Effect of a nonzero temperature on quantum critical points in itinerant fermion systems

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I reexamine the work of Hertz on quantum phase transitions in itinerant fermion systems. I determine when it is permissible to integrate out the fermions and analyze the critical phenomena via an effective bosonic theory in which only fluctuations of the ordering field are explicitly retained. By solving appropriate scaling equations I obtain the different regimes of behavior of the correlation length and free energy in the disordered phase of the effective bosonic theory. The results in many cases differ from those of Hertz, but make contact with more recent work on the dilute Bose gas. I briefly discuss the relevance of the results to heavy-fermion materials.

I. INTRODUCTION

Fermion systems that undergo low-temperature magnetic-nonmagnetic phase transitions are of current interest. Many "heavy-electron" metals either undergo magnetic transitions at low temperatures[1] or may easily be induced to undergo such transitions by alloying.[2] Several authors have proposed that high-T_c CuO_2 superconductors are at or near a magnetic instability.[3,4] In all of these cases one may suppose the physics is dominated by a zero-temperature critical point, at which the ground state of the system changes from ordered to disordered as some parameter in the Hamiltonian is varied. Zero-temperature transitions have been studied by many authors. In a pioneering paper, Hertz[5] showed that in quantum systems at $T=0$ statics and dynamics are inextricably mixed, so that in contrast to $T>0$ phase transitions the value of the dynamic exponent, z, affects the static critical behavior. In particular he showed that the scaling dimension of a quartic interaction in a model with dynamic exponent z in spatial dimensionality d was $[4-(d+z)]$. He also showed that in Fermi-liquid-like systems the particle-hole continuum of fermion excitations will often *overdamp* the mode associated with the ordering field, leading to a dynamic exponent $z>1$. Hertz also considered the quantum-classical crossover occurring at $T>0$; however, his results on this topic are in many respects incorrect. A correct treatment of the crossover from quantum to classical critical behavior in the dilute interacting Bose gas in $d>2$ was given by Weichman, Rasolt, Fisher, and Stephen.[6] In this problem there is no particle-hole continuum but the dynamic exponent $z=2$. The dilute interacting Bose gas in $d=2$ was studied by Fisher and Hehenberg.[7] Zero-temperature properties of the dilute interacting Bose gas with variable interaction range have been studied by Kolomeisky and Straley.[8]

In this paper I study the scaling behavior of some itinerant fermion systems near a $T=0$ magnetic phase transition. I assume it is possible to integrate out the fermions and thereby reduce the problem to the study of an effective bosonic theory describing fluctuations of the ordering field. As discussed below, this approach is not useful for "$2k_F$" density-wave instabilities and superconducting transitions, but is, I believe, useful for transitions in heavy-fermion materials. For simplicity I restrict myself to the disordered phases of the models I consider. In Refs. 6–8 the focus is on the ordered phase and ordering transition, although some results in the disordered phase are also given. I obtain results for the specific heat, the susceptibility associated with the ordering field and the correlation length, and I discuss the various crossovers. The results may be viewed as an extension of the results of Refs. 6 and 7 to transitions in fermion systems and to the disordered phases and as a correction of some of the results of Ref. 5. An alternative approach to magnetic phase transitions in fermion systems is a self-consistent one-loop approximation which has been applied to a variety of relatively realistic models by Moriya and coworkers.[9] The results presented here amount to a renormalization-group derivation of the results of this formalism and of the limits of its applicability in the case of spatial dimension $d=3$. In the case $d=2$ I show that additional logarithms occur.

This paper is organized as follows: in Sec. II I give the formalism and assumptions and derive renormalization-group equations. In Sec. III I solve the equations in the $d=3$ case. In Sec. IV I solve them in the $d=2$ case. In Sec. V I explain the connection of my results to those of Moriya. In Sec. VI and Figs. 2 and 3 I summarize the results I have obtained. Sections II and VI may be read independently by readers not interested in the technical aspects of the calculations. Appendices give details of various calculations.

II. FORMULATION

I use the conventional approach to critical phenomena: first, integrate out all degrees of freedom except those associated with the ordering field, so that the partition function is expressed as an appropriately weighted sum over configurations of the order parameter. Then, progressively eliminate longer and longer wavelength components of the ordering field until a fixed point is reached.

One may question the validity of integrating out the low-energy excitations. I argue that one may always for-

mally integrate out the fermions. If the resulting action is an analytic function of the ordering field then integrating out the fermions is justified *a posteriori*; if the resulting action is nonanalytic in a nontrivial way, then integrating out the fermions is not appropriate. It is shown in Appendix E that in models without nesting and for which the ordering wave vector Q is not an extremal spanning vector of the Fermi surface (this is defined more precisely in Appendix E but roughly means $Q \neq 2k_F$), the action is not singular at least to quartic order—and this is all one needs to analyze the fixed points considered here. At least in uranium heavy-fermion systems the ordering wave vector is generally not determined by a Fermi-surface instability.[1] The situation in $CeCu_6$ is at present less clear,[2] but the present treatment may be relevant. In $La_{2-x}Sr_xCuO_4$ at superconducting concentrations the magnetism is apparently a "$2k_F$" effect,[10] while in $YBa_2Cu_3O_{6+x}$ the magnetic fluctuations apparently do not occur at an extremal spanning vector.[11]

The first step is most easily accomplished via standard functional integral techniques;[5] the results may also be obtained by more conventional methods.[9] I give the results first, and then comment on them. The partition function is expressed as a functional integral over a field $\boldsymbol{\phi}$, which one may think of as the fluctuating part of the magnetization density (in the case of a ferromagnet) or staggered magnetization density (in the case of an antiferromagnet), thus

$$Z = Z_0 \int \mathcal{D}\boldsymbol{\phi}_k \exp -S_{\text{eff}}[\{\boldsymbol{\phi}_k\}] \tag{2.1}$$

with Z_0 the partition function of the noncritical degrees of freedom. The effective action for the critical fields has different forms in different cases. One may write ($\mathbf{h}$ is an externally applied magnetic field)

$$S_{\text{eff}}[\boldsymbol{\phi}] = S^{(0)} + \beta \sum_{k,n} \mathbf{h}_n(\mathbf{k}) \cdot \boldsymbol{\phi}_{-n}(-\mathbf{k}) + S^{(2)}_{\text{eff}}[\boldsymbol{\phi}] + S^{(4)}_{\text{eff}}[\boldsymbol{\phi}] + \cdots . \tag{2.2}$$

Here $S^{(0)}$ is the ϕ-independent part of the action. In the case of a ferromagnetic transition in a nondisordered material one has[5,9]

$$S^{(2)}_{\text{eff}}[\boldsymbol{\phi}] = \beta V \sum_n \int \frac{d^d k}{(2\pi)^d} \left[\delta + k^2 + \frac{|\Omega_n|}{k} \right] \times \boldsymbol{\phi}_n(\mathbf{k}) \cdot \boldsymbol{\phi}_{-n}(-\mathbf{k}) , \tag{2.3a}$$

while for an antiferromagnetic transition one has

$$S^{(2)}_{\text{eff}}[\boldsymbol{\phi}] = \beta V \sum_n \int \frac{d^d k}{(2\pi)^d} [\delta + k^2 + |\Omega_n|] \times \boldsymbol{\phi}_n(\mathbf{k}) \cdot \boldsymbol{\phi}_{-n}(-k) . \tag{2.3b}$$

In either case

$$S^{(4)}_{\text{eff}}[\boldsymbol{\phi}] = u\beta V^4 \sum_{n_1 \cdots n_4} \int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_4}{(2\pi)^d} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \delta_{n_1 + n_2 + n_3 + n_4} [\boldsymbol{\phi}_{n_1}(\mathbf{k}_1) \cdot \boldsymbol{\phi}_{n_2}(\mathbf{k}_2)][\boldsymbol{\phi}_{n_3}(k_3) \cdot \boldsymbol{\phi}_{n_4}(k_4)] . \tag{2.4}$$

Here $\beta = 1/k_B T$, V is the volume of the system, and $\Omega_n = 2\pi n k_B T$ is a Matsubara frequency. In the antiferromagnetic case k is measured from the ordering wave vector. The ellipsis in Eq. (2.2) denotes terms with higher powers of ϕ, k, or Ω_n than written. The bare length and energy scales have been chosen to make the coefficients of k^2 and $|\Omega_n|$ (or $|\Omega_n|/k$) unity.

The model involves the following parameters: volume V (which I set to infinity in what follows), temperature T, δ and u, which come from the underlying electron Hamiltonian, and cutoffs Λ for momentum integrals and Γ_k for frequency sums involving the ϕ-ϕ correlator at momentum k. Except in Appendix A and just above Eqs. (2.6), where explicit cutoffs are retained for clarity, I choose units such that Λ is set to 1, while Γ_k is set to 1 for the antiferromagnet and to k for the ferromagnet. δ is the control parameter: it may be made to vary from small positive to small negative values by varying some parameter in the Hamiltonian such as the interaction strength or carrier concentrations. u is supposed to have a negligible dependence on Hamiltonian parameters over the range of interest. In a Fermi liquid, one expects both δ and u to vary with temperature as T^2 at low T.[9,12] This T dependence is negligible in comparison to the T dependences considered here, so will not be explicitly written. For completeness, in Sec. VI and below Eq. (3.8) I briefly mention what happens in high dimensions when the T dependence of δ becomes important.

