Threshold critical dynamics of driven interfaces in random media

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The dynamics of driven interfaces in random media are analyzed, focusing on the critical behavior near the depinning threshold. The roughening exponent in the critical region is shown to be independent of the type of disorder, in contrast to the equilibrium static behavior where there are two different universality classes corresponding to random-bond and random-field disorder. This critical dynamic roughening exponent is argued to be equal to its equilibrium static random-field value: $\zeta_c = \epsilon/3$ to *all* orders in ϵ in $5-\epsilon$ dimensions. All other critical exponents are obtained in terms of z, the dynamic exponent, which is calculated to $O(\epsilon)$ to be $z = 2 - 2\epsilon/9 + O(\epsilon^2)$. The results agree fairly well with recent numerical simulations. For random-field disorder, the same results have been obtained earlier by Nattermann *et al.* [J. Phys. (France) II **2**, 1483 (1992)] to $O(\epsilon)$. The results above threshold are used, together with scaling laws, to yield conjectures on the critical behavior as threshold is approached from below. In particular, the probability that the diameter of an "avalanche" exceeds l decays as $l^{-\kappa}$ just below threshold with $\kappa = d - 3 + \zeta$.

I. INTRODUCTION

Interfaces between two phases which reside in a disordered medium control many of their dynamic properties. Well-known examples are domain walls in random magnets[1–3] and interfaces between two fluids in porous media.[4–7] The phase separation in these systems is controlled by domain walls, which are driven by their surface tension, but impeded by having to jump thermally over impurity barriers. In this paper we will be concerned with a different regime, the dynamics of a single interface which separates the system into two halves and is driven by an external field which exerts a force per unit area F on the interface. With such driven interfaces, the effects of thermal fluctuations can often be ignored at long length scales. Nevertheless, when the driving force F is large, the interface has a large mean velocity with which it advances through the medium, causing the forces exerted by the impurities on the interface to fluctuate rapidly in time and space, roughly mimicking thermal noise. The behavior in this regime is expected to be well represented by the Kardar-Parisi-Zhang (KPZ) equation.[8] When, on the other hand, the driving force is small and the temperature low, the interface is pinned by the impurities in one of many metastable configurations, and exhibits hysteretic behavior. As the driving force is increased, in the absence of thermal fluctuations, there is a sharp threshold force F_T below which the interface is pinned and above which the interface moves forward. This depinning threshold may be treated as a dynamical phase transition and analyzed as a critical phenomenon.[9] Closely related threshold phenomena have been studied extensively in the context of sliding charge-density waves (CDWs)[9–13] and also control critical current behavior of flux lattices in superconductors pinned by impurities.[14] By analogy with these systems, we conjecture that the mean velocity v of the interface near threshold behaves as

$$v \sim (F/F_T - 1)^\beta , \quad (1)$$

defining the exponent β. When v is small, since the interface is almost pinned, it moves forward in a jerky, irregular manner. The dynamics are complicated, but can be characterized by diverging correlation lengths and times. In this paper, we shall focus on the critical dynamics in this regime just above threshold.

We model the dynamics of the interface with a purely relaxational equation of motion arising from a Hamiltonian that is a sum of two terms—a surface energy for the interface that is proportional to its area, and a random pinning potential from the random impurities in the medium.[15] In a d-dimensional medium, we represent the interface by its "height" $h(x,t)$, which is its position in the direction of the applied force F. This height is a function of the time t and of the $d-1$ component vector x representing the transverse directions. We shall assume that h is a single-valued function of x at any time t; thus we ignore overhangs in the interface and bubbles that could be pinched off from it, the effects of which we shall briefly return to later. The Hamiltonian of the interface is then

$$\mathcal{H} = \int d^{d-1}x \{ (\nabla h)^2 + V(h(x);x) \} , \quad (2)$$

where we have neglected higher-order gradient terms in an expansion of the surface energy. This leads to the deterministic equation of motion,

$$\partial_t h(x,t) = -\delta\mathcal{H}/\delta h = \nabla^2 h(x,t) + Y(h(x,t);x) + F , \quad (3)$$

where $Y(h;x) = -\partial_h V(h;x)$ and the extra term F arises from the external driving force. Here and henceforth we ignore thermal fluctuations. The explicit x dependence of V and Y reflects the quenched randomness due to the impurities in the medium. The random force $Y(h;x)$ is taken to have zero mean, and correlations of the form

$$\langle Y(h;x) Y(h';x') \rangle = \delta^{d-1}(x-x') \Delta(h-h') , \quad (4)$$

where the angular brackets $\langle\ \rangle$ represent an average over realizations of the randomness, and Δ falls off rapidly when its argument is large. An implicit cutoff in the δ

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function at scales of order the impurity separation or other microscopic scales, is assumed.

In general, when the interface is moving at a finite velocity v, there will be nonequilibrium terms in the equation of motion that cannot be expressed as derivatives of any Hamiltonian. Thus the true high-velocity behavior of the interface cannot always be obtained from an analysis of Eq. (3). In particular, a "KPZ term" $(\nabla h)^2$ will have to be added to Eq. (3) in the high-velocity regime.[8] However, for the critical dynamics, which we are interested in here, Eq. (3) should be adequate, as will be explained later. The crossover from the critical to the high-velocity behavior will also be discussed later in this paper.

It is important to distinguish between two different classes of disordered systems, for which the appropriate forms of $Y(h;x)$ are different. For a fluid invading a porous medium, the pinning energy is the sum of the contributions from all the impurities in the region invaded by the fluid. On the other hand, for systems like a domain wall in a ferromagnet with exchange disorder, where there is a symmetry relating the two sides of the interface, the pinning energy arises only from the impurities in the vicinity of the interface. The first case, which also occurs in certain magnetic systems, corresponds to random-field (RF) disorder,[1] and the second to random-bond (RB) disorder.[2,3] On long length scales,

$$\Delta(h-h')\sim -\partial_h^2\delta(h-h') \tag{5a}$$

for random-bond disorder, owing to the short-range correlated nature of V, while

$$\Delta(h-h')\sim\delta(h-h') \ , \tag{5b}$$

for random-field disorder. The static behavior in equilibrium, i.e., at a finite temperature and zero driving force, is known to depend strongly on whether the disorder is of the random-field or random-bond type. We shall show in this paper that, for the driven fluctuationless threshold phenomena of interest here, the critical behavior is essentially independent of the type of disorder, in striking contrast to the behavior in equilibrium.

In both the driven and equilibrium situations, interfaces in random media are not flat on long length scales, but rough, due to the effects of the random forces. The deviations from a flat ideal interface can be characterized by a roughness exponent ζ defined by[16]

$$\langle[h(x,t)-h(0,t)]^2\rangle\sim|x|^{2\zeta} \ . \tag{6}$$

The different phases and regimes of the system will be characterized by distinct exponents ζ: $\zeta_{\rm eq}$ in the equilibrium static phase, ζ_+ in the driven moving phase, ζ_c in the critical regime around the zero-temperature depinning transition at F_T, and ζ_- in a driven system below threshold. For the equilibrium static phase, in less than five dimensions, $\zeta_{\rm eq}^{\rm RF}$ is believed to be exactly equal to $(5-d)/3$ for random-field disorder;[1] for random-bond disorder, numerical solution of the renormalization-group fixed point equations in $d=5-\epsilon$ dimensions yields $\zeta_{\rm eq}^{\rm RB}\approx 0.2083\epsilon+O(\epsilon^{3/2})$,[2] with the exact result $\zeta_{\rm eq}^{\rm RB}(d=2)=2/3$.[3] Below threshold, at zero temperature, ζ_- will depend on the history of the system, and we will not discuss it in detail. Although experience with CDWs and other systems[11–13,17] suggests that some of the exponents at the depinning transition could be different on the two sides of the transition, ζ_c can be obtained from the interface fluctuations at threshold, and must therefore be the same above and below F_T. We use this result to make conjectures about the behavior just *below* threshold in the last section.

In the moving phase of the driven system it is necessary to add nonequilibrium terms to Eq. (3) to obtain ζ_+, as mentioned earlier.[8] At any finite distance above threshold, the roughness exponent crosses over from ζ_c on length scales shorter than the correlation length, to ζ_+ on long length scales. (In the unphysical case of more than three dimensions, there can be a dynamic phase transition at a critical force $F_D>F_T$ from one value of ζ_+ to another.)

