy under the assumption, but will investigate this in future. In what follows we take the soliton profile to be rotationally symmetric, a "hedgehog" profile, and to have only one free parameter, the soliton size. With these approximations we derive an approximate free energy, which is exact in the limit of small and large sizes.

In Sec. I we start with a formal expression for the thin-film effective action. We perform the functional integral over the ^{3}He fermion fields and linearize the spectrum of the Green's function. This reduces the functional trace to an integral over quantum-mechanical ones and greatly simplifies calculation. In Sec. II we derive an approximate expression for the free energy that is exact for both small and large solitons. The paper ends with a conclusion.

II. THE ^{3}He-A FREE ENERGY WITH A LINEARIZED SPECTRUM

The soliton free energy relative to the uniform configuration can be expressed as the ratio of two path integrals over the four component Grassmann fields f and $f^\dagger$ as[1]

$$e^{-\beta(F-F_0)}=\frac{\int Df\,Df^\dagger e^{-S}}{\int Df\,Df^\dagger e^{-S_0}}\ , \tag{3}$$

where $f^\dagger=(\psi^\dagger,i\sigma_2\psi)$ and β is the inverse temperature. The action takes the form

$$S=\int_0^\beta d\tau\int d^2\mathbf{x}\left[\frac{1}{2}f^\dagger(\partial_\tau+H)f+\frac{1}{3g}A_{ai}^\dagger A_{ai}\right]\ , \tag{4}$$

$$H=\tau_3\epsilon(\mathbf{p})-\begin{bmatrix}0 & \{p_i,\sigma_a A_{ai}\}\\ \{p_i,\sigma_a A_{ai}^\dagger\} & 0\end{bmatrix}\ , \tag{5}$$

where $\epsilon(\mathbf{p})=\mathbf{p}^2/2m^2-\mu$ and μ is the chemical potential. Note that $a=1,2,3$ and $i=1,2$; also, as stated in the introduction, we have neglected excitations perpendicular to the film. That is, $\mathbf{p}=(p_1,p_2)$. The τ matrices act in particle-hole space and the σ matrices act in spin space. In the reference action, S_0, the spin vector is a constant.

Performing the path integral and substituting for the ^{3}He-A order parameter, Eq. (1), with $\hat{l}=-\hat{\mathbf{z}}$ we obtain

$$E-E_0=-\frac{1}{2}\int\frac{d\omega}{2\pi}\,\mathrm{Tr}\ln\left[\frac{i\omega+H}{i\omega+H_0}\right], \quad (6)$$

$$H=\tau_3\epsilon(\mathbf{p})-(\Delta/2k_F)\{\tau_i p_i,\sigma_a d_a\}\ . \quad (7)$$

Here, we have restricted ourselves to zero temperature, so that the soliton free energy becomes the soliton creation energy, $E-E_0$. We can easily generalize to arbitrary temperatures through the substitution $\int(d\omega/d\pi)\to(1/\beta)\sum_n$ and $\omega\to\omega_n=(\pi/\beta)(2n+1)$. The remaining trace is over the spatial, spin, and particle-hole degrees of freedom.

Because of the kinetic term ϵ, the momenta in the problem are peaked about the Fermi momentum, that is, $k\sim\sqrt{2m\mu}=k_F$, so it is a good approximation to linearize the spectrum.[5,6] For example, consider the trace of some function of H

$$\mathrm{Tr}C(H)=\int\frac{d^2k}{(2\pi)^2}d^2x\ \mathrm{tr}C\left[\tau_3\left[\epsilon(k)+\frac{ik_i\partial_i}{m}-\frac{\partial_i^2}{2m}\right]-\frac{\Delta}{2k_F}\{\tau_i(k_i+i\partial_i),\sigma_a d_a\}\right]. \quad (8)$$

Introducing the new set of variables, $\mathbf{x}=\zeta\hat{\mathbf{k}}+\eta\hat{\mathbf{b}}$ with $\hat{k}=(\cos\theta,\sin\theta)$ and $\hat{\mathbf{b}}=(-\sin\theta,\cos\theta)$, and neglecting derivatives in favor of powers of k_F,[7] we have

$$\mathrm{Tr}C(H)=\frac{m}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\int d\zeta\, d\eta\int d\epsilon\,\mathrm{tr}C[\tau_3(\epsilon+iv_F\partial_\zeta)-\Delta\tau_i\hat{k}_i\sigma_a d_a]\ . \quad (9)$$

Note also that we have made the approximation

$$\int\frac{d^2k}{2\pi}\simeq\frac{m}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\int_{-\infty}^{\infty}d\epsilon\ .$$

Inspection of Eq. (9) shows that we can combine the integrals over ζ and ϵ into a quantum-mechanical trace as follows:

$$\mathrm{Tr}C(H)=mv_F\int_0^{2\pi}\frac{d\theta}{2\pi}\int d\eta\,\mathrm{Tr}_\zeta C(\tau_3 v_F p_\zeta-\Delta\tau_i\hat{k}_i\sigma_a d_a)\ . \quad (10)$$

