Generalized elasticity theory of quasicrystals

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The classical theory of elasticity describing three- and lower-dimensional systems is generalized to higher-dimensional spaces. The elastic properties of quasicrystals can be derived from this theory, appropriately. The practical application is given to pentagonal, octagonal, dodecagonal, and icosahedral quasicrystals. The explicit form is obtained for all elastic equations including Hooke's law, equilibrium equation, etc., in all the cases mentioned above.

I. INTRODUCTION

Since the first quasicrystal structure was observed in 1984,[1] great progress has been made in the study of the elastic properties of quasicrystals.[2–7]

According to the Landau theory, the mass density $\rho(\mathbf{r})$ for a d-dimensional quasicrystal can be expressed in terms of a Fourier series,

$$\rho(\mathbf{r})=\sum_{\mathbf{G}\in L_R}\rho_{\mathbf{G}}e^{i\mathbf