For the case of random-field disorder, the critical behavior just above threshold has been obtained to $O(\epsilon)$ in $d=5-\epsilon$ by Nattermann *et al.*,[18] using a method developed by us for CDWs[13] to deal with short-distance singularities that affect a conventional low-frequency analysis. They show that the roughness exponent, $\zeta_c^{\rm RF}$, is equal to $\epsilon/3+O(\epsilon^2)$. In this paper, we analyze the critical behavior above threshold for random-bond disorder, and we show that $\zeta_c^{\rm RB}=\zeta_c^{\rm RF}$. We also argue that

$$\zeta_c^{\rm RB}=\zeta_c^{\rm RF}=(5-d)/3 \tag{7}$$

to *all* orders in $\epsilon=5-d$, so that to all orders in ϵ, $\zeta_c^{\rm RF}=\zeta_c^{\rm RB}=\zeta_{\rm eq}^{\rm RF}$. Thus $\zeta_c^{\rm RB}$ is different from $\zeta_{\rm eq}^{\rm RB}$ even to lowest order in ϵ, in contradiction to the claim by Nattermann *et al.*[18] that $\zeta_c^{\rm RB}=\zeta_{\rm eq}^{\rm RB}$ to $O(\epsilon)$. Indeed, we argue that the result Eq. (7) is generally a lower bound.

The behavior at threshold and just above will be the main focus of this paper. In addition to the above results, we will define a correlation length ξ as the characteristic length scale parallel to the interface, and relate the exponent ν with which it diverges at threshold to ζ_c by using a symmetry of Eq. (3). A characteristic time scale, $\tau\sim\xi^z$, will also be defined. All of the exponents will be related to ζ_c and z, so that with the result from Eq. (7) for ζ_c, z is the only unknown exponent; this we calculate to $O(\epsilon)$ in $d=5-\epsilon$, recovering the result which Nattermann *et al.*,[18] obtained for the random-field case.

An outline of the rest of this paper is as follows. In the remainder of Sec. I, we present a physical argument for the equivalence of random-field and random-bond disorder for the depinning transition. In Sec. II, the formalism for the ϵ expansion introduced by us for CDWs (Ref. 13) is reviewed, and the results derived for the interface dynamics near threshold for the random-field case. In Sec. III we return to random-bond disorder and general considerations. Section IV analyzes the long-distance behavior above threshold, while comparison with numerical simulations and future prospects are discussed in Sec. V. Finally, in the last section, we use scaling laws and analogies with CDWs to make some conjectures about the scaling behavior as F is increased to threshold from below.

We now attempt to motivate the equivalence between random-field and random-bond disorder in the moving phase from simple arguments. Random-bond disorder may be viewed as a special case of short-range correlated disorder, for which a "conservation law" forces the Fourier transform $\Delta(\mu)$ of the force-force correlations $\Delta(h)$ to be $O(\mu^2)$ at small μ, instead of the more general $O(1)$ behavior for random-field disorder. In the moving phase, the important quantity that affects the behavior is the force-force correlation as a function of *time,* rather than h. (In the perturbation expansion we will carry out, this appears as a related quantity, the mean-field correlation function of the interface height at different times.) In the critical regime, the interface tends to get stuck in regions where the force Y from the impurity potential is negative, while moving quickly through regions where Y is positive, the details of which do not play an essential role. Thus, the fact that $Y=-\partial_h V$, which results in Eq. (5a), is no longer seen, and behavior similar to the random-field case results.

This can be shown for a simple example, that of linear ratcheted impurity potentials, shown in Fig. 1: $V(h;x)$ is shown as a function of h for some particular value of x. The potential has sections of constant positive slope f_p, giving rise to a retarding force with randomly distributed sharp downward steps. The behavior for this potential is trivial above the threshold at $F_T=f_p$ since the ratchet constraint does not affect the dynamics. For random-bond disorder, the size of the downward steps is such that the impurity potential just after the step is always zero (or distributed in some way independently about zero). This results in $V(h+h')-V(h')$ remaining of order 1 for large $|h|$. Random-field disorder corresponds to the more general case, where the step sizes are randomly distributed, so that the potential difference $V(h+h')-V(h')$ between two points separated by a distance h has mean zero with $\sim\sqrt{|h|}$ random fluctuations. As the interface moves through this impurity potential, which it will do for $F>F_T$, it experiences a *constant* retarding force $-f_p$, with no effect from the steps. It is thus clear that the dynamics of the interface with such an impurity potential must be the same for random-field and random-bond disorder. (Note that in equilibrium with no driving force, the thermal distribution of the interface positions will be affected by the sizes of the steps in the potential, yielding different behavior for the two cases.)

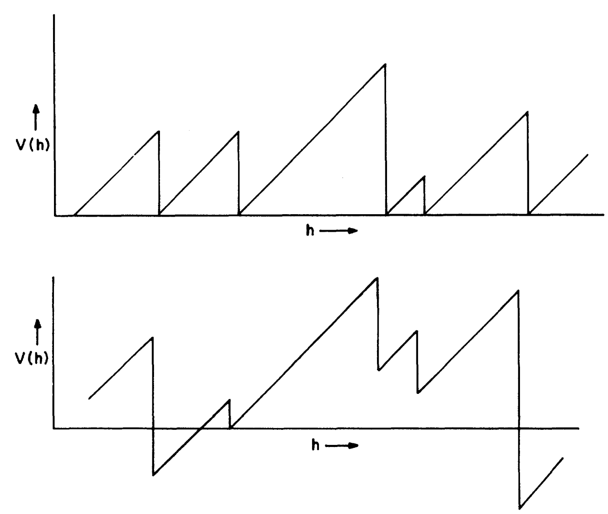

FIG. 1. Ratcheted pinning potential, with segments with constant slope interspersed with randomly located downward steps. In the lower figure, the sizes of the steps are random and uncorrelated with others, leading to $\sim\sqrt{h}$ random fluctuations in the height of the potential at long distances, corresponding to random-field disorder. The upper figure, with all the minima having the same potential, corresponds to random-bond disorder.

II. RANDOM-FIELD DISORDER AND ϵ EXPANSION

In this section we analyze the behavior for random-field disorder systematically, reviewing the results for random-field disorder obtained by Nattermann *et al.*[18] Our method and notation here is slightly different from theirs, being closely related to our earlier analysis for CDWs. All the critical exponents in this section agree with the previous $O(\epsilon)$ results of Nattermann *et al.*[18], but in addition, the roughness exponent ζ (and thereby, the correlation length exponent ν) are obtained here to all orders in ϵ.

Equation (3) can be analyzed using the formalism of Martin, Siggia, and Rose (MSR).[19] Introducing an auxiliary field $\hat{h}(x,t)$, we construct a generating functional Z, which can be written as

$$Z=\int[dh][d\hat{h}]\exp\left[\int d^{d-1}x\,dt\,i\hat{h}(x,t)\times[\partial_t h-\nabla^2 h-F-Y(h;x)]\right], \tag{8}$$

where an integration over $\hat{h}$ leaves us with a product of δ functions that imposes the solution of Eq. (3).

Just above threshold, when v is small, $h(x,t)$ changes almost adiabatically, staying close to some local minimum in the potential that arises from the elastic interactions within the interface and from the impurity pinning.[9] Occasionally, the local minimum in which a region of the interface is moving disappears; the interface then moves forward rapidly to another local minimum with an instantaneous velocity of $O(1)$. The qualitative nature of the dynamics thus consists of slow, smooth motion interspersed with sharp jumps whose duration on the scale of the background slow dynamics becomes smaller and smaller as $v\to 0$.

The generating functional in Eq. (8) can be expressed as an expansion about a mean-field solution, as done by Sompolinsky and Zippelius.[20] In mean-field theory, Eq. (2) is modified by making the local elastic interactions of the interface, represented by the $(\nabla h)^2$ term in the Hamiltonian, infinite ranged. Equation (3) is then modified to

$$\partial_t h=vt-h+Y(h;x)+F\ . \tag{9}$$

Here the mean velocity of the interface is v, and the mean field enters the local equation of motion as vt. The force F has to be adjusted so that the consistency condition $\langle h(x,t)\rangle=vt$ is satisfied at all times (or equivalently, v is adjusted at fixed $F > F_T$). Equation (9) may be viewed as arising from an effective potential experienced by the interface, given by $\frac{1}{2}(vt-h)^2+V(h;x)-hF$.

The nature of the mean-field solution, and therefore the expansion of Eq. (8) that it leads to, depends on the form chosen for the pinning potential $V(h)$. In particular, depending on whether the maxima in $V(h)$ are sharp, with linear cusps, or are smooth, the mean-field critical behavior is different.[9,11] This difference arises because, in mean-field theory for a smooth pinning potential, when a local minimum of the effective potential seen by the interface disappears, the interface accelerates for a long time (that diverges as $v \to 0$) before jumping forward with a velocity of $O(1)$. In contrast, for linear cusped potentials, the interface starts moving fast as soon as the minimum disappears, so that the divergent time scale associated with the acceleration time is absent; this results in $v \sim (F-F_T)$, so that $\beta_{MF}=1$. In any finite dimensionality d, the sharp jumps of different regions of the interface cause impulses on other regions that they are elastically coupled to; we call these "kicks." As d decreases, a region receives such kicks from fewer and fewer neighbors, leading to a very irregular local force. As for CDWs,[11] we expect that in sufficiently low dimensionality, the interface will typically be kicked over the top of its local effective potential, so that the acceleration time[9] present in the mean-field dynamics for smooth potentials is destroyed. The equivalence between the depinning for interfaces in smooth and cusped potentials with random-bond disorder has been verified numerically for $d=2$.[21] From the anlaysis for CDWs, we believe that this equivalence will be valid for all $d<5$, i.e., all physically relevant spatial dimensionalities. We thus carry out the expansion around the mean-field limit with linear cusped potentials.