The dependence on $|\Omega_n|$ in Eqs. (2.3) arises because the spin modes described by $\boldsymbol{\phi}$ have been assumed to lie inside the particle-hole continuum of the Fermi liquid and are therefore overdamped (see Fig. 1). This will always be the case for a ferromagnetic transition (case F in Fig. 1), because the magnon in the ordered state has a k^2 dispersion, while at small k the upper ω boundary of the particle-hole continuum is given by $\omega = v_F k$. The overdamping of the mode will occur for an antiferromagnetic transition if the ordering wave vector is inside the particle-hole continuum (case A in Fig. 1), but will not occur if it is outside (case B in Fig. 1). If the ordering wave vector is at the edge of the continuum (case C in Fig. 1) the situation is more complicated. Some wave vectors are damped and some are not, and also as shown in Appendix E the coefficient u diverges as $T \to 0$. This paper does not apply to this case.

The difference between Eqs. (2.3a) and (2.3b) arises[5] because in a ferromagnet the fluctuations of the order parameter are conserved, so that the damping rate Γ_k must vanish as $k \to 0$, while in the antiferromagnet the fluctuations are not conserved. For a clean ferromagnet one expects $\Gamma_k \sim k$. In a disordered ferromagnet one might expect impurity scattering to lead to diffusive behavior

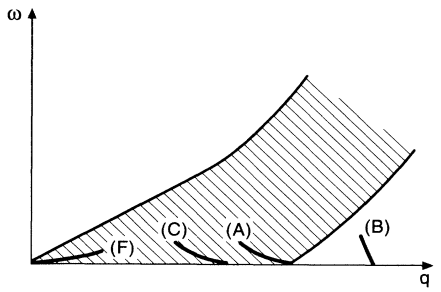

FIG. 1. Plot of (q,ω) plane showing location of particle-hole continuum (shaded region) and places (indicated by heavy lines) where important critical fluctuations may exist. In the ferromagnet (F) and "case C" antiferromagnet the modes are inside the particle-hole continuum and overdamped leading to dynamic exponent $z>1$; in "case B" antiferromagnet the modes are outside and may be underdamped, possibly giving dynamic exponent $z=1$; in "case A" antiferromagnetic modes with $Q>2k_F$ may be underdamped while those with $Q<2k_F$ are overdamped at mean-field level, and a different analysis, not given in this paper, is required.

($\Gamma_k \sim k^2$) at sufficiently long wavelengths, as pointed out in Ref. 5. However, the critical point studied here is in the Gaussian universality class, so $\nu=\frac{1}{2}$ and disorder is a relevant perturbation[13] near critical points. Only critical points in nondisordered materials will be explicitly considered here, although formally extending the results to $z=4$ would be straightforward.

Hydrodynamic considerations also imply that for $T>0$, a ferromagnetic mode at sufficiently long wavelengths will be diffusive. However, in a Fermi liquid this "hydrodynamic" behavior occurs only for wave vectors $k<k_{\rm hydro}\sim T^2$.[12] At low temperatures this scale is so low that the hydrodynamic region makes no contribution to the phenomena of interest here. This point will be discussed further in subsequent sections.

In Ref. 4 a different model (the Shraiman-Siggia[14] model of lightly doped antiferromagnets) of spins coupled to fermions was considered. The model was arranged so that the fermions did not overdamp the spin modes, so $z=1$. The results of this paper do not apply to the model of Ref. 4. However, I believe that the model considered in Ref. 4 does not apply to any situation in which spin fluctuations are coupled to a Fermi liquid, and in particular does not apply to cases A and C of Fig. 1. The point is that the assumption $z=1$ implies that the spin-wave velocity, c, remain finite at the transition. Now the spin-wave velocity is given by the hydrodynamic relation[15]

$$c^2=\rho_s/\chi_\perp \tag{2.5}$$

where ρ_s is the spin stiffness of the antiferromagnet and $\chi_\perp$ is the transverse spin susceptibility of the spin system in the ordered phase. Now generically ρ_s vanishes at the $T=0$ transition; in the model of Ref. 4, $\chi=0$ in the disordered phase, and $\chi_\perp$ vanishes at the same rate as ρ_s as the $T=0$ transition is approached from the ordered side, so c remains finite at the transition. In the Fermi liquid-like situations considered here $\chi_\perp$ is finite in the disordered phase and is highly unlikely to vanish at the transition. Equation (2.5) then implies c will vanish at the transition, so $z>1$.

Having specified the model, I now define the renormalization-group transformation I wish to use. There are many different possible formulations; the one used here is most closely related to that in Sec. III of Ref. 5, although some important differences exist and will be discussed below. The model is specified by the parameters δ, u, T and the energy and momentum cutoffs Γ and Λ. I perform the functional integral over modes with wave vectors k satisfying $\Lambda\geq k\geq\Lambda/b$ $(b>1)$ (note this involves summing over all Matsubara frequencies for these modes) and then rescale momenta to restore the cutoff Λ to the original value, field ϕ to keep the coefficient of the k^2 term unity and temperature T to keep the coefficient of $|\Omega_n|$ equal to unity (or $1/k$). This produces a model identical in form to Eq. (2.2) but with parameters δ', u', T', a ϕ-independent contribution to S, a momentum cutoff equal to the original momentum cutoff Λ, and a different frequency cutoff Γ'. The easiest way to see that the frequency cutoff has changed is to notice that it must enter Eqs. (2.2), (2.3), and (2.4) as a cutoff $n_{\rm max}=\Gamma/2\pi T$ on Matsubara sums. The procedure described above does not change $n_{\rm max}$, but does change T; therefore Γ must change. Although it is not necessary, it is convenient also to integrate out modes with energies between the new cutoff Γ' and the old cutoff Γ. The technical details are given in Appendix A. The result, in the limit $b\rightarrow 1^+$, is a set of renormalization-group equations:

$$\frac{dT(b)}{d\ln b}=zT(b)\ , \tag{2.6a}$$

$$\frac{d\delta(b)}{d\ln b}=2\delta(b)+2u(b)(n+2)f^{(2)}(T(b),\delta(b))\ , \tag{2.6b}$$

$$\frac{du(b)}{d\ln b}=[4-(d+z)]u(b) -4u(b)^2(n+8)f^{(4)}(T(b),\delta(b))\ , \tag{2.6c}$$

$$\frac{dS_0(b)}{d\ln b}=(d+z)S(b)_0+f^{(0)}(T(b),\delta(b))\ , \tag{2.6d}$$

Expressions for the $f^{(n)}(T)$ are given in Appendix A. In these equations and throughout the paper, rescaled quantities have explicit b dependence indicated, e.g. $T(b)$, while quantities without b dependence (e.g., T) refer to physical quantities.

Equations (2.6) are very similar to those derived by Hertz,[5] except that he did not write an explicit equation for the temperature, and have the same physical content as those of Refs. 6–8.

Equations (2.6) have an unstable Gaussian fixed point at $T=u=\delta=0$. In subsequent sections I will solve for the renormalization-group flows near this fixed point. I conclude this section with a few general remarks. First, from Eq. (2.6c) it is clear that if $(d+z)\geq 4$, the interaction u initially decreases under scaling. Thus, as first pointed out by Hertz,[5] if $(d+z)\geq 4$ then in some parameter regimes near the $T=\delta=u=0$ critical point one expects the exponents and amplitudes to be those of the Gaussian model. However, from Eq. (2.6a) one sees that T increases under renormalization, while from the expressions for $f^{(n)}(T)$ in Appendix A one sees that for $T(b)$ greater than the upper frequency cutoff, the form of

the scaling equations changes. It will be explicitly shown in subsequent sections that in the $T(b)>1$ regime the appropriate scaling variable measuring the strength of the interactions is the quantity $v(b)=T(b)u(b)$, which to linear order has scaling dimension $(4-d)$, as expected in a d-dimensional classical theory. It will also be shown that (as is intuitively very plausible) the regime $T(b)>1$ can be reached by scaling only if excitations at wavelengths of order the correlation length or longer have energies less than kT, which means these modes should be treated classically. This quantum-classical crossover was noticed by Hertz,[5] who employed a different method of scaling and made a qualitative argument that one should distinguish a quantum regime in which the Matsubara frequencies are treated as a continuum and a classical regime in which one only retains the lowest $(n=0)$ Matsubara frequency in any sums. However, Hertz obtained incorrect values for exponents over much of the classical regime. In fact, for a wide range of parameters, *including* those corresponding to fixing the microscopic parameters at the critical $T=0$ values and lowering the physical temperature to zero, the critical behavior is that of the classical Gaussian model, because by the time the condition $T(b)>1$ becomes satisfied, the interaction has scaled to a negligibly small value.

The qualitative considerations presented here concerning the crossover to the classical regime also show why the "hydrodynamic" effects which on sufficiently long scales change, e.g., the dynamic exponent in the ferromagnetic problem from $z=3$ to $z=4$ are not relevant to the static critical behavior. The point is that the hydrodynamic scale beyond which, say, the spin mode becomes diffusive scales as T^p with $p\geq 2$ set by electron-electron interactions,[12] so it is necessarily longer than the momentum scale at which classical scaling occurs, which will be shown below to be $T^{2/z}$. But once the classical regime is reached, the dynamics becomes unimportant.