Applying the same analysis to the free energy, we arrive at

$$E-E_0=-\frac{1}{2}mv_F\int\frac{d\omega}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\int d\eta\,\mathrm{Tr}_\zeta\left[\frac{i\omega+H_L}{i\omega+H_{0L}}\right]$$
$$=-\frac{1}{4}mv_F\int\frac{d\omega}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\int d\eta\,\mathrm{Tr}_\zeta\left[\frac{\omega^2+H_L^2}{\omega^2+H_{0L}^2}\right], \quad (11)$$

where H_L is the argument of C in Eq. (10). The last line in Eq. (11) follows as the determinant is real. That is, we get the same answer if we replace the argument of the trace with its Hermitian conjugate. For convenience, we give the forms for the squared Hamiltonians

$$H_L^2=v_F^2p_\zeta^2+v_F\Delta\tau_i\epsilon_{ij}\hat{k}_j\sigma_a\partial_\zeta d_a+\Delta^2\ ,$$
$$H_{0L}^2=v_F^2p_\zeta^2+\Delta^2\ . \quad (12)$$

III. SCALE-DEPENDENCE OF THE SOLITON CREATION ENERGY

In this section we will look at how the creation energy varies with the soliton size. To do this we will assume a profile and introduce a length scale ξ so that the spin field is $\mathbf{d}=\mathbf{d}(\mathbf{x}/\xi)$. In particular, we will eventually choose the sherically symmetric "hedgehog" profile, with $d_1+id_2=e^{i\arctan(y/x)}\sin F$ and $d_3=\cos F$ with $F=2\arctan(r/\xi)$.[8] But, for the moment we will replace the energy functional by a function of the soliton size, ξ.

Performing the cannonical transformations $\zeta\to\xi\zeta$, $p_\zeta\to p_\zeta/\xi$ and rescaling $\eta\to\xi\eta$, $\omega\to(\Delta/\lambda)\omega$, with the dimensionless scale factor $\lambda=\xi\Delta/v_F$, the creation energy, Eq. (11), takes the form

$$E-E_0=-\frac{1}{2}\mu\int\frac{d\omega}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\int d\eta\,\mathrm{Tr}_\zeta(1+G\lambda V)\ , \quad (13)$$

$$G^{-1}=\omega^2+p_\zeta^2+\lambda^2\ , \quad (14)$$

$$V=\tau_i\epsilon_{ij}\hat{k}_j\sigma_a\partial_\zeta d_a\ . \quad (15)$$

Expanding in powers of V and performing the momentum integrals we find[9]

$$E-E_0=-\frac{1}{2}\mu\lambda\int\frac{d\omega}{2\pi}\int_0^{2\pi}\frac{d\theta}{2\pi}\int d\eta\sum_{n=1}^{\infty}\frac{-1^{n+1}}{n}\left[\frac{1}{2\sqrt{1+\omega^2}}\right]^n\times\int d\zeta_1\cdots d\zeta_n\exp[-\lambda\sqrt{1+\omega^2}(|\zeta_1-\zeta_2|+\cdots+|\zeta_n-\zeta_1|)]\times\mathrm{tr}[V(\zeta_1,\eta,\theta)\cdots V(\zeta_n,\eta,\theta)]\ . \quad (16)$$

This equation gives us a nice way of isolating the small scale behavior of the creation energy. At first glance it would appear that to each order in λ one has to sum up an infinite series in powers of V. This is indeed the case for a one-dimensional soliton, as in polyacetelene.[9] But, in the present case $\int d\zeta\, V=0$, so that there are no factors of V unaccompanied by propagators from the exponent. This means that to leading order in λ only the $n=2$ term has to be considered. Now the large-scale behavior of the creation energy can be obtained through the derivative expansion. This expansion will start with a term quadratic in the derivatives of **d**, which again comes from the $n=2$ term in the expansion equation (16). Hence a good approximation to the creation energy, that captures both the large and small scale behaviors, is[10]

$$E-E_0 \simeq (E-E_0)_{(2)} = \frac{1}{4}\mu \int \frac{d\omega}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \int d\eta \, \mathrm{Tr}_\zeta (G\lambda V)^2 . \quad (17)$$

To calculate $(E-E_0)_{(2)}$ we need the Fourier transform of V. Tacking the hedgehog profile with $d_1+id_2 = e^{i\theta}(\zeta+i\eta)\sin F$, $d_3=\cos F$, and the radial profile being $F=2\arctan(r)$, we find that the Fourier transforms of the spin fields are

$$\tilde{d}_1+i\tilde{d}_2=2i[(k_1+ik_2)/k]K_1(k)$$

and $\tilde{d}_3=2K_0(k)$, where $K_{0,1}$ are Bessel functions.[8] We could always take another reasonable profile in our approximation, but do not expect there to be a significant change in the results. In particular, the fact that the creation energy is zero and has a positive gradient at $\lambda=0$ is independent of profile choice. This can be seen by an expansion of $E-E_0$ along the same lines as in Ref. 9. Substituting the profile into Eq. (17) and doing the θ, η, and $\zeta_{1,2}$ integrals we obtain