After averaging over impurity configurations, an expansion around the mean-field solution for linear cusped potentials yields a generating functional from Eq. (8) that has the low-frequency form[13]

$$\overline{Z}=\int [dH][d\hat{H}]\exp\left[-\int dt\, d^{d-1}x\{[F-F_{MF}(v)]\hat{H}(x,t)+\hat{H}(x,t)(\partial_t-\nabla^2)H(x,t)\}\right.$$
$$\left.+\frac{1}{2}\int dt_1 dt_2 d^{d-1}x\, \hat{H}(x,t_1)\hat{H}(x,t_2)C[vt_1-vt_2+H(x,t_1)-H(x,t_2)]\right] . \quad (10)$$

Only the terms that are important in obtaining the critical behavior of the system have been retained in Eq. (10); various other irrelevant terms that do not affect the behavior at long length and time scales have been discarded. Here $H(x,t)$ is like a coarse-grained version of $h(x,t)-vt$, and $i\hat{H}(x,t)$ is like a coarse-grained version of the auxiliary field $\hat{h}(x,t)$.[22] In Eq. (10), we have chosen to shift the point around which the expansion is carried out away from the mean-field solution; the velocity v is the correct velocity at the force F, rather than the mean-field approximation to it. This results in the linear term in $\hat{H}$ in Eq. (10), where F is the force at which the velocity is v in the true short-range coupled system given by Eq. (3), while $F_{MF}(v)$ is the force in the mean-field approximation. The consistency condition, $\langle H\rangle=\langle h\rangle-vt=0$, is satisfied by adjusting $F-F_{MF}(v)$. The function C is the mean-field correlation function, obtainable (in principle) by solving Eq. (9) for $h_{MF}(t)$ as follows:

$$C(v\tau)=\langle [h_{MF}(t)-vt][h_{MF}(t+\tau)-vt-v\tau]\rangle , \quad (11)$$

where the average is over the random potential which yields independence of t by the statistical translational invariance. On long lengths scales, response and correlation functions of Eq. (3) are equal to the expectation values of various operators[13,20] evaluated using Eq. (10):

$$\frac{\partial}{\partial\varepsilon(x_{n+1},t_{n+1})}\cdots\frac{\partial}{\partial\varepsilon(x_{n+m},t_{n+m})}\langle h(x_1,t_1)\cdots h(x_n,t_n)\rangle_c$$
$$\approx\langle \hat{H}(x_{n+1},t_{n+1})\cdots\hat{H}(x_{n+m},t_{n+m})H(x_1,t_1)\cdots H(x_n,t_n)\rangle_c . \quad (12)$$

The left-hand side of the equation is the nonlinear response of a general (truncated) correlation function of h to a perturbing force $\varepsilon(x,t)$ added to Eq. (3), while the right-hand side is evaluated using Eq. (10). (For the case $m=0$, this yields correlation functions.)

As for CDWs,[13] the upper critical dimension can be obtained by considering the full form of the generating functional $\overline{Z}$, including all the terms not explicitly shown in Eq. (10), and requiring that under a rescaling to longer length and time scales all non-Gaussian terms should decay away. This yields the result $d_c=5$. At $d=5$, there is an infinite set of marginal operators, which can all be absorbed into a single marginal function that is the function $C(v\tau)$ in Eq. (10). We perform a renormalization-group (RG) analysis of the problem in $5-\epsilon$ dimensions, retaining only the Gaussian terms and the marginal function, which are shown in Eq. (10). In a perturbation expansion, the term $\hat{H}(\partial_t - \nabla^2)H$ is treated as the propagator, and the operator $\hat{H}\hat{H}C$ gives the vertices in the diagrams. [The first term in Eq. (10) merely serves to cancel all tadpole insertions in diagrams and can be ignored for most purposes.] The renormalization-group analysis is carried out by rescaling x, t, and the fields, together with an integration of high-momentum modes within an infinitesimal shell of the upper cutoff. (Modes of all frequencies within the momentum shell are integrated out at each stage.) We perform the scale change $x = bx'$, $t = b^z t'$, $\hat{H} = b^{\theta+1-d}\hat{H}'$, and $H = b^{\zeta}H'$, anticipating the scaling of the surface roughness as in Eq. (6), with $\zeta = \zeta_c$.

The important difference between interfaces and CDWs is that H is not a periodic phase variable, so that there is no analog to the $\phi \to \phi + 2\pi$ periodicity that was present there. This results in the nontrivial roughening exponent ζ, which is zero[13] for CDWs. However, by considering the scaling of different nonlinear response functions, it is still possible to show that vt and H must scale in the same way,[13] so that

$$v = b^{\zeta - z} v' \ . \tag{13}$$

The physical significance of ζ is that regions of size the correlation length ξ will undergo local jumps of magnitude $\delta h \sim \xi^{\zeta}$, which diverges at threshold for $\zeta > 0$. We define a correlation length exponent ν by $\xi \sim (F - F_T)^{-\nu}$; Eq. (13) then implies that the velocity exponent β is given by

$$\beta = (z - \zeta)\nu \ , \tag{14}$$

derived earlier in Ref. 18. With $b = e^l$, and infinitesimal l, the scaling of the $F - F_{\rm MF}(v)$ term yields

$$\frac{d}{dl}[F - F_{\rm MF}(v)] = (z + \theta)[F - F_{\rm MF}(v)] + \text{const} \ , \tag{15}$$

which implies

$$\nu = 1/(z + \theta) \ . \tag{16}$$

An exact relation between the exponents ζ and ν can be obtained from a symmetry of the equation of motion [Eq. (3)], an additional static force $\varepsilon(x)$ can be exactly compensated by the change $h(x,t) \to h(x,t) + \nabla^{-2}\varepsilon(x)$, if its spatial average is zero. Although this changes the forces Y from the impurities, the distribution of Y is not affected, so that after averaging over realizations of the randomness, the change in Y is inconsequential. The static response function in the moving phase is thus of the form

$$\chi(q, \omega = 0) \equiv \frac{\partial h(q; \omega = 0)}{\partial \varepsilon(q)} \sim \frac{1}{q^2} \ . \tag{17}$$

Since ε scales like a force, this implies that $\zeta + 1/\nu = 2$, so that

$$\nu = 1/(2 - \zeta) \ , \tag{18}$$

as obtained earlier in Ref. 18.

We have thus obtained all the exponents in terms of two unknown exponents, ζ and z. In order to obtain these we have to consider the renormalization of the function $C(v\tau)$. To one-loop order, the renormalization equation for C is very similar to that obtained in Ref. 13, and is given by

$$\frac{\partial C(v\tau)}{\partial l} = [\epsilon + 2\theta + 2(z-2)]C(v\tau) + \zeta v\tau C'(v\tau) - \frac{1}{8\pi^2}\{C'(v\tau)^2 + [C(v\tau) - C(0)]C''(v\tau)\} \ , \tag{19}$$

where the primes on C denote derivatives with respect to the argument $v\tau$, which scales as H. This equation is the same as in Ref. 13, except for the term proportional to ζ which arises from the rescaling of H. The boundary conditions that determine the fixed-point solution to this equation are, however, quite different; C is no longer a periodic function, but must instead decay away at infinity. We look for a fixed-point solution C^* to this equation, obtained by setting $\partial C/\partial l$ to zero and adjusting ζ.

With $C^*(u)$ decaying sufficiently rapidly for large values of its argument, we can obtain the correct $\zeta = \zeta_c^{\rm RF}$ directly from Eq. (19). Integrating from $-\infty$ to ∞ we obtain

$$[\epsilon + 2\theta + 2(z-2) - \zeta]\int_{-\infty}^{\infty} C^*(u)\,du = 0 \ .$$

Provided that $\int C^* \neq 0$, we thus obtain $\zeta = 2\theta + 2(z-2) + \epsilon$, whence, using Eqs. (16) and (18),

$$\zeta_c^{\rm RF} = \epsilon/3 \ , \tag{20}$$

which we shall later show is, in fact, true to all orders in ϵ. [To $O(\epsilon)$, this result has been obtained in Ref. 18.] With this value of ζ, an implicit fixed-point solution to Eq. (19) can also be obtained[2] as follows:

$$C^*(u) - C^*(0) - C^*(0)\ln[C^*(u)/C^*(0)] = 4\pi\epsilon u^2/3 \ , \tag{21}$$

with $C^*(0)$ arbitrary (corresponding to an overall rescaling of variables). This has a linear cusp at the origin: $C^*(u) - C^*(0) \sim |u|$ and decays as $\exp[-4\pi\epsilon u^2/3C^*(0)]$ for large u. The cusp in C^* at the origin is due to the jumps in sections of the interface[13] during which the instantaneous velocities become of $O(1)$, which is much greater than the $O(v)$ velocities with which $C(v\tau)$ is scaled. This cusp already occurs in mean-field theory for the random potential with linear upward cusps and is a general feature if there are jumps which constitute a finite fraction of the total motion of each segment of the interface.