III. SOLUTION OF SCALING EQUATIONS IN $d=3$, DISORDERED PHASE

In this section I use the scaling equations (2.6a)–(2.6d) to obtain information about the disordered phases of the models defined by Eq. (2.1) near their $T=0$ critical point, in the case $d=3$. Because $z=2$ or 3, $4-(d+z)<0$ and u is always irrelevant. I therefore linearize the equations in u. Further, the dependence of $f^{(2)}(T,\delta)$ on δ will not be important. Thus I consider the limiting equations, valid for $\delta, u \ll 1$,

$$\frac{dT(b)}{d\ln b}=zT(b)\ , \tag{3.1a}$$

$$\frac{d\delta(b)}{d\ln b}=2\delta(b)+2u(b)(n+2)f^{(2)}[T(b)]\ , \tag{3.1b}$$

$$\frac{du}{d\ln b}=[4-(d+z)]u(b)\ . \tag{3.1c}$$

These equations may easily be solved, yielding

$$T(b)=Tb^z\ , \tag{3.2a}$$

$$u(b)=ub^{4-(d+z)}\ , \tag{3.2b}$$

$$\delta(b)=b^2\left[\delta+2(n+2)u\int_0^{\ln b}dx\, e^{[2-(d+z)]x}f^{(2)}[Te^{zx}]\right]\ . \tag{3.2c}$$

Scaling stops when $\delta(b)\sim 1$. One must distinguish two regimes: $T(b)\ll 1$ and $T(b)\gg 1$. If $\delta(b)\sim 1$ occurs with $T(b)$ small, then an expansion in powers of T about $T=0$ is clearly possible. Hertz[5] refers to this as the quantum (or $T=0$) regime. To obtain the condition on T for the occurrence of the quantum regime, I set $T=0$ in (3.2a), perform the x integral, which is dominated by x near 0, set $\delta(b)=1$, solve for b, substitute the result into (3.1a) and demand $T(b)\ll 1$. The result is

$$1\gg T/r^{z/2} \tag{3.3a}$$

with

$$r=\delta+\frac{2(n+2)uf^{(2)}(T=0)}{z+d-2}\ . \tag{3.3b}$$

However, if the inequality (3.3a) is reversed, the situation is different. It is convenient to divide the scaling into two regimes: $T(b)\ll 1$ and $T(b)\gg 1$. It is shown in Appendix B that for $T\gg 1$, $f^{(2)}(T)\simeq CT$. If one rewrites the scaling equations in terms of $v=(uT)$, they become for $T\gg 1$

$$\frac{d\delta(b)}{d\ln b}=2\delta(b)+2(n+2)Cv(b)\ , \tag{3.4a}$$

$$\frac{dv(b)}{d\ln b}=(4-d)v(b)-4(n+8)Dv(b)^2\ . \tag{3.4b}$$

The details are given in Appendix D. The initial conditions $\bar{\delta}$ and $\bar{v}$ for these equations are obtained by evaluating (3.2b) and (3.2c) at $T(b)=1$ and are (B is given in Appendix B)

$$\bar{\delta}=T^{-2/z}[r+BuT^{(d+z-2)/z}]\ , \tag{3.5a}$$

$$\bar{v}=uT^{(d+z-4)/z}\ . \tag{3.5b}$$

If $r\ll T^{2/z}$, both $\bar{\delta}$ and $\bar{v}$ are much less than unity. We may therefore linearize (3.4b) and solve the equations, obtaining

$$\delta(b)=(\bar{\delta}+C\bar{v})e^{2\ln b}-C\bar{v}e^{\ln b}\ , \tag{3.6a}$$

$$v(b)=\bar{v}e^{\ln b}\ . \tag{3.6b}$$

If $\delta(b)=1$ with $v(b)$ small, scaling stops and one may do perturbation theory in $v(b)$. If $v(b)$ becomes of order 1 with $\delta(b)$ small, scaling changes and non-Gaussian behavior may occur. The condition for Gaussian behavior $[\delta(b)=1, v(b)\ll 1]$ is, from (3.6) and (3.5), in $d=3$

$$\mathcal{K}=\frac{uT^{1+1/z}}{[r+(B+C)uT^{1+1/z}]^{1/2}}\ll 1\ . \tag{3.7}$$

Equation (3.7) is the usual Ginsburg criterion.[16] Equation (3.7) is only violated within a window $\Delta T\sim[T_c]^2$ of the true transition temperature $T_c(r,u)$ which therefore to a very good approximation is given by

$$T_c(r,u)=[r/(B+C)u]^{z/z+1}\ . \tag{3.8}$$

Equation (3.8) for T_c is inconsistent with Ref. 5. In Ref. 5 it is stated that $T_c(r,u)\sim r^{\bar{\beta}}$ with arbitrary exponent $\bar{\beta}$. In fact we see that T_c may be computed essentially exactly. Further, by using measurements of the correlation length and Eqs. (3.9) and (3.11) below to fix the r and T scales and the quantity $u\,(B+C)$ one may obtain a parameter-free prediction of T_c for a given system.

Away from the true transition and for $T>r^{z/2}$, the behavior is found to be in the universality class of the classical Gaussian model, in disagreement with the result of Ref. 5. The present conclusion agrees with the result of Weichman *et al.*[6] which they obtained for a boson model with $d=3$, $z=2$. In particular one may write equations such as (3.7) in terms of the variable $x=r/T^{(d+z-2)/z}$. The exponent of T in this equation defines a Gaussian–non-Gaussian crossover exponent ψ which for $d=3$, $z=2$ we find to be $\psi=\frac{3}{2}$, in agreement with Ref. 6. In describing the crossover occurring when (3.7) is violated I prefer the term "Gaussian–non-Gaussian" to the term "quantum classical" used in Ref. 6 because in the whole regime $|r|<T^{2/z}$, the asymptotic behavior is described by a classical model (relevant modes have energies $<k_BT$) although this classical model has the same critical behavior as in the quantum ($r>T^{2/z}$) regime. Also, note that if $(d+z-2)/z\geq 2$ (i.e., $d\geq 2+z$) the T dependence of the bare parameter δ becomes more important than the T dependence calculated here, and the exponent $(d+z-2)/z$ should be replaced by 2.

I now turn to the physical properties, namely specific heat, correlation length and order-parameter susceptibility. The characteristic frequency ω^* observed, e.g., in an inelastic neutron-scattering experiment scales in all cases as $\omega^*\sim\xi^{-z}$. Consider first the quantum regime $T<(r^{z/2})$. An expansion in powers of $T/r^{z/2}$ is obviously possible. The $T=0$ results for the correlation length and susceptibility are the usual ones for the Gaussian model, e.g.,

$$\xi=r^{-1/2}\ . \tag{3.9}$$

To obtain the free energy F one integrates (2.6d) from $b=0$ to $b=r^{-1/2}$ and adds $b^{-(d+z)}$ times the free energy of the Gaussian model at scale $b=r^{-1/2}$. The details are given in Appendix B. The specific-heat coefficient $\gamma=C_V/T$ is given by differentiating Eqs. (B7) twice and is, for the ferromagnet,

$$\gamma=\frac{4K_3A_2}{\pi}\ln(1/r)-\frac{64K_3}{\pi}\frac{T^2}{r^2}\ln(1/T)\ , \tag{3.10a}$$

while for the antiferromagnet

$$\gamma=\gamma_0-\frac{16A_2K_3}{\pi}r^{1/2}-\frac{128K_3A_4}{3\pi}\frac{T^2}{r^{3/2}}\ . \tag{3.10b}$$

K_3, $A_{2,4}$ are numbers defined in Eqs. (B4) and (B6b), respectively. Note that for the antiferromagnet the leading term, γ_0, in the low-T and r expansion for γ is not divergent as $r\to 0$; only the coefficient of the first correction in r or T diverges.

It is tempting to speculate about the relevance of (3.10b) to the heavy-fermion compounds.[1,2,17] Many of these display Fermi-liquid behavior (including $F\sim T^2$) at low T; however, in some cases they may be driven antiferromagnetic by 2–3 % doping[2] and in at least some cases the leading correction to Fermi-liquid behavior are anomalously large.[17]

Now consider the "classical" regime $r<T^{2/z}$. Two subregimes exist, according as r is larger or smaller than $BuT^{1+1/z}$. In the former subregime an expansion in powers of T is possible, but the expansion parameter is $(B+C)uT^{1+1/z}/r$. The correlation length is given by Eq. (3.9). However, if $r<(B+C)uT^{1+1/z}$, the correlation length is given by

$$\xi^{-2}=(B+C)uT^{1+1/z}\ . \tag{3.11}$$

The free energy is computed in Appendix B and has two interesting contributions: one due to fluctuations on the scale of the thermal de Broglie wavelength and one given by $kT\xi^{-d}$ as expected near a classical critical point. The first term leads to the following behavior of the specific-heat coefficient, $\gamma=C_V/T$:

$$\gamma=\frac{8K_dA_2}{\pi}\ln 1/T\quad (d=3,\ z=3)\ , \tag{3.12a}$$

$$\gamma=\gamma_0-\alpha T^{1/z}\quad (d=3,\ z=2)\ . \tag{3.12b}$$

The other term gives a contribution

$$\gamma=\frac{[(B+C)uT^{1/z}]^2}{[r+(B+C)uT^{1+1/z}]^{1/2}} \tag{3.13}$$

which becomes important only very near the true transition $r=-BuT^{1+1/z}$. The results for the free energy differ somewhat from the results for boson models.[6] The difference is due to the presence, in the problem considered here, of the particle-hole continuum of low-energy excitations.