$$(E-E_0)_{(2)} = \frac{1}{4}\mu\lambda^2 \int \frac{d\omega}{2\pi} \int \frac{dp}{2\pi}\, dq\, dl \frac{16 l^2[K_1^2(\sqrt{q^2+l^2})+K_0^2(\sqrt{q^2+l^2})]}{(\omega^2+p^2+\lambda^2)[\omega^2+(p-l)^2+\lambda^2]} , \quad (18)$$

where the momenta q and l are conjugate to ζ and η, respectively. The ω and p integrals can be readily done using the Feynman trick. This leaves the following double integral:

$$(E-E_0)_{(2)} = \frac{16\mu\lambda^4}{\pi} \int_0^\infty ds\, s[K_0^2(\lambda s)+K_1^2(\lambda s)] \int_0^{\pi/2} d\phi \frac{\cos\phi}{\sqrt{4/s^2+\cos^2\phi}} \left\{ \ln\left[\cos\phi+\sqrt{4/s^2+\cos^2\phi}\right] - \ln\frac{2}{s} \right\} , \quad (19)$$

where $s=\sqrt{q^2+l^2}$ and $\tan\phi=q/l$. This integral was done numerically for various values of λ and the results are shown in Fig. 1. This shows that the creation energy to second order in V is a monotonically increasing function of the soliton size tending to a constant at spatial infinity. With $(E-E_0)_{(2)} \rightarrow \mu$ as $\lambda \rightarrow \infty$. This last limit can also be obtained from the leading term in the derivative expansion of the creation energy. After a tedious but straightforward expansion of the creation energy in powers of derivatives of the spin field we obtain

$$E-E_0 = \frac{\mu}{4\pi} \int d\eta\, d\zeta (\partial_\zeta d_a)^2 + \frac{\mu}{24\pi\lambda^2} \int d\eta\, d\zeta [\{(\partial_\zeta d_a)^2\}^2 - (\partial_\zeta^2 d_a)^2] + O(\lambda^{-4}) . \quad (20)$$

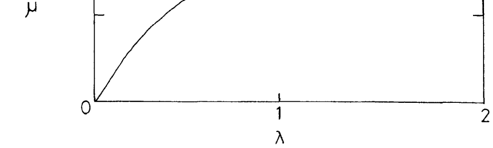

FIG. 1. Plot of the creation energy to order V^2 against the skyrmion size, as measured by the dimensionless scale λ.

Of course it is a trivial matter to go back to the original **x** coordinates at this stage. One does this by substituting $\hat{\mathbf{k}}\cdot\partial$ for ∂_ζ and integrates over $\hat{\mathbf{k}}$. Substituting for our choice of profile, we find after long calculation

$$E-E_0=\mu-\mu(1/9\lambda^2)+O(\lambda^{-4}) . \quad (21)$$

This behavior at large λ is in contrast to the case of Belavin-Polyakov solitons, where the creation energy is an increasing function of size for small solitons and a decreasing function for large solitons.[8]

IV. CONCLUSION

We have shown that within the purely two-dimensional limit skyrmions in films of ^{3}He-A have no physical size. It would be more desirable to incorporate the effects of the film thickness, but this ruins the validity of linearization as the effective chemical potential becomes small for large values of the quanta of perpendicular momenta. We hope to investigate the effects of film thickness in future, but see no reason why they should change the basic shape of the creation energy.

For the case of Belavin-Polyakov solitons, quantum fluctuations in the soliton field actually give rise to a shrinking effect.[8] This, combined with the tendency to expand at large λ, gives the soliton a finite size for a sufficiently large number of fermion species. However, in the present case there is no large scale expansion effect. Further, superfluid ^3He is well described within mean-field theory and quantum fluctuations in the order parameter should be negligible.

ACKNOWLEDGMENT

This work was supported by the Japan Society for the Promotion of Science.

[1]H. Kleinert, Fortchr. Phys. **26**, 565 (1978).

[2]T. H. R. Skyrme, Proc. R. Soc. London **260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).

[3]G. E. Volovic and V. M. Yakovenko, J. Phys. Condensed Matter **1**, 5263 (1979).

[4]G. Williams, Z. Phys. B **85**, 205 (1991).

[5]For a comprehensive review see J. W. Serene and R. Rainer, Phys. Rep. **101**, 221 (1983).

[6]D. Waxman and G. Williams, Ann. Phys. **220**, 274 (1992).

[7]This is valid so long as the scale of spatial variation of the soliton is not as small as the coherence length. If the soliton profile varied over scales of the order of the coherence length then the topological notions will break down anyway.

[8]B. Moussallam, Phys. Rev. B **43**, 3325 (1991).

[9]D. Waxman and G. Williams, J. Phys. A **24**, 3611 (1991).

[10]For a survey of the two-point approximation see I. Adjali, I. J. R. Aitchison, and J. A. Zuk, Nucl. Phys. **A537**, 457 (1992), and references therein.