The validity of the fixed-point solution [Eq. (21)] relies on several factors. First, $\int C$ must be nonzero. This is certainly true initially for random-field disorder and Eq. (19) implies that this is preserved by the RG flows. Second, $C(u)$ must fall off more rapidly than $1/u$. Examination of the RG flow equations shows that long-range power-law tails cannot be generated by the nonlinear terms and, if they are not present initially, as they will not be for short-range correlated random forces, they will not be generated. Note however that there exists a family of fixed-point solutions to Eq. (19) with different values of ζ, C_ζ^*, with $\int C_\zeta^* = 0$ and, for most ζ, power-law tails. These are not relevant for the present problem; indeed we shall argue in the next section that even for random-bond disorder, the same fixed point [Eq. (21)] is obtained.

Before proceeding further, it is instructive to contrast the behavior of the nonequilibrium threshold in the presence of a driving force to that for the equilibrium static behavior in the absence of the driving force. For the equilibrium static behavior, RG equations have been derived[2] to lowest order in $5-\epsilon$ dimensions for the potential-potential correlation function $R(h-h') \equiv \langle V(h;x)V(h';x')\rangle$. For random-field disorder this grows as $|h-h'|$ for large separation, and the fact that this behavior is preserved under renormalization, fixed the static equilibrium roughening exponent $\zeta_{\rm eq}^{\rm RF} = \epsilon/3$. Taking two derivatives of the RG flow equation for $R(h)$ yields an equation for $R''(h) = \Delta(h)$ which is identical to Eq. (19). We believe that this is essentially a coincidence which is likely to be valid only at $O(\epsilon)$. Note however, that even at $O(\epsilon)$ it is only the *form* and not the interpretation of the RG flow equation that is the same: for the static equilibrium case, the force must always be the derivative of a potential, a property that must be preserved under renormalization. For the driven nonequilibrium behavior, on the other hand, C is related to the force-force correlations essentially as a function of time, which will in general, not be expressible as the derivative of a potential (or at least not one with short-range correlations even if that were initially the case). This follows from the physical argument in Sec. I that the details of the force in regions through which the interface passes rapidly do not play a significant role.

This contrast would arise in a rather subtle way if we were to directly average the MSR equation [Eq. (8)] over randomness. This would yield a generating functional very similar to Eq. (10), but with the force-force correlations $\Delta[vt - vt' + \delta h(t) - \delta h(t')]$ [with $h(t) = vt + \delta h(t)$] replacing C in Eq. (10). This function should be interpreted here as the *temporal* force correlations. Since directly working with the MSR equation is equivalent to an expansion around a freely moving interface with no force from the impurities, the initial value of this function is the same as $\Delta(h)$; under renormalization, however, this is no longer true. We must thus distinguish between the force correlations as a function of h, $\Delta(h)$, and as a function of time, $\tilde{\Delta}(vt)$; the renormalized $\tilde{\Delta}$ need not be the second derivative of a well-behaved potential-potential correlation function. This difference will play an essential role in the random-bond case discussed in Sec. III.

We now return to the random-field case and argue that Eq. (20) is valid to all orders in ϵ. In order for this to be true, all loop contributions to Eq. (19) must be total derivatives, and thus vanish when integrated over $(-\infty, \infty)$. Although this is manifestly true for the first-order terms, which are shown explicitly in Eq. (19), it is necessary to prove this for the multiple-loop contributions, which could in general produce $O(\epsilon^2)$ or higher-order corrections to ζ_c.

The fact that all loop corrections to Eq. (14) are total derivatives is easiest to see in the frequency domain, where it is equivalent to the statement that loop corrections to the $\langle HH \rangle$ correlation function vanish at zero frequency. The analysis here is very similar to that of Appendix A in Ref. 13. Figure 2 shows a typical contribution to the correlation function. Each vertex in the diagram comes from an operator

$$\hat{H}(x,t_1)\hat{H}(x,t_2)C[vt_1 - vt_2 + H(x,t_1) - H(x,t_2)] .$$

There are thus two outgoing lines at each vertex, for the two $\hat{H}$ operators. A Taylor expansion of the function C in powers of H yields an arbitrary number of factors of H, each of which is represented by an incoming line at the vertex. The fact that there are two time indices asso-

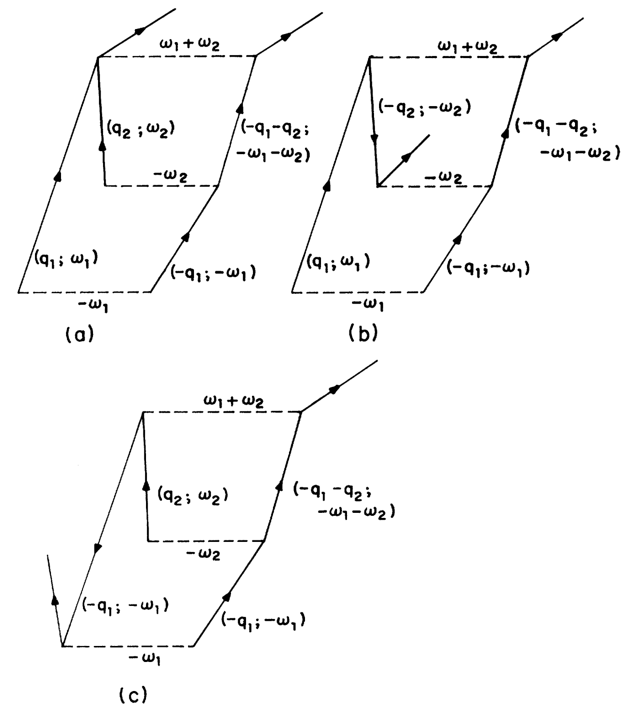

FIG. 2. Set of diagrams contributing to the renormalization of the correlation function at two-loop order. (b) and (c) are generated by moving one of the external lines in (a) over all the other half-vertices in the tree connected to it. The arrows indicate the direction of time. The external lines have zero frequency and momentum; the frequencies and momenta of the internal lines are shown in each diagram. A specific internal line can have different frequencies in the different diagrams, resulting in different factors of $(q^2 - i\omega)^{-1}$ from the propagators. (Although the signs of the various internal momenta are different in the different diagrams, a propagator depends only on the magnitude of the momentum flowing along it.) The dashed lines connect two half-vertices. The incoming lines at each vertex give rise to factors of the frequency flowing from one side to the other of the vertex on which the weight of the vertex depends. The product of these factors is $-\omega_2(\omega_1+\omega_2)^3$, $\omega_2^2(\omega_1+\omega_2)^2$, and $\omega_1\omega_2(\omega_1+\omega_2)^2$, respectively, for the three diagrams. These can be seen to add up to zero.

ciated with each vertex is represented by dividing it into two half-vertices, connected by a dashed line. If a frequency ω flows along a dashed line, the vertex gives rise to a factor of $C(\omega)=\int d\tau\, C(v\tau)\exp(i\omega\tau)$ in the diagram. Also, each incoming line at a vertex gives rise to a derivative of the function C, which in the frequency domain corresponds to a factor of the frequency flowing from one-half of the vertex to the other. (Since a vertex is local in space, the momentum flowing along a dashed line does not give rise to any such factors.) An outgoing line from a half-vertex is either contracted with an incoming line at some other half-vertex through a propagator, or is an external line of the diagram. Causality requires that in any contraction $\langle \hat{H}(x_1,t_1)H(x_2,t_2)\rangle$, t_1 should be less than t_2; this can be satisfied in any diagram provided that there are no closed loops leading from a half-vertex back to itself.

By proceeding backwards from either of the two external lines of the diagram, it is possible to obtain a tree of internal lines that are connected to it. By moving an external line successively over all the half-vertices of the tree it is connected to (see Fig. 2), we obtain a set of diagrams, none of which have closed loops (so that all of them satisfy causality), with the same loop momenta for each diagram. The lines in different diagrams have different frequencies, so that the factor from the propagator associated with each line $[(q^2-i\omega)^{-1}]$ is different in the different diagrams. Since $C(v\tau)$ has a factor of v in its argument, the frequencies of the propagators all scale with v, while the momenta are $O(1)$, so that this difference is inconsequential for the low-frequency renormalization of interest here. [This is equivalent to the procedure in Ref. 13 of replacing propagators by δ functions in time, which is necessary even to obtain a functional renormalization equation like Eq. (19)]. We are left with a set of diagrams whose only important difference is the number of incoming lines at each half-vertex. Since each incoming lines gives rise to a factor of the frequency flowing along the dashed line connecting the two halves of a vertex, the sum of the diagrams generated by moving an external line has a factor of the total frequency flowing into the tree connected to the (moving) external line. This is equal to the frequency flowing out along the external line, which we have taken to be zero, since we are interested in the zero-frequency corrections to the correlation function. We thus obtain the desired result, that all higher-order terms in Eq. (19) are total derivatives with respect to $v\tau$.