IV. SOLUTION OF SCALING EQUATIONS IN $d=2$

A. Gaussian case, $z>2$

Assume first $z>2$, so the $T=0$ fixed point is Gaussian. Then the solutions (3.2) are still valid, as is the criterion (3.3) for the quantum-classical crossover. However, the results for the "classical" regime in which the inequality in (3.3a) is reversed must be reconsidered. To see this, evaluate (3.6) in $d=2$ at the scale $T(b)=1$, assuming that the renormalized $\delta(b)<1$. I adopt the notation $\bar{\delta},\bar{v}$ for the values of δ,u at $T(b)=1$ and find

$$\bar{v}=uT^{(z-2)/z}\ , \tag{4.1a}$$

$$\bar{\delta}=T^{-2/z}[r+BuT]\ . \tag{4.1b}$$

To continue scaling, one may use the linearized form of classical equations (3.4a) and (3.4b) because (4.1a) guarantees that $\bar{v}$ is small and by assumption $\bar{\delta}$ is small. Note that in contrast to the usual classical two-dimensional (2D) models,[18] where interaction terms $\phi^6,\phi^8,\ldots$ are as relevant as the ϕ^4 term, in the present case the scaling from the $T=0$ fixed point ensures that only the ϕ^4 term needs to be considered. Further, the initial conditions

(4.1) imply that $\bar{v},\bar{\delta} \ll 1$, so the linearized scaling equations may be used. Solving (3.4a) and (3.4b) in $d=2$ yields

$$\delta(b)=\bar{\delta}e^{2\ln b}+C\bar{v}\ln b e^{2\ln b} , \tag{4.2a}$$

$$v(b)=\bar{v}e^{2\ln b} . \tag{4.2b}$$

As before, scaling changes when either $\delta(b)$ or $u(b)$ becomes of order unity. If $\delta(b)\sim 1$ with $v(b)\ll 1$, then scaling stops and we may compute by perturbation theory about the Gaussian model. If $v(b)\sim 1$ with $\delta(b)\lesssim 1$, then the form of the scaling equations changes. From (4.1) and (4.2) one sees that the condition for Gaussian behavior is

$$\frac{r+BuT}{uT}+\tfrac{1}{2}C\ln[uT^{(2-z)/z}]\gg 1 . \tag{4.3}$$

When this condition is violated one expects non-Gaussian behavior. Note that at $r=0$ (i.e., when approaching the $T=0$ fixed point by varying temperature) the condition is satisfied. In this case the asymptotic behavior is Gaussian; however, the expansion parameter is only $1/\ln[uT^{(2-z)/z}]$.

The non-Gaussian regime is beyond the scope of this paper. However, if (4.3) is satisfied, we must still consider two subregimes. The first is the "perturbative classical" subregime defined by the condition

$$\delta(b)=1\gg v(b)\ln b , \tag{4.4a}$$

i.e.,

$$\bar{\mathcal{H}}=\frac{\frac{1}{2}CuT\ln[T^{2/z}/(r+BuT)]}{r+BuT}\ll 1 . \tag{4.4b}$$

In this regime the correlation length ξ is given by

$$\xi^{-2}=r+BuT=\mathcal{O}(\bar{\mathcal{H}}) . \tag{4.5}$$

On the other hand, in the regime in which (4.3) holds but $\bar{\mathcal{H}}\gg 1$ one finds that the classical rescaling parameter b is given by the solution of the equation

$$1=C\bar{u}\ln b e^{2\ln b} . \tag{4.6a}$$

Solving this equation iteratively leads to

$$\ln\xi=-\tfrac{1}{2}\ln[Au/T] -\tfrac{1}{2}\ln(-\tfrac{1}{2}\ln[AuT^{(2-z)/z}])+\cdots . \tag{4.6b}$$

As in Sec. III, I obtain the free energy by integrating Eq. (2.6d). The leading behavior of the specific-heat coefficient γ comes from the classical Gaussian region and is

$$\gamma=\alpha T^{2/z-1} . \tag{4.7}$$

B. Marginal case, $z=2$

I now consider the behavior of the marginal case $d=2$, $z=2$. The solution of the scaling equations even near the $T=0$ critical point proceeds along the same lines as in previous cases, but is more complicated. The details are given in Appendix C. The scaling equations for $z=2$, $d=2$ are essentially those introduced and solved previously by Fisher and Hohenberg[7] in a study of the dilute Bose gas. Reference 7 focused on the regime $\delta\leq\delta_c$. Where there is overlap, the results derived here agree with those of Ref. 7. Near the $T=0$ fixed point, the nonlinear terms in (2.6c) must be retained. The solution is

$$u(b)=\frac{u}{1+[2(n+8)/\pi^2]u\ln b} . \tag{4.8}$$

Using (4.8) in (2.6b) gives, in the quantum regime,

$$\delta(b)=e^{2\ln b}[r+\pi^2 a_1 T^2 e^{-2\ln b}/\ln b] . \tag{4.9}$$

Here a_1 is a number and r is defined in Eq. (C7). Note that the coefficient of the T^2 term is universal, i.e., independent of u. From (4.9) and (3.2a) one finds that the condition for the existence of the quantum regime is

$$r>T . \tag{4.10}$$

If (4.10) is satisfied then the correlation length is given by

$$\xi^{-2}=r , \tag{4.11}$$

while the specific-heat coefficient γ is

$$\gamma=\frac{4A_2}{\pi^2}\ln 1/r . \tag{4.12}$$

If (4.10) is violated then one finds, at the scale $T(b)=1$,

$$\bar{\delta}=\frac{[r+GT/\ln 1/T]}{T} , \tag{4.13a}$$

$$\bar{v}=\frac{\pi^2}{n+8}\frac{1}{\ln 1/T} . \tag{4.13b}$$

To continue scaling one inserts (4.13) into (4.2). As before, three regimes exist. In the perturbative Gaussian regime defined by

$$\frac{G\pi^2}{\ln 1/T}<r/T<1 , \tag{4.14}$$

one finds that the correlation length is still given by (4.11), while the specific-heat coefficient is

$$\gamma=\frac{4}{\pi^2}A_2\ln 1/T . \tag{4.15}$$

Next, there is the classical Gaussian regime, defined by

$$\left\{G\pi^2+\frac{r\ln 1/T}{T}+c\ln\left[\frac{n+8}{n^2}\ln 1/T\right]\right\}>\pi^2/n+8 . \tag{4.16}$$

In this regime, which for $r<0$ only exists in the irrelevantly low-temperature range $\ln[\ln 1/T]\gg 1$, the correlation length satisfies the equation

$$[T\xi^2]\ln[T\xi^2]=\frac{2n+16}{\pi^2}\ln 1/T ; \tag{4.17}$$

i.e., $\xi^2\sim 1/T$ up to logarithmic terms. Note that the ratio of the energy of a fluctuation at scale ξ to kT is only $1/\ln(1/T)$ (at $r=0$); thus in the $d=2$, $z=2$ case the fluctuations are only weakly classical at $r=0$. The condition $\ln[\ln 1/T]\gg 1$ for the validity of the perturbation theory in this regime was obtained in Ref. 7.

To understand the regime $r<0$ where (4.16) is violated one must deal with nonlinear terms in v in the scaling equations (3.4a) and (3.4b). I reserve a detailed discussion of this issue for a subsequent paper. Here I simply note that in the limit of a large number of spin components, n, the classical 2D scaling equations for the interaction v have a fixed point at $v=v^*\sim 1/n$. In Eq. (3.4b) for δ one substitutes v^* for v and then solves. The result is

$$\xi^{-2}=(r+KT) \tag{4.18}$$

where K is a universal number computed at large n in Appendix D. Of course, the divergence of ξ at $r=-KT$ implied by (4.18) is an artifact of the $n\to\infty$ limit. Rather, where the right-hand side of (4.18) vanishes sets the scale at which amplitude fluctuations of the field ϕ begin to become frozen out, and a nonlinear-σ model description becomes appropriate. The universal scale at which the crossover occurs is reminiscent of the results of Ref. 4, where a model with $z=1$ was studied and the crossover occurred at $r\sim T^2$. Note that in the region where $|r|/T$ is of order unity, the energy of an excitation at scale ξ is kT (times a factor of order unity). Thus the behavior in this region is not strictly classical and may be in many respects similar to that found in Ref. 4, where a model with $z=1$ was found (in the analogous region) to yield a correlation length such that excitations at scale ξ had energies of order kT. In Ref. 4 this behavior was conjectured to be universal in the region $|r|<T^{z/2}$ for models with $(d+z)<4$; the results presented here support that conjecture.

V. RELATION TO "SCR" FORMALISM

The "SCR" (self-consistently renormalized) formalism of Moriya is a self-consistent one-loop approximation for the models presented in Eqs. (2.3) and (2.4). It has been used to analyze realistic models; the results agree well with many experiments.[9] In this method one approximates the theory by the Gaussian model, Eq. (2.2), with the parameter δ replaced by a parameter $\delta_{\rm SCR}$ defined via the self-consistent equation (written here for the antiferromagnet, $z=2$).

$$\delta_{\rm SCR}=\delta+2(n+2)u\int_0^1\frac{d^dk}{(2\pi)^d}\int_0^1\frac{d\varepsilon}{\pi}\coth\frac{\varepsilon}{2T}\times\frac{\varepsilon}{(\delta_{\rm SCR}+q^2)^2+\varepsilon^2}\ . \tag{5.1}$$

In $d=3$ this procedure does not give the crossover to true critical in the regime controlled by the Gaussian fixed point. It does yield the same exponents as found here for the regime controlled by the Gaussian fixed point, albeit with different coefficients. For example, if the $T=0$ critical point were approached by varying T (with the $T=0$ parameters tuned to their critical values) Eq. (1) would give

$$\xi^{-2}=(n+2)uT^{3/2} \tag{5.2a}$$

while the treatment in Sec. III gives

$$\xi^{-2}=2(n+2)u(B+C)T^{3/2}\ . \tag{5.2b}$$

In $d=2$, $z=3$ the SCR procedure does not yield the logs generated by scaling in the classical region and of course does not give the universal behavior found for $d=2$, $z=2$.