Note that this derivation requires the use of the $v\rightarrow 0$ limit, and is thus valid only for the critical behavior. In the moving phase, at a nonzero velocity, there could be $O(v)$ (or higher-order) corrections to this result. Also, we have assumed here that the singularity that is present at the origin of the fixed-point solution, C^*, does not affect the low-frequency form of C^* at any order; this is argued to be plausible for the CDW case in Ref. 13. Physically, this is because the cusp in C arises from the sharp jumps in h, which are $O(1)$, and are expected to remain so under renormalization (in the limit $v\rightarrow 0$), instead of becoming more singular or rounded out.

In order to calculate z we need, in addition to the fixed point C^*, the renormalization of the $\hat{H}(x,t)\partial_t H(x,t)$ term in Eq. (10). At one-loop order, this is renormalized by a contraction of the quadratic term in the expansion of $C[vt-vt'+H(t)-H(t')]$ in powers of H:[13]

$$[C''(vt_1-vt_2)\langle \hat{H}(t_2)H(t_1)\rangle(t_1-t_2)]\hat{H}(t_1)\partial_{t_1}H(t_1)\ . \tag{22}$$

In this contraction, since the contraction of $\hat{H}$ with H forces t_1-t_2 to be $O(1)$ or smaller, and we are interested in the $v\rightarrow 0$ limit, we can replace $C''(vt_1-vt_2)$ by $C''(0)$. However, the fixed-point solution C^* has a δ function at the origin in its second derivative, so that $C''(0)$ appears to be ill defined. In Ref. 13, where a periodic solution C^* was obtained to Eq. (19) (with $\zeta=0$), it was shown that C^* can be interpreted as the correlation function for a simple fixed-point distribution of potentials V; a detailed analysis of the exact correlation and response functions for these potentials showed that $C''(0)$ should be interpreted as $C''(0^+)$. Physically, this is because the jumps, which cause the cusp in C^*, only have effects *after* they occur. The appearance of $C''(0^+)$ is thus essentially due to causality. The only feature of the shape of the pinning potentials that is necessary for this result is that the maxima should be linear cusps. Although we have not obtained an explicit distribution of (nonperiodic) potentials V for the interface problem, the same linear cusped structure, which was shown to be generic in $4-\epsilon$ dimensions for CDWs, should be valid here, so that the singularity at the origin can be dealt with in the same way. Expanding $C^*(v\tau)$ as $c_0+c_1|v\tau|+c_2(v\tau)^2$ around the origin and substituting in Eq. (19), together with Eqs. (16) and (18), we obtain a correction to the $\hat{H}\partial_t H$ term from Eq. (22) equal to $[(\epsilon-\zeta)/3+O(\epsilon^2)]\hat{H}(t)\partial_t H(t)$. From the scaling dimension of the $\hat{H}\partial_t H$ operator, this yields

$$\theta+\zeta=(\epsilon-\zeta)/3+O(\epsilon^2)\ , \tag{23}$$

so that

$$z=2-(\epsilon-\zeta)/3+O(\epsilon^2)=2-2\epsilon/9+O(\epsilon^2)\ , \tag{24a}$$

and from Eqs. (14) and (18),

$$\beta=1-\epsilon/9+O(\epsilon^2)\ . \tag{24b}$$

Note that Eqs. (18) and (20) are exact to all orders in ϵ, whereas Eq. (24) is merely an $O(\epsilon)$ calculation, with, in general, higher-order corrections. The exponent identities derived here for ν and β [Eqs. (14) and (18)] have been obtained earlier by Nattermann *et al.*,[18] as have Eqs. (20) and (24) for ζ and z to $O(\epsilon)$. These authors did not, however, extend Eq. (20) past lowest order.

III. BOUNDS AND RANDOM-BOND DISORDER

We now consider general arguments that go beyond the ϵ expansion. It is possible to obtain Eq. (20) heuristically by using a scaling argument directly on the equation of motion, as done by Hentschel and Family.[23] For random-field disorder, $Y(h;x)$ has short-range correlations in h and x; if we average the random force over the characteristic length scales of the system, h_L perpendicular to the interface at scale L parallel to the interface, balancing the first and the second term on the right-hand side of Eq. (3) yields $h_L/L^2 \sim 1/(L^{d-1}h_L)^{1/2}$, from which we obtain $h_L \sim L^{(5-d)/3}$, which is equivalent to Eq. (20). Implicit in this argument is the assumption that the strength of the noise and the coefficient of the $\nabla^2 h$ term in Eq. (3) are not singularly renormalized. As argued above [Eq. (17)], the $\nabla^2 h$ term is unrenormalized, while the strength of the disorder, related to the function C, flows to a fixed-point value. This by itself is not sufficient. What is needed is that the scaling of C be like that of a trivial short-range random force correlation. An examination of the linear terms in Eq. (19) shows that, formally, $C(v\tau)\sim\delta(v\tau)$, corresponding to the trivial scaling of a δ function correlated force, is a fixed point of the linear zeroth-order RG flow equation if $\zeta=\epsilon/3$. The fact that this scaling is not changed by the nonlinear terms—which effectively act only to broaden out the δ function—is what is needed to make the simple power counting argument work. Note that similar arguments for the equilibrium static interface problem also yield the presumably exact $\zeta_{\rm eq}^{\rm RF}=\epsilon/3$, but for the random-bond case they yield [using $x\delta'''(x)=-3\delta''(x)$] $\zeta_{\rm eq}^{\rm RB}=\epsilon/5$, which is not correct. Thus considerable care must be used in such "Flory-type" scaling arguments, although they do often yield good approximations and in some cases exact results. In the present case, these simple arguments would yield $z=2$, which is not correct even at lowest order in ϵ due to the nontrivial renormalization of the $\partial_t h$ term obtained in the previous section.

Another route to understanding whether the result $\zeta_c=(5-d)/3$ is exact makes use of rigorous bounds on correlation lengths defined via finite-size scaling in disordered systems.[24] Specifically, we consider the width of the distribution of threshold fields, $\delta F_T(L)=F_T(L)-F_T(L=\infty)$ in systems of size $L^{\zeta}\times L^{d-1}$. From Ref. 24 we know that the width of the distribution is

$$\Delta F_T(L)\ge c/(\mathrm{Vol.})^{1/2}=cL^{-(\zeta+d-1)/2} .$$

We can define a finite-size scaling correlation length exponent with the correct choice ζ, by $\Delta F_T(L)\sim L^{-1/\nu_{\rm fs}}$, implying $\nu_{\rm fs}(\zeta+d-1)\ge 2$. Using Eq. (18), and assuming the conventional result that $\nu_{\rm fs}=\nu$, this yields

$$\zeta_c \ge \frac{5-d}{3} \tag{25a}$$

and

$$\nu \ge \frac{3}{d+1} . \tag{25b}$$

Although we believe that these bounds, which are saturated by the Flory-type scaling arguments mentioned above, are in fact correct, considerable caution is in order. For the case of CDWs, an analogous argument yields $\nu_{\rm fs}\ge 2/d$; this correctly describes the finite-size variations in threshold fields, but the dynamical behavior just above threshold is characterized by a correlation length exponent $\nu=1/2$: i.e., $\nu_{\rm fs}\ne\nu<2/d$.[11,12] Physically these two distinct exponents are associated with the fact that in the periodic steady state above threshold in CDWs, the system keeps feeling the same random potential, so that the velocity correlations "know" about the random potential at arbitrary distances away, causing the finite-size scaling (and also the scaling below threshold) to involve several different exponents. Formally, there is an extra operator (the uniform part of the force-force correlations), which is relevant at the critical RG fixed point C^*, causing the finite-size variations in F_T and yielding $\nu_{\rm FS}\ge 2/d$ distinct from ν.[13] This type of behavior cannot occur in the interface system since the interface continually moves into regions of "new" random potential; thus the dynamics must be determined more locally, suggesting that, indeed, $\nu=\nu_{\rm fs}$. This is supported by the absence of an extra operator in the RG equations in the interface case. We thus conjecture that there is a single correlation length exponent ν which will be obtained both above and below threshold.

Later in this section we will return to the behavior of dynamic correlations above threshold. But we are now brought naturally to the random-bond problem by the simple observation that the above argument is independent of the type of disorder provided it has *only* short-range correlations. (All that is needed is that the threshold can be crossed smoothly by increasing or decreasing the concentration of stronger or weaker pinning impurities.) We thus expect that for the random-bond case also, $\zeta_c^{\rm RB}\ge(5-d)/3$. This is in contrast to the static equilibrium result, $\zeta_{\rm eq}^{\rm RB}=0.2083\epsilon+O(\epsilon^{3/2})<\epsilon/3$ (Ref. 25).