VI. CONCLUSION

In this paper I have used renormalization-group techniques to study the Gaussian or marginal fixed points occurring when itinerant fermion systems in 2 or 3 spatial dimensions undergo $T=0$ phase transition at wave vectors Q which do not span the Fermi surface. I have given a detailed treatment of the model introduced by Hertz,[5] and have corrected his treatment of the classical regime. As emphasized by Hertz,[5] the results depend crucially upon the value of $d+z$, where d is the spatial dimension and z the dynamic exponent of the $T=0$ transition. In the models considered here the particle-hole continuum Fermi excitations overdamps the mode associated with the ordering transition and implies to $z=3$ for the clean ferromagnet and $z=2$ for the antiferromagnet. If $(d+z)>4$, the $T=0$ transition is above its upper critical dimension and the results are nonuniversal in that they depend upon the value of the coefficient u of the ϕ^4 term in the Hamiltonian. As microscopic parameters are varied at $T=0$, the correlation length diverges as $\xi\sim r^{-1/2}$ ($r=0$ at the transition). The results for $T>0$ depend on whether $d>2$ or $d=2$.

The results for the case $d>2$ are summarized in Fig. 2, which depicts the different regions of behavior occurring in the (T,δ) plane. At $T=0$ a phase transition in the Gaussian universality class occurs as δ is varied through a critical value δ_c. For simplicity I have not considered the ordered phase in this paper, because the order may affect the fermion spectrum, and too many special cases arise. The behavior for low T and $\delta<\delta_c$ of a model of superfluid helium was studied in Ref. 6. In the disordered phase at low T (region I in Fig. 2) the physics is quantum in the sense that fluctuations on the scale of ξ have energies much greater than kT. It is also quantum in the sense that the specific heat, C_V, takes the Fermi liquid form $C_V/T=\gamma+a_1T^2\ln T+a_2T^2+\cdots$. This means among other things that in this region the soft spin fluctuations do not qualitatively change the fermion propagators. As the temperature is raised beyond the quantum-classical crossover scale $T\sim\delta^{z/2}$, the physics changes because the energy of modes on the scale ξ becomes less than kT. This quantum-classical crossover was discussed by Hertz,[5] who also showed that the quantum-classical crossover exponent was $z/2$. The physics of the classical region $T>|\delta-\delta_c|^{z/2}$ was not correctly described in previous work.[5] It is necessary to distinguish two regimes. One, region II, I call the perturbative classical subregime. In this subregime the correlation length ξ is still given by the $T=0$ expression $\xi^{-2}\sim|\delta-\delta_c|$ (with corrections of order $T^{(z+1)/z}$), even though modes at the scale of ξ have energies less than T. The specific heat is dominated by fluctuations with ener-

gies of order kT; for $z=3$ this leads to $C_V/T \sim \ln 1/T$ while for $z=2$, $C_V/T \sim \gamma_0 + \alpha T^{1/2}$. Thus for the antiferromagnet, the correction to the leading Fermi-liquid behavior $C_V/T=\gamma$ becomes nonanalytic. This result may be relevant to the heavy electron metals, many of which are near antiferromagnetic instabilities.[1,2,17] In these materials the leading correction to the low-temperature Fermi-liquid result $\gamma=$const is often very large and difficult to understand within Fermi-liquid theory.[17] In region III the correlation length is controlled by T rather than $|\delta-\delta_c|$; the specific heat and corrections to the fermion propagators will be as in region II. Although the critical phenomena in regions II and III are classical in the sense that modes on the scale of the correlation length have energies less than T; they are in the same Gaussian universality class as in region I. For example, the exponent relating the order parameter susceptibility χ to the correlation length takes the Gaussian model value 2. Also (although I have not presented the calculation here) the amplitude ratio constructed by comparing the magnetic field dependence of χ and ξ to the $|\delta-\delta_c|$ dependence (regions I, II) or temperature dependence (region III) always takes the Gaussian model value of 1.[19] In Ref. 5 it was incorrectly claimed that for $T>\delta^{z/2}$ the critical phenomena would be in a non-Gaussian 3D universality class. The reason that classical Gaussian behavior is found over such a wide range is that by the time one has scaled the problem onto an effective classical problem the interaction (which was irrelevant at the $T=0$ fixed point) has become so small that one may neglect it, unless one is very close to the true transition temperature $T_c(\delta)$. The dotted line labeled T_G in Fig. 2 indicates the "Ginzburg temperature;" between this and T_c, true three-dimensional critical fluctuations are important. The critical region is very narrow at low T. The crossover to true critical behavior is described for $2+z-d>2$ by the Gaussian–non-Gaussian exponent $\psi=(d+z-2)/z$, in agreement with results for the dilute interacting Bose gas.[6] However, in region III, but not in regions I or II, the ϕ^4 coupling u is a "dangerous irrelevant operator," so that it enters the expressions, e.g., for ξ and $T_c(\delta)$. Further, I have calculated $T_C(\delta)$ in terms of parameters which may be fixed by measurements of $\lim_{T\to 0}\xi(\delta=\delta_c,T)$ and $\lim_{\delta\to\delta_c}\xi(\delta,T=0)$.

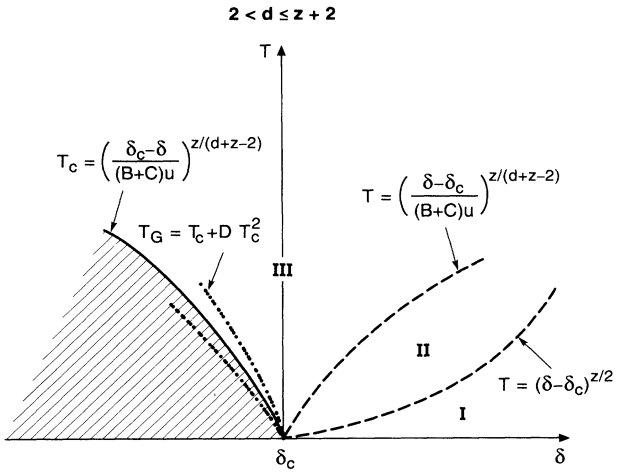

FIG. 2. Phase diagram in plane of temperature T and control parameter δ, showing different regimes of behavior for dimensions larger than two. The shaded area marks the ordered region. The behavior in this regime depends on whether the model has Ising, xy, or higher symmetry and on whether the fermion spectrum is full, or partially gapped, and is not discussed in the text. The unshaded area is discussed in the text. Region I is the disordered quantum regime; here thermal effects are negligible, the correlation length is given by Eq. (3.9) and the specific-heat coefficient by Eq. (3.10). Region II is the perturbative classical regime; here the correlation length is given by Eq. (3.9) but the specific-heat coefficient is given by Eq. (3.12). Region III is the classical Gaussian region; here the important cutoff is T, not $\delta-\delta_c$, and the relevant modes have energies less than kT. The correlation length is given by Eq. (3.11) and the specific-heat coefficient by (3.12). The solid line gives the dependence of the transition temperature T_c on $(\delta-\delta_c)$. The dotted lines give the dependence of the Ginsburg temperature T_G on parameters. Inside the dotted lines true three-dimensional classical critical phenomena occur.

The results quoted here were derived under the assumption that the T dependence of the bare parameters could be neglected. In a Fermi liquid, one expects T^2 corrections to physical quantities, including δ.[12] I write $\delta=\delta_0+\delta_2 T^2$. The T dependence of δ will be more important than the T dependence coming from the dangerous irrelevant operator u if $d>2+z$. In this case one should replace the exponent "$(d+z-2)/z$" by "2" and the quantity "$(B+C)u$" by "δ_2" in all formulas.

I turn now to the two-dimensional case, summarized in Fig. 3. The qualitative physics is very similar to that in $d=3$, but fluctuation effects are stronger, especially in the classical regime. Further, in the case $d=2$, $z=2$ the $T=0$ transition is at its upper critical dimension. In all cases the corrections to C_V/T (and therefore to the fermion propagators) are larger. Define $T_+(\delta)$ to be the upper boundary of region II and $T_-(\delta)$ to be the lower. For $d>2$ or $z>2$, $R=\lim_{\delta\to\delta_c} T_+(\delta)/T_-(\delta)$ diverges as $(\delta-\delta_c)^{(z-2)/z}$ while at $d=2$, $z=2$, $R\sim \ln 1/(\delta-\delta_c)$, so region II is almost squeezed out of existence in this case. I conjecture it does not exist for $d=2$, $z<2$. It was not found in a model with $d=2$, $z=1$.[4] Further, for $d>2$ the expansion parameter giving corrections to Gaussian behavior in region III was a power of T. In $d=2$ the expansion parameter is $[\ln T]^{-1}$ for $z>2$ for $z=2$ if $(\delta-\delta_c)/T>0$, while for $z=2$ and $\delta\le\delta_c$ the expansion parameter is $(\ln[\ln(T)])^{-1}$ (a similar $\ln[\ln(T)]^{-1}$ expansion was found in the study of the dilute interacting 2D Bose gas[7]). Thus the true critical behavior will be difficult to access and, indeed, my results for $z=2$, $d=2$, $\delta\le\delta_c$ are of essentially no practical relevance. To obtain useful results in this regime another technique is required. In this paper I obtained a few such results using a large-n technique similar to that of Ref. 4, but clearly further work is required. As in Fig. 3 the dotted line gives the Ginzburg temperature below which classical fluctuations become important and non-Gaussian behavior may occur. Below the dashed line the physics is

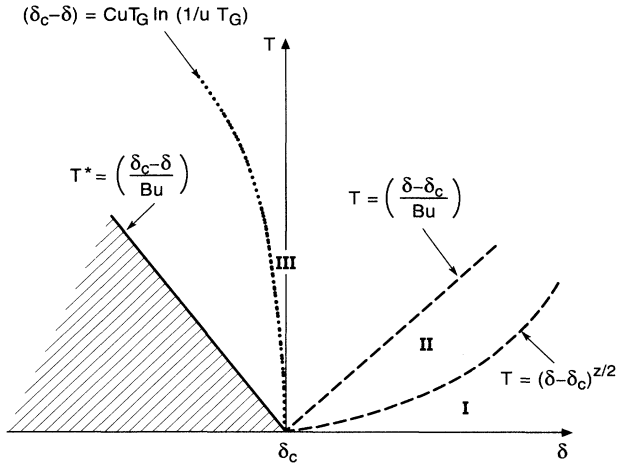

FIG. 3. Phase diagram in plane of temperature T and control parameter δ for two spatial dimensions. Region I is the quantum regime. The correlation length is given by Eq. (3.9) and the specific-heat coefficient by Eq. (B7c) ($z>2$) or (C8) ($z=2$). Region II is the perturbative classical regime. The correlation length is given by Eq. (4.5) ($z>2$) or (4.11) ($z=2$) and C_V/T by Eq. (4.7) ($z>2$) or (4.15) ($z=2$). In region III is the classical Gaussian regime. The correlation length is given by Eq. (4.6b) ($z>2$) or (4.17) ($z=2$). The dotted curve defines the Ginsburg temperature, below which nontrivial two-dimensional classical fluctuations occur. The solid curve $T^*(\delta)$ (the prefactor of which has not been calculated here) gives the crossover to a regime dominated by fluctuations about a $T=0$ ordered state. The behavior in this regime will depend on whether the model has Ising, xy, or higher symmetry and on whether the fermions are fully or partially gapped, and has not been discussed here.

dominated by thermal fluctuations about a $T=0$ ordered state; the behavior here depends on whether the model has Ising[20] or XY (Ref. 21) symmetry (in which case a $T>0$ transition will occur) or Heisenberg or higher symmetry [in which case long-range order is believed to occur only at $T=0$ (Ref. 22)], and on the effect of the order on the fermion spectrum. These issues were not considered here. Results for the bosonic XY model were given in Ref. 7.

One crucial assumption underlying the present treatment is that the effective action equation (2.2) is an analytic function of ω (or ω/k), T, k, and δ. As pointed out by Hertz[5] this assumption breaks down in cases in which perfect nesting is important, such as the half-filled Hubbard model or the superconductor without pair breaking. As shown here it also breaks down in clean systems when the ordering wave vector spans the Fermi surface. The results obtained here need not apply in those cases. Another deficiency of the present treatment is that it does not include the effects of disorder, which are present in many experimentally interesting systems and which are relevant in the critical phenomenon sense.

Note added in proof. Recently, I became aware of related work by Continentino,[23] who gave a qualitative scaling analysis of the transitions considered here. Continentino's results differ from those presented here; I believe the differences occur because Continentino assumes that the hyperscaling relation holds. Hyperscaling has been shown not to apply to models above their upper critical dimension.[24] All of the models considered here are above their upper critical dimension except for the $d=2$ antiferromagnet, and the results I have presented for $(d+z)>4$ are inconsistent with hyperscaling.

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APPENDIX A: DERIVATION OF RENORMALIZATION-GROUP EQUATIONS

In this Appendix the renormalization-group equations (2.6a)–(2.6d) are derived. The procedure is similar to that of Hertz[5] but the frequency cutoff and the temperature dependence of the terms in the equation is analyzed more carefully. Before plunging into the full calculation it is instructive to consider the Gaussian model, defined by Eq. (2.3a) or (2.3b). The free energy corresponding to Eq. (2.3a) is

$$F_{\text{Gauss}} = V\int_0^{\Lambda}\frac{d^dk}{(2\pi)^2}\int_0^{\Gamma_k}\frac{d\varepsilon}{\pi}\coth\frac{\varepsilon}{2T} \times\tan^{-1}\left[\frac{\varepsilon/\Gamma_k}{(\delta+k^2)}\right]. \tag{A1}$$

Here $\Gamma_k=\Gamma k$ for the ferromagnet and $\Gamma_k=\Gamma$ for the antiferromagnet. Now to perform a classical renormalization-group transformation one eliminates from the problem modes with wave vectors k satisfying $\Lambda\geq k\geq\Lambda/b\,(b>1)$ and then rescales. Eliminating the rapidly spatially varying modes yields

$$F = V\int_0^{\Lambda/b}\frac{d^dk}{(2\pi)^d}\int_0^{\Gamma_k}\frac{d\varepsilon}{\pi}\coth\frac{\varepsilon}{2T}\tan^{-1}\frac{\varepsilon/\Gamma_k}{(\delta+k^2)} + F_k'\ln b \tag{A2a}$$

with

$$F_k' = V\Lambda^dK_d\int_0^{\Gamma_\Lambda}\frac{d\varepsilon}{\pi}\coth\frac{\varepsilon}{2T}\tan^{-1}\frac{\varepsilon/\Gamma_\Lambda}{(\delta+\Lambda^2)}. \tag{A2b}$$

Here Γ_Λ is the frequency cutoff when the magnitude of the wave vector $k=\Lambda$. Now in (A.2a) I rescale lengths, setting $k=k'/b$. This requires rescalings of $\delta=\delta'/b^2$, $\varepsilon=\varepsilon'/b^z$, $T=T'/b^z$, and volume $V=V'b$ leading in the limit $b\to1$ to

$$F = V'b^{-(d+z)}\int_0^{\Lambda}\frac{d^dk'}{(2k)^d}\int_0^{\Gamma_{k'}b}\frac{d\varepsilon'}{\pi}\coth\frac{\varepsilon'}{2T'} \times\tan^{-1}\frac{\varepsilon'/\Gamma_{k'}}{(\delta'+k')} + F_k'\ln b\,. \tag{A3}$$

Thus to complete the transformation we must eliminate energies ε' satisfying $\Gamma_{k'}b>\varepsilon'>\Gamma_{k'}$, or, in the original language, $\Gamma_k\geq\varepsilon\geq\Gamma_k/b$, generating a new contribution. $F'(\varepsilon)\ln b$ given by

$$F'_\varepsilon = \frac{V}{\pi}\int_0^\Lambda \frac{d^d k}{(2\pi)^d}\coth\frac{\Gamma_k}{2T}\tan^{-1}\frac{\Gamma_k}{(\delta+k^2)} \ . \tag{A4}$$

The Gaussian-model calculation illustrates another point. The model was formulated in Sec. II in terms of a sum over discrete Matsubara frequencies. One could impose the energy cutoff directly on the Matsubara sum, for example, retaining only frequencies $\Omega_n < \Gamma$. However, it is not clear how to continuously vary a cutoff on a sum over discrete frequencies. Hertz[5] assumes one may treat them as a continuum. I believe that it is conceptually more straightforward to write the free energy in terms of integrals over continuous real frequencies and impose the cutoff on these integrals.

One may evaluate the free energy of the model defined by Eqs. (2.3a) and (2.4) as an expansion in powers of u. The result is

$$F = F_{\rm Gauss} + un(n+2)[I(\Lambda,\Gamma,\delta,t)]^2 + 4u^2n^2(n+8)[I(\Lambda,\Gamma,\delta,T)]^2 J(\Lambda,\Gamma,\delta T) + \cdots \tag{A5}$$

where $F_{\rm Gauss}$ was defined in Eq. (A1), and

$$I(\Lambda,\Gamma,\delta,T) = V\int_0^\Lambda \frac{d^d k}{(2\pi)^d}\int_0^{\Gamma_k}\frac{d\varepsilon}{\pi}\coth\left[\frac{\varepsilon}{2T}\right]\frac{\varepsilon/\Gamma_k}{(\varepsilon/\Gamma_k)^2+(\delta+k^2)^2} \ , \tag{A6a}$$

$$J(\Lambda,\Gamma,\delta,T) = V\int_0^\Lambda \frac{d^d k}{(2\pi)^d}\int_0^{\Gamma_k}\frac{d\varepsilon}{\pi}\coth\left[\frac{\varepsilon}{2T}\right]\frac{(\varepsilon/\Gamma_k)(\delta+k^2)}{[(\varepsilon/\Gamma_k)^2+(\delta+k^2)^2]^2} \ , \tag{A6b}$$

and the ellipsis denotes terms not important for the present discussion.