Although the equilibrium force-force correlation function $\Delta(h)$ is governed by Eq. (19) to $O(\epsilon)$, it satisfies the condition $\int_{-\infty}^{\infty}dh\,\Delta(h)=0$, so that the result [Eq. (20)] is not applicable to $\zeta_{\rm eq}^{\rm RB}$. However, as we have discussed earlier, the interface moves much more quickly through regions in which the impurity potential tends to push it forward than those in which the potential has a retarding effect. The conservation law for random-bond disorder, that $Y(h)$ integrated over h for a large range must be of $O(1)$, is not correct when we consider the force as a function of time; thus $C(vt)$, which is closely related to the temporal force-force correlation function $\tilde{\Delta}$, satisfies $\int C\ne 0$, and Eq. (20) is valid for $\zeta_c^{\rm RB}$. The physical manifestation of this effect was seen in the simple example of the ratcheted potential discussed in Sec. I. We now analyze the behavior for a general form for the impurity potential.

For any impurity potential, when some segment of the interface reaches a local instability in the effective potential in which it moves, it jumps forward to a new position. In the $v\to 0$ limit, the criterion determining the new position of this segment is that the total force on the segment (including the pinning force and the elastic force from its neighbors) must be the same in the new position as in the old one just before the jump. Note that this condition is expressed in terms of the forces rather than the pinning potential, and is insensitive to the behavior of the pinning potential in the region across which the interface jumps quickly. In particular, it is possible to shift the pinning potential uniformly, starting from some point inside this region, by a random amount without affecting the dynamics in the limit $v \to 0$. This process can be repeated at each of the jumps. The new potential has $\sim\sqrt{x}$ fluctuations at large distances from the sum of the random shifts; this corresponds to random-field disorder, but has the same dynamics as the old random-bond potential. In terms of the forces, this transformation results in a destruction of the conservation law that requires the Fourier transform $\Delta(\mu)$ to be $O(\mu^2)$ at small μ while leaving the temporal correlation function $\tilde{\Delta}$ unchanged. Note that this mapping from one potential to another, while leaving the dynamics unchanged, is not valid for the equilibrium static system, where the probability of the interface being at some particular location depends on the potential at that point.

This collapse of random-bond to random-field behavior can be seen explicitly for a realization of disorder that is more realistic than the ratchet model with its trivial dynamics. We consider a random-bond pinning potential, which consists of successive parabolic segments joined together, forming linear cusps at their maxima [see Figure 3(a)]. The different parabolic segments are taken to have varying widths, but the slope at the cusps is taken to be unity. The change in slope across a cusp is thus always equal to 2. All the segments are taken to be symmetric around their minima, so that the potential is the same at the cusps joining the different segments. We can evaluate the mean-field correlation function for this potential explicitly. For a parabolic segment of the form $\frac{1}{2}mh^2$, the change in slope at the cusp can be seen from Eq. (9) to result in the interface jumping forward by an amount $\delta h = 2/(m+1)$ in the limit $v \to 0$. Subsequent to the jump, the interface moves smoothly forward, satisfying the condition $vt - h - mh = \text{const}$. Thus $h - vt$ decreases linearly as a function of vt, with a slope of $m/(m+1)$. This lasts for an interval of $v\delta t = 2/m$. Figure 3(b) shows $h - vt$ as a function of vt for the realization of disorder in Fig. 3(a). It is easy to verify that, with a sequence of such random linear segments in $h - vt$, for which the mean of $h - vt$ within any segment depends on m, the evolution of $\int dt[h - vt]$ will have $\sim\sqrt{vt}$ fluctuations, so that the integral of C is not zero and Eq. (20) is valid. (Note, however, that for high velocities, the integral of C will be small.)

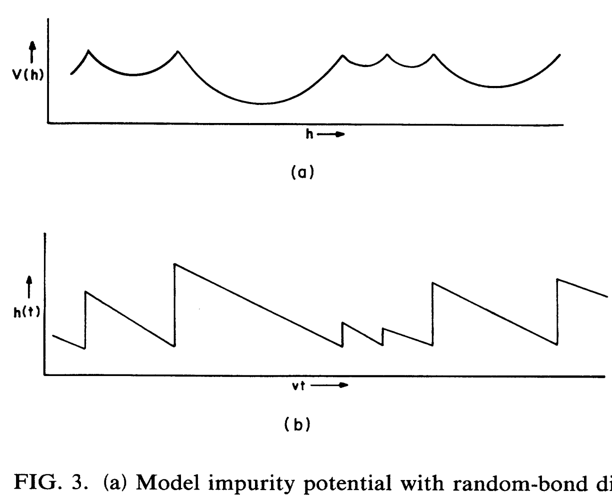

FIG. 3. (a) Model impurity potential with random-bond disorder, consisting of parabolic segments of different widths. The slopes at the cusps which join the different segments are all ± 1. (b) Plot of the mean field $h_{\text{MF}} - vt$ against vt for the potential shown in (a) in the limit of $v \to 0$. The area under a triangular wedge of width l is proportional to $l^2/(l+1)$; for a random distribution of l, this results in $\sim\sqrt{vt}$ fluctuations in $\int dt[h - vt]$. At nonzero v, the jumps will be rounded on a time scale of order 1, corresponding to $vt \sim v$.

We thus see that, with random-bond disorder, the mean-field correlation function that is the starting point for the renormalization flow of Eq. (19) already has the form characteristic of random-field disorder. This is in agreement with our arguments that, for dynamics, random-bond disorder is merely a special case of random pinning forces (most simply manifested as random-field disorder) with the same physical behavior.

As discussed earlier, it is possible to directly average the MSR generating functional, without expanding around the mean-field solution, in which case the renormalized force-force correlation function $\tilde{\Delta}$ plays a role very similar to C. In this method, $\tilde{\Delta}(vt)$ has a bare value (before renormalization) that is the same as $\Delta(h)$. Although we have argued that after renormalization, $\int\tilde{\Delta} \neq 0$, so that Eq. (20) is valid for ζ_c^{RB}, the mechanism by which this occurs is rather subtle, since formally all the loop terms in Eq. (19) are total derivatives for $v \to 0$, so that with $\zeta = \epsilon/3$, the integral of $\tilde{\Delta}$ should naively remain unchanged. However, the high-frequency singularities are not well controlled in this approach: unlike in an expansion around mean-field theory, where the different fields in an operator are time ordered in a manner that eliminates these singularities, a direct average of Eq. (8) results in the fields in the vertex operator occurring at the same time. Although the low-frequency loop terms in Eq. (19) are all total derivatives, and thus cannot change $\int\tilde{\Delta}$, these high-frequency singularities should renormalize $\int\tilde{\Delta}$ to a nonzero value, driving it towards the same fixed point as C. We have not, however, verified this directly.

The equivalence of random-bond and random-field disorder is a result of the motion of the interface, and should be true at any finite velocity v. In the $v \to \infty$ limit, the interface moves smoothly and h and vt are essentially equivalent. The force $Y(h;x)$ then appears as spatially and temporally varying noise, $Y(vt;x)$. The conservation law present in the case of random-bond disorder thus becomes effective in the infinite velocity limit. However, its effects should be destroyed in a perturbation expansion in powers of $1/v$; the calculation is fairly complicated, and we have not carried it out.

IV. CORRELATIONS AND CROSSOVERS ABOVE THRESHOLD

We now consider the scaling behavior of correlation functions in the critical regime, i.e., f small, x and t large. The linear response to an extra force applied to the interface scales as

$$\chi(q,\omega) \sim f^{-2\nu} X(q\xi, \omega\xi^z) \; . \tag{26a}$$

For static forces, we have $\chi(q,0)\sim 1/q^2$ for all (small) q, so that $X(u,0)\sim 1/u^2$. For q and ω both small, we have

$$\chi(q,\omega)\sim\sigma/(-i\omega+Dq^2)\ , \tag{26b}$$

so that we can define a dynamic exponent in the moving phase of $z_+=2$. Here $\sigma\sim f^{\beta-1}$ is the differential "conductivity" of the interface; the diffusion coefficient scales as $D\sim f^{\nu(z-2)}$ so that, since $z<2$, the behavior becomes superdiffusive near threshold. In the opposite limit $q\xi\gg 1$ and $\omega\xi^z\gg 1$, χ is independent of f, yielding

$$\chi(q,\omega)\sim q^{-2}\hat{X}(\omega/q^z)\ . \tag{26c}$$

The mean response to an instantaneous kick at one point thus falls off as $\langle\delta h(x,t)\rangle\sim x^{-(d-3+z)}Y(t/x^z)$.

At finite velocity the random forces act essentially as white noise on long length and time scales; thus we expect that the Fourier transform of the height-height correlations will behave as $1/(\omega^2+D^2q^4)$ for small q and ω. This implies that for $x\gg\xi$, the equal-time roughness scales as $\sim x^{3-d}+$const., i.e., $\zeta_+=0$ for $d>3$ and $\zeta_+=(3-d)/2$ for $d<3$. In general, the singular part of the correlations scales as

$$\langle[h(x,t)-h(0,0)]^2\rangle\sim x^{2\zeta_c}\rho(x/\xi,t/\xi^z)\ , \tag{27}$$

with $\rho(0,0)=$const. and $\rho(u,0)\sim u^{2(\zeta_+-\zeta_c)}$ for $u\to\infty$.

Near threshold the velocity-velocity correlations,

$$\Gamma(x,t)\equiv\left\langle\left[\frac{\partial h(x,t)}{\partial t}-v\right]\left[\frac{\partial h(0,0)}{\partial t}-v\right]\right\rangle$$

$$\sim f^{2\beta}G(x/\xi,t/\xi^z)\ , \tag{28a}$$

are also interesting. At equal time, $\Gamma(x,0)\sim\xi^{d+1-2z+2\zeta}/x^{d+1}$ for $x\gg\xi$. For $x\ll\xi$ and $t\ll\xi^z$, the behavior of $\Gamma(x,t)$ is determined by the properties of "avalanches"[11,12] which occur in the interface just above threshold (and as F is increased below threshold). We expect these avalanches to have linear spatial extent l parallel to the interface varying up to $O(\xi)$, with motion normal to the interface δh of order l^ζ, leading to the total moment of the avalanches scaling as $l^{d-1+\zeta}$. The instantaneous local velocities in this regime tend to be either $O(1)$ (during an avalanche) or very small. If, as we expect, most of the motion occurs in jumps where $\partial h/\partial t$ is $O(1)$, then we should have

$$\Gamma(x,t)\sim\frac{v}{x^{z-\zeta}}\gamma(t/x^z) \tag{28b}$$

on scales small compared to ξ, with the overall factor of v the probability that a given segment is moving at a given time. More understanding of the avalanche process is needed, however, to analyze whether this is, in fact, the correct behavior.

At long distances and low frequencies, outside of the scaling region, the behavior of the model Eqs. (2) and (3) should be correctly given by linear diffusion with white noise. This is not true, however, for more realistic models in $d\le 3$, and is due to a special symmetry of the model; if we consider the dynamics of a tilted interface with $h(x,t)=\eta x+\tilde{h}(x,t)$ with $\tilde{h}$ small, we can change variables to $\tilde{h}$ without changing the statistical properties of the potential, implying that correlations of the $\tilde{h}$ are the same as h with $\eta=0$ [this is a similar reason to that which yielded the simple form of the static response function Eq. (17)]. In more realistic models, we would expect that such a tilted interface would have a velocity normal to itself which would be the same as the velocity v_0 of the $\eta=0$ interface.[8] This implies a velocity of $v_0/\sqrt{1+|\eta|^2}$ in the h direction. Therefore, a better model might include a factor of $(1+|\nabla h|^2)^{-1/2}$ multiplying the left-hand side of Eq. (3), or, essentially equivalent, terms of the form $(\nabla h)^2\partial h/\partial t$ and higher-order terms previously not allowed by the extra symmetry. These are irrelevant near the critical fixed point in $5-\epsilon$ dimensions to $O(\epsilon)$, and we expect that as long as $\zeta_c<1$ (so that $\nabla h\ll 1$ on long length scales), they will be irrelevant in all $d>2$ (although they can renormalize the velocity by finite amounts).[26] In the moving phase, on the other hand, these terms have the form $v(\nabla h)^2$ on shifting $h\to h-vt$ and such a term is known to be relevant for $d\le 3$ as studied by KPZ.[8] At sufficiently long scales, for $x\gg\xi_{\rm KPZ}$, this term will generally change both ζ_+ and z_+ which becomes equal to $2-\zeta_+$.[27] In two dimensions, exact results yield $\zeta_+=\frac{1}{2}$ (unchanged from the simple diffusion result) but $z_+=\frac{3}{2}$ (Ref. 8) while in three dimensions numerical estimates[28] yield $\zeta_+\approx 0.3$ and $z_+\approx 1.7$.

We thus expect that for the physical dimensions $d=2$ and 3, the behavior just above threshold will be correctly given by our earlier analysis, since the neglected terms are irrelevant. However, the $\partial h/\partial t(\nabla h)^2$ term is dangerously irrelevant, and for long enough scales $x>\xi_{\rm KPZ}\gg\xi$ the behavior will cross over to that of the KPZ equation. In two dimensions, we expect $\xi_{\rm KPZ}\sim\xi^\delta$ with $\delta\ge 1$, while in three dimensions, $\xi_{\rm KPZ}$ presumably grows as an exponential of ξ, since the KPZ nonlinearity is marginally relevant in the moving phase.[8]

For the unphysical case of $d>3$ the asymptotic behavior near threshold will not be affected by the extra nonlinear terms, [8] and $\zeta_+=0$, but for strong pinning there may be a second, dynamic transition at a higher field $F_D>F_T$ above which the behavior changes sharply to that with $\zeta_+>0$. This scenario might well occur even for $d>5$, for which the critical behavior at the threshold F_T is trivial.

V. COMPARISON WITH NUMERICS, EXPERIMENTS, AND OTHER WORK

We now turn to a comparison with experiments and numerical simulations. Much of this work has been done in two dimensions for which, substituting Eq. (20), we predict a roughness exponent of $\zeta_c=1$. In modeling the interface by Eq. (3), we have neglected the effects of overhangs in the interface, which can lead to bubbles being pinched off and left behind as the fluid moves forward. For $\zeta<1$, these will be exponentially suppressed on long length scales, so that this "solid on solid" approximation is valid, and the interface has a well-defined orientation on long length scales. For $\zeta=1$, however, overhangs will be marginal; a more careful analytical or numerical treatment is then needed to ascertain their role, which may well be to destroy the interface on large scales.

Numerical simulations directly on Eq. (3), with random-bond disorder,[29] have yielded critical exponents $\zeta_c^{RB}=0.97\pm0.05$ and $\nu=1.05\pm0.1$ for $d=2$, in agreement with our predictions of $\zeta_c=\nu=1$. Note that these results are inconsistent with ζ_c^{RB} being equal to the static roughness exponent ζ_{eq}^{RB}, which for $d=2$ is exactly equal to 2/3. The velocity exponent β is found to be small ($\beta\approx0.3$); the data are also well fit by a form $v\sim1/|\ln(F/F_T-1)|$. A naive extrapolation of the $O(\epsilon)$ value of β from Eqs. (14), (18), (20), and (24) to $d=2$ yields $\beta=\frac{1}{3}$; higher-order corrections may well reduce this further.

We now briefly explore the possibility that the dynamic exponent $z=1$ in two dimensions, which would result in $\beta=0$, necessitating the analysis of possible logarithmic behavior. With $\zeta=1$, avalanches of size ξ extend a distance $\delta h\sim\xi$ normal to the interface. If these avalanches progress to completion in one stage without the disturbance needing to propagate back and forth, this would suggest $z=1$. To analyze the resulting behavior of the velocity, we need to go back to the RG equation.

The RG equations derived so far have been in the $v\to0$ limit; at any finite v, there are corrections, which are likely to be regular in v. For instance, the loop terms in Eq. (19) are total derivatives only for $v\to0$. Since this was required to obtain $\zeta_c=\epsilon/3$, there will be corrections to the scaling of the H field that are regular in v. The same is true for the scaling of time, for which the dynamic exponent z also has regular corrections. Since the H field scales as vt, we expect, from these sources, an effective RG equation for $v(l)$ of the form

$$\frac{dv}{dl}=\beta v+O(v^{n+1}) , \tag{29}$$

where n is some positive integer. We can take the sign of the second term to be positive on physical grounds. For $\beta=0$, integrating Eq. (29) out to $v(l^*)=v^*$ yields $v\sim1/(l^*+\text{const.})^n$, which implies an asymptotic form

$$v\sim1/|\ln(F/F_T-1)|^{1/n} . \tag{30}$$

In practice, even when $\beta\neq0$, but is small, the $O(v^{n+1})$ term in Eq. (29) will result in almost logarithmic behavior over a fairly large range of v. The actual power of the logarithm depends on the value of n. One might naively expect $n=2$, due to the invariance required under the change $v\to-v$, but since we are dealing with singular functions here, this has to be analyzed carefully; preliminary calculations indicate that, instead, $n=1$, yielding $v\sim1/\ln f$. We note that temporal correlations in $h(x,t)$ will be free of logarithmic corrections, and can therefore be used to obtain z (and thereby β by scaling).

Experiments[5] and numerical simulations[6] of a preferentially wetting fluid in a two-dimensional porous medium yield an apparent roughness exponent which increases with decreasing velocity, apparently approaching a value of $\zeta_c\approx0.8$, somewhat less than the predicted value. Whether this is due to the relatively small range of sizes or is a result of new physics in the fluid invasion problem associated with the motion of the fluid (the force is not exerted uniformly on the interface) we leave for future inquiry. It is also possible that this discrepancy is the result of the breakdown of our solid on solid assumption, although one would then expect ζ_c to be greater than 1 (since for $\zeta_c<1$ our model should be valid), in disagreement with the experimental results. It is possible that the inclusion of overhangs, which are marginal in the solid on solid approximation, only leads to logarithmic corrections to the predicted $\zeta_c=1$; such logarithmic factors could easily lead to apparent roughening exponents significantly less than 1 over the range of f covered by the experimental and numerical work.