Now in Eq. (A5) I separate out of each term the regions given by $\{\Lambda \geq k \geq \Lambda/b, \Gamma \geq \varepsilon \geq 0\}$ and $\{\Lambda \geq k \geq 0, \Gamma_k \geq \varepsilon \geq \Gamma_k/b^2\}$, linearize in the quantity $(b-1)$, rescale and compare the result with what one obtains in perturbation theory in u with renormalized parameters. Note that nontrivial renormalization of the coefficient of the k^2 or $i\omega$ terms is not necessary to order u^2. The result is Eqs. (2.6a)–(2.6d) with (setting $V=\Lambda=\Gamma=1$)

$$f^{(0)}(T) = F'_k + F'_\varepsilon \ , \tag{A7a}$$

$$f^{(2)}(T) = \frac{2}{\pi}\int_0^{k'}\frac{d^d k}{(2\pi)^d}k^{z-2}\coth\frac{k^{z-2}}{2T}\frac{1}{1+(\delta+k^2)^2} + K_d\int_0^1\frac{d\varepsilon}{\pi}\coth\frac{\varepsilon}{2T}\frac{\varepsilon}{\varepsilon^2+(\delta+1)^2} \ , \tag{A7b}$$

$$f^{(4)}(T) = \frac{2}{\pi}\int_0^1\frac{d^d k}{(2\pi)^d}k^{z-2}\coth\frac{k^{z-2}}{2T}\frac{(\delta+k^2)}{[\Gamma^2+(\delta+k^2)^2]^2} + K_d\int_0^1\frac{d\varepsilon}{\pi}\coth\frac{\varepsilon}{2T}\frac{\varepsilon(\delta+\Lambda^2)}{[\varepsilon^2+(\delta+\Lambda^2)^2]^2} \ . \tag{A7c}$$

The only difference between the ferromagnet and the antiferromagnet is the factor k^{z-2} in and before the coth.

APPENDIX B: SOLUTION OF SCALING EQUATIONS, $d+z>4$

In this appendix technical details of the solution of the scaling equations are given. There are two subsections.

1. Quantum regime $T < r^{z/2}$

a. Solution of Eq. (3.2c)

One should expand $f^{(2)}$ [Eq. (A7b)] in powers of T. Only even powers occur and the most efficient way to generate the expansion is to write $f^{(2)}(T) = f^{(2)}(T=0) + Tdf^{(2)}/dT$. One may determine the rescaling, b, at which $\delta(b)=1$ by retaining only the $T=0$ term in the expansion of $f^{(2)}$ and solving the equation $\delta(b)=1$ for b. The result is

$$b^2 = 1/r \tag{B1}$$

with

$$r = \delta_0 + \frac{2(n+2)uf^{(2)}(T=0)}{d+z-2} \ . \tag{B2}$$

$f^{(2)}(T=0)$ depends upon d and z. From (A7b) one finds

$$(d=3,\ z=3):\ f^{(2)}(T=0) = K_3\ln 2/\pi \ , \tag{B3a}$$

$$(d=3\ ,\ z=2):\ f^{(2)}(T=0) = \frac{K_3}{2\pi}\left[\ln 2 + 4\int_0^1 dk\frac{k^2}{1+k^4}\right] \ , \tag{B3b}$$

$$(d=2\ ,\ z=3):\ f^{(2)}(T=0) = \frac{K^2}{2\pi}\left[\ln 2 + 4\int_0^1 dk\frac{k}{1+k^4}\right] \ , \tag{B3c}$$

where K_d is defined via

$$\int \frac{d^d k}{(2\pi)^d} f(k) = K_d \int k^{d-1} dk f(k) . \quad \text{(B4)}$$

b. Evaluation of free energy

One integrates (2.6d) with initial condition $S_0 = 0$ to the scale b defined in (B1) and (B2) adds the free energy of the Gaussian model at scale b, then multiplies the whole by $b^{-(d+z)}$ to undo the trivial rescalings and get a free-energy density. Consider first the solution to (2.6d); formally this may be written as

$$b^{-(d+z)} S_0(b) = \int_0^{\ln b} dx e^{-(d+z)x} f^{(0)}(T e^{zx}) . \quad \text{(B5)}$$

In the quantum (low-T) regime one expands $f^{(0)}$ and integrates term by term. The term of order T^0 in $f^{(0)}$ gives a negligible contribution to S. The terms of order T^2 and T^4 may be most easily derived by writing $f^{(0)}(T) = f^{(0)}(0) + T df^{(0)}/dT$. For $d=3$, $z=2,3$ and $d=2$, $z=3$ the only important contributions come from F_k' [Eq. (A2)] and one finds

$$f^{(0)}(T) - f^{(0)}(0) = 2\frac{4K_d}{\pi}[A_2 T^2 - \tfrac{4}{3} A_4 T^4] \quad \text{(B6a)}$$

with

$$A_n = \int_0^\infty dx \frac{x^n}{\sinh^2(x)} . \quad \text{(B6b)}$$

Substituting (B6a) into (B5), integrating and adding the contribution from the Gaussian model at scale $\delta(b)=1$ gives, for the leading singular (r-dependent) terms in F,

$$(d=3\ ,\ z=3):\quad F = -\frac{2K_3 A_2}{\pi} T^2 \ln 1/r + \frac{16K_3}{3\pi}\frac{T^4}{r^2}\ln 1/T , \quad \text{(B7a)}$$

$$(d=3\ ,\ z=2):\quad F = -\frac{8K_3 A_2 T^2}{\pi} r^{1/2} + \frac{32K_3 A_4}{9\pi}\frac{T^4}{r^{3/2}} , \quad \text{(B7b)}$$

$$(d=2\ ,\ z=3):\quad F = -\frac{8K_2 A_2}{\pi}\frac{T^2}{r^{1/2}} . \quad \text{(B7c)}$$

2. Classical regime $T > r^{z/2}$

a. Solution of Eq. (3.2c)

It is convenient to break the scaling up into two regimes: $T(b) < 1$ and $T(b) > 1$. This introduces multiplicative errors of order unity coming from imprecise treatment of the crossover regime $T(b) \sim 1$, but makes the analysis simpler. A more accurate treatment could be performed. To solve the equation for $T(b) < 1$ I again separate $f^{(2)}(T) = f^{(2)}(T=0) + T df^{(2)}/dT$. The term $f^{(2)}(T=0)$ is easily integrated.

To deal with the T dependence I use (2.6a) to change variables from x to T; the result is, at the scale $T(b)=1$,

$$\delta = T^{-2/z}\left[r + 2(n+2)uBT^{(d+z-2)/z}\right] \quad \text{(B8)}$$

with

$$B = \frac{1}{z}\int_0^1 dT T^{(2-d-2z)/z}[f^{(2)}(T) - f^{(2)}(0)] . \quad \text{(B9)}$$

To continue scaling we need the high-T limit of $f^{(2)}(T)$. From Eq. (A7b) this is seen to be

$$\lim_{T\to\infty}\frac{f^{(2)}(T)}{T} = C = \frac{4K_d}{\pi}\int_0^1 dk \frac{k^{d-1}}{1+k^4} + \frac{K_d}{2} . \quad \text{(B10)}$$

The rescaling factor needed to make $\delta(b)=1$ is given by combining (3.6a) and (B8) and is, in $d=3$,

$$b^* = [r + (B+C)uT^{1+1/z}]^{-1/2} . \quad \text{(B11)}$$

In $d=2$ extra logarithms occur as discussed in Sec. IV.

b. Calculation of free energy

As before, I proceed from (B5). It is convenient to divide the integral into two regimes: $0 \le x \le z^{-1}\ln 1/T$ (quantum regime) and $z^{-1}\ln 1/T \le x \le \ln b^*$. In the first regime the T-independent part of $f^{(0)}$ gives a negligible contribution, while the T-dependent part may be written

$$\int_0^{z^{-1}\ln 1/T} dx\, e^{-(d+z)x} f^{(0)}(T e^{zx}) = T^{(d+z)/z}\int_T^1 dT\, T^{-2-d/z}[f^{(0)}(T) - f^{(0)}(0)] \quad \text{(B12)}$$

$\lim_{T\to 0}[f^{(0)}(T) - f^{(0)}(0)] \sim T^2$, so for $d=z=3$ a logarithm occurs. For $d=3$, $z=2$ the T^2 term diverges at the lower end, which leads to finite renormalization of the T^2 term in the free energy. The leading behavior for $d=3$, $z=2$ or $d=2$, $z=3$ is $T^{(d+z)/z}$. In the regime $z^{-1}\ln 1/T \le x \le \ln b^*$ one may approximate

$$f^{(0)}(T) = DT \quad \text{(B13)}$$

with

$$D = \frac{2K_d}{\pi}\int_0^1 d\varepsilon \frac{1}{\varepsilon}\tan^{-1}\varepsilon + \frac{2K_d}{\pi}\int_0^1 dk k^{d+1-z}\tan^{-1}\frac{1}{k^2} . \quad \text{(B14)}$$

Using (B13) in (B5) and adding the free energy of the Gaussian model gives for $d=3$, $z=3$.

$$F = \frac{4K_3 A_2}{\pi} T^2 \ln 1/T - ETb^{*-d} \quad \text{(B15)}$$

with

$$E = D + \frac{2K_d}{\pi}\int_0^1 dk\, k^{d-1}\int_0^1 d\varepsilon \tan^{-1}\frac{\varepsilon}{k^{(z-2)}} . \quad \text{(B16)}$$

For $d=3$, $z=2$ or $d=2$, $z=3$,

$$F = \alpha T^{(d+z)/z} - ETb^{*-d} . \quad \text{(B17)}$$

Here α is a number of order the sum of D and the appropriate terms in (B12). Note that the second term is the classical free energy kT/ξ^d expected for a model with $\xi = b^*$. This term makes a negligible contribution to the thermodynamics except very near the true transition line.

APPENDIX C: MARGINAL CASE

In this appendix I give details of the solution of the scaling equations in the marginal case $d=2, z=2$.

1. Quantum regime $T<r$

a. Solution of (2.6)

Near the $T=0$ fixed point Eq. (2.6c) becomes

$$\frac{du}{d\ln b}=-4(n+8)f^{(4)}(T)u^2 . \tag{C1}$$

This equation may be solved, yielding

$$u(b)=\frac{u}{1+4(n+8)g(b)u} \tag{C2}$$

with

$$g(b)=\int_0^{\ln b} dx f^{(4)}(Te^{2\ln b}) . \tag{C3}$$

Combining (C3) and (A7c) and noting $K_2=1/2\pi$ gives

$$g(b)\cong\frac{\ln b}{2\pi^2} . \tag{C4}$$

Equation (2.6b) may now be solved as before, yielding

$$\delta(b)=e^{2\ln b}\left[\delta_0+\int_0^{\ln b} dx\, e^{-2x}(n+2)u(x)f^{(2)}(T)\right] . \tag{C5}$$

Expanding $f^{(2)}$ for small argument and integrating gives

$$\delta(b)=e^{z\ln b}r+\beta T^2 \tag{C6}$$

with

$$r=\delta_0+\int_0^{\infty} dx\frac{2(n+2)ue^{-2x}}{1+[2(n+8)/\pi^2]ux} . \tag{C7}$$

b. Evaluation of free energy

I follow the steps outlines in Sec. B 1 *b*. Because $d+z=2z$ a resonance occurs in (B5). The result is

$$F=\frac{2}{\pi^2}A_2T^2\ln 1/r . \tag{C8}$$

2. Classical regime $T>r$

a. Solution of Eq. (2.6c)

As in Appendix B 2 a, I break the scaling up into two regimes, $T<1$ and $T>1$. It is also convenient to expand $f^{(2)}(T)$ as

$$f^{(2)}(T)=\sum_{m=0}^{\infty} a_m T^{2m} . \tag{C9}$$

For $n\geq 1$, the integral in (C5) is dominated by its upper limit and is the exponential integral $E_n(x)=\int_1^x dx\, e^{nx}/x$. Approximating this by the first term in its large x expansion gives, at the scale $T=1$,

$$\bar{\delta}=[r/T+G/\ln 1/T] \tag{C10}$$

with

$$G=\frac{\pi^2(n+2)}{n+8}\sum_{m=1}^{\infty}\frac{a_m}{4m-2} . \tag{C11}$$

Similarly, from (C2),

$$\bar{u}=\frac{\pi^2}{2(n+8)}\frac{1}{\ln 1/T} . \tag{C12}$$

One may continue scaling for $T>1$ as before, if $\bar{u}$ is sufficiently small.

b. Evaluation of free energy

The treatment is essentially identical to that in Appendix B, Sec. 2 b. Because $d+z=2z$ a resonance occurs. Again the leading term comes from the fluctuations at the scale of kT. The result is

$$F=\frac{2A_2}{\pi^2}T^2\ln 1/T . \tag{C13}$$

The scale b^* is proportional to $\ln 1/T$ in the Gaussian regime where the present treatment is valid, so I neglect it.

APPENDIX D: LARGE-n EXPANSION, $d=2, z=2$

The large n limit of the model (2.2) is taken so that (un) remains finite. The diagrams generated by the renormalization procedure of Appendix A have one loop, which gives a factor of n from the spin sum. The terms which survive in the large n limit therefore are those with the minimum number of vertices (each of which gives a factor $1/n$). These are precisely the terms retained, for different reasons, in Appendix A. I therefore solve the large n limit of the equations already obtained. In the quantum regime $T<r$ the solution goes through as before—one may simply take the large n limit of the results of Appendix C, Sec. 1. However in the classical regime, the nonlinear terms in the scaling equation for $v=Tu$ are required. By combining (2.6c) and (A7c) I find

$$\frac{dv}{d\ln b}=2v-2v^2/v^* \tag{D1}$$

with the fixed point value v^* given (to leading order in $1/n$) by

$$v^*=\lim_{T\to\infty}\frac{f^{(4)}(T)}{2nT}=\pi^2/[(\ln 2+1/2+\pi/4)n] . \tag{D2}$$

Thus Eq. (D1) has a fixed point at $v=v^*\sim 1/n$. I now solve Eq. (2.6b) in the $T>1$ regime with v set equal to v^* and initial condition $\delta=\bar{\delta}$ given by Eq. (C10). The result is

$$\delta(b)=e^{2\ln b}[r/T+2nCv^*] . \tag{D3}$$

C was defined in Eq. (B10); in $d=2, C=1/2\pi^2$. Setting $\delta(b)=1$ and recalling the rescaling needed to get to the classical regime gives the result quoted in the text, with

$$K=\frac{1}{\ln 2+1/2+\pi/4} . \tag{D4}$$

APPENDIX E: COEFFICIENT OF ϕ^4 term

In this appendix I show that in a weak-coupling expansion the quantity u defined in Eq. (2.4) is finite in the limit as $T \to 0$ in two or three spatial dimensions, as long as the ordering wave vector is not an external spanning vector of the Fermi surface. The term "extremal spanning vector" will be defined more precisely below.

In a weak-coupling expansion for a magnetic instability at wave vector Q the quantity u is given by (ignoring numerical factors)

$$u = \int d^d k T \sum_{i\omega_n} \frac{1}{(i\omega_n - \varepsilon_k)^2 (i\omega_n - \varepsilon_{k+Q})^2} . \tag{E1}$$

Here $i\omega_n = (2n+1)\pi T$ is a fermion Matsubara frequency. Care is required in evaluating this integral because if it is possible to put both k and $(k+Q)$ on the Fermi surface then the denominator vanishes. At a nonzero temperature the necessarily nonzero value of $i\omega_n$ implies that $u < \infty$; however, as $T \to 0$ the Matsubara frequencies become a continuum, and at $T=0$ singularities may occur. To deal with the singularities I assume $T > 0$, do first the k integrals and then the ω sums, and then take the $T \to 0$ limit.

I turn now to the evaluation of (E1). It is convenient to write

$$\varepsilon_{k+Q} = \varepsilon_k + f(|k|, |Q|, \mu) \tag{E2}$$

where μ is the cosine of the angle between k and Q. The interesting case is when it is possible to put both k and $k+Q$ on the Fermi surface, i.e., when there is a k and μ^* with $-1 \le \mu^* \le 1$ for which $\varepsilon_k = f(\mu^*) = 0$. If there is no way to put both k and $k+Q$ on the Fermi surface (as in case "*B*" in Fig. 1) then the mode is underdamped. If the condition $f(\mu^*) = 0$ can be met, then three cases arise:

(a) If $\mu^* = 1$ then k is parallel to $k+Q$ and we are considering $Q = 0$. In this case, as discussed in Ref. 5, u is proportional to the second derivative of the density of states at the Fermi level and is in general nonzero (although it vanishes in the particular case of free fermions in $d = 2$).

(b) $-1 < \mu^* < 1$ and $\partial f / \partial \mu^* \neq 0$. In this case Q is not an extremal spanning vector of the Fermi surface. A specific example would be $Q < 2k_F$ for free fermions in 2 or 3 spatial dimensions. In this case one may evaluate the angular integral in (E1) and it is apparent that the region near $\mu = \mu^*$ does not make an appreciable contribution to the integral. One is left with

$$u = \int d\varepsilon_k N(\varepsilon_k) f(\varepsilon_k) T \sum_{i\omega_n} \frac{1}{(i\omega_n - \varepsilon_k)^2} . \tag{E3}$$

Here $f(\varepsilon_k)$ is the result of performing the angular integral. It has an analytic dependence on $i\omega_n$ which has been suppressed, and is a smooth function of ε_k near $\varepsilon_k = 0$. The integral over ε_k may now be done; again the result is not dominated by the region near $i\omega_n = 0$ and is a smooth function of $i\omega_n$. The sum over $i\omega_n$ may now be done; the result is obviously not infrared divergent.

(c) Either $\mu^* = -1$ or $df/d\mu^* = 0$. In this case Q is an extremal spanning vector of the Fermi surface because if $\varepsilon_k = 0$ then $\mathbf{k}+\mathbf{Q}$ is always on one side of the Fermi surface and the μ integral is cut off only by T. A simple example is $Q = 2k_F$ for free fermions in two or three dimensions, where $\mu^* = -1$. The $\mu^* = -1$ case corresponds to the $2k_F$ spin or charge-density wave which occurs often in practice. In the three-dimensional $\mu^* = -1$ case one finds after doing the angular integral

$$u = \int d\varepsilon_k N(\varepsilon_k) T \sum_{i\omega_n} \frac{1}{(i\omega_n - \varepsilon_k)^2 (i\omega_n + A\varepsilon_k)} . \tag{E4}$$

Here A is a number of order unity. The quantity in (E4) obviously diverges as $1/T$ for small T. In two spatial dimensions the factor $(i\omega_n + A\varepsilon_k)^{-1}$ becomes $(i\omega_n + A\varepsilon_k)^{-3/2}$ so the ultimate divergence is $(1/T)^{3/2}$. The divergence would be more severe in models with nesting. The difficulties arising in the nested case were discussed by Hertz.[5] The divergence would be cut off in a model with a finite electron mean free path at $T = 0$ (due, e.g., to impurity scattering). I note also that if Q is an extremal spanning vector of the Fermi surface then for $Q > 2k_F$ the mode is not damped, while for $Q < 2k_F$ the mode is damped. This would lead to a more complicated structure in the quadratic term $S^{(2)}$.

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