In three dimensions, numerical simulations of interface motion in ferromagnets with random-field disorder,[4] which should be in the same universality class as fluid invasion, yield the critical roughness exponent $\zeta_c=0.67\pm0.03$, in good agreement with Eq. (20). The correlation length exponent ν has also been obtained below threshold: $\nu=0.75\pm0.05$. This is in agreement with the prediction from Eqs. (18) and (20): $\nu=3/4$. As discussed earlier, the possibility that there are several different ν's cannot be ruled out although, in contrast to CDWs, there does not seem to be any reason for this to be the case here.

If the random forces from the impurities are much stronger than the elastic forces that tend to keep the interface smooth, the interface breaks up.[6] The motion of the interface forward at any point just depends on the local impurity forces. The depinning transition occurs when the regions with weak impurity forces, where the interface does not get pinned, percolate through an infinite system. Elastic theories of the type we have considered here are no longer applicable.[30] The crossover to percolationlike behavior is characterized by a lack of any definite orientation of the interface at long length scales ($\zeta=1$). Recent experiments on the capillary absorption of fluids into two- and three-dimensional porous media,[7] although giving rise to interfaces that are smooth on long length scales, have been analyzed in terms of a percolationlike theory in which the fluid experiences no resistance to passing through a region in the direction opposite to the overall flow. In two dimensions, the depinning transition was argued to be a directed percolation transition, with the interface being pinned when strong impurities that stop its motion span the entire system from left to right without backsteps (since overhangs of the interface are not allowed in the model). Since the transverse wandering of directed percolation clusters is much less than their length, this leads to the interface having a definite orientation, as in our case, even though elastic theories are not applicable. Similar considerations, with the directed percolation of two-dimensional surfaces, were used to explain the results in three dimensions.

Finally, we compare our results with another recent theoretical prediction for elastic interface motion. Parisi[31] has conjectured that ζ_c/z is exactly equal to $\epsilon/4$ for random-field disorder. By comparing with our expression for ζ_c/z obtained from Eqs. (20) and (24), we see that this is not correct even to $O(\epsilon)$. However, a truncation of Eq. (24) at $O(\epsilon)$ yields results for ζ_c/z fairly close to $\epsilon/4$ in $d=2$ and 3, perhaps accounting for the agreement with Parisi's numerical results[31] which use a variant of Eq. (3).

In this paper, we have analyzed a model for the critical dynamics of random interfaces just above the depinning transition. We have obtained the critical exponents to $O(\epsilon)$ in $5-\epsilon$ dimensions, and reduced by scaling laws all of the exponents to two (ζ_c and z) in general d. We have shown that the critical behavior is the same for random-bond and random-field disorder, in contrast to the static equilibrium behavior. We have also argued that the roughness exponent ζ_c should be equal to $\epsilon/3$ to all orders in ϵ (with possible nonperturbative corrections, which must be non-negative as $\epsilon/3$ is a lower bound). Although the result $\zeta_c=\epsilon/3$ agrees with numerical simulations in two and three dimensions, a rigorous proof that ζ_c is not modified by high-frequency singularities is lacking. In two dimensions, numerical simulations on models close to ours yield good agreement with our results. But those on more realistic models as well as experiments seem to yield results somewhat different from the ones we have obtained here, although the causes of these discrepancies are not clear. For the case of random-field disorder, all the exponent identities obtained in this paper, as well as the $O(\epsilon)$ calculation of the remaining unknown exponents ζ_c and z, agree with the earlier results of Ref. 18. In addition, we have argued that the result $\zeta_c=\epsilon/3$ should be correct to all orders in ϵ. Also, we have seen that random-bond and random-field disorder belong to the same universality class, which disagrees with Ref. 18.

VI. AVALANCHES AND SCALING BELOW THRESHOLD

So far in this paper we have focused on the behavior of driven interfaces above threshold. In this final section we make use of some of our resutls to speculate about the critical behavior as the threshold is approached from below by gradually increasing the driving force. The analysis closely parallels that carried out elsewhere for charge-density waves,[11,32] although we expect the interface case to be *simpler* due to the absence of the dangerously irrelevant operator at threshold which occurs for CDWs and complicates matters considerably.[13,32]

As the force is increased towards threshold local instabilities occur resulting in "avalanches" with a distribution of sizes. The evolution of the spatially averaged interface position as F increases is dominated by these discontinuous local jumps which have amplitudes which are the "moment," $\int \Delta h_{\mathrm{aval}}$, of the corresponding avalanches. Although the behavior is irreversible below threshold, it has been argued for CDWs[11,32] that as F_T is approached by increasing F *monotonically* the behavior is universal in the sense of critical exponents and scaling functions. The distribution of avalanches sizes is conjectured to obey a scaling form near threshold.

$$\text{Prob(diameter of avalanche} > l) \approx \frac{1}{l^{\kappa}}\hat{p}(l/\xi_-) \tag{31}$$

with $\xi_- \sim (F_T-F)^{-\nu_-}$ a correlation length and the scaling function $\hat{p}$ decaying rapidly for $l \gg \xi_-$. For discrete models of the interface, the mean number of avalanches triggered by a small increase in F is not singular as $F \nearrow F_T$ and we conjecture that this is true generally.[33] Since the typical Δh's of interface motion of sections of linear size l scale as l^{ζ} just above threshold, we expect this also to be the case just below threshold for $l<\xi_-$. Thus the moment of an avalanche should scale as $l^{d-l+\zeta}$.[34] The mean polarizability on increasing F

$$\chi_{\uparrow} = \frac{d\langle h \rangle}{dF_{\uparrow}} \sim (F_T-F)^{-\gamma} \tag{32}$$

is then obtained from the avalanche distribution and scaling of the moments, yielding

$$\gamma=(d-1+\zeta-\kappa)\nu_- \; . \tag{33}$$

Right at threshold, the roughness of the interface as a function of position scales as l^{ζ}. We therefore expect that the total change in $\langle h \rangle$ of a section of size l^{d-1} as F is increased from 0 to F_T will also scale as l^{ζ}. By scaling, we then have $\chi_{\uparrow} \sim \xi_-^{\zeta}/(F_T-F)$ yielding $\gamma=1+\zeta\nu_-$ and hence $\kappa=d-1-1/\nu_-$.

Finally, we can use the result from the scaling of χ above threshold that $\chi \sim (\text{length})^2$. This arises from the statistical rotational symmetry of the interface which can also be used directly for finite-sized regions below threshold to argue that $\chi_{\uparrow} \sim \xi_-^2$, and therefore $\gamma=2\nu_-$ yielding

$$\nu_- = \frac{1}{2-\zeta} = \nu \; , \tag{34}$$

i.e., the correlation length scales in the same way above and below threshold. (This is *not* the case for CDWs due to the dangerously irrelevant operators.)[11,13,32]

From Eq. (34) and the earlier results, we obtain

$$\kappa=d-3+\zeta \; . \tag{35}$$

Thus the calculations presented in this paper above threshold combined with scaling laws yield the prediction

$$\kappa=2-\tfrac{2}{3}\epsilon=\tfrac{2}{3}(d-2) \tag{36}$$

to all orders in $\epsilon=5-d$ and probably exact. Note that for $d=2$, this yields $\kappa=0$ so that large avalanches of size $\xi \times \xi$ are likely and the description breaks down, as consistent with the expectation from the result $\zeta(d=2)=1$ as discussed earlier.

To our knowledge, Eq. (36) is the first analytical renormalization-group result for the distribution of "avalanches" in a nontrivial nonequilibrium model. It relies, however, on assumptions about scaling laws relating quantities above and below threshold. Further analysis along the lines of Narayan and Middleton's[32] recent work on CDWs may well be able to provide analytic justification of the results.

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[26]An $O(\epsilon)$ calculation at the critical fixed point yields the scaling of this operator as $2\zeta_c - z - 2\epsilon/9 + O(\epsilon^2) = 2\zeta_c - 2 + O(\epsilon^2)$. However, an exact calculation for general d is still lacking, so that it is possible that this term becomes relevant before d is decreased to 2, i.e., when ζ_c is still less than 1. There should also be a term of the form $\nabla^2 h(\nabla h)^2$ on the right-hand side of Eq. (3); in the absence of singular corrections to the RG equations this is not renormalized by loop corrections, and thus scales with dimensions $2\zeta_c - 2$ for all d.

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[34]Naively, a similar argument for CDWs would imply that the moment of an avalanche should scale with its diameter with an exponent $2+d/2$, which does not agree with the numerical results even for automaton models. We believe this is likely to be because of complications caused by the dangerously irrelevant operator present there, although the mechanism for this is not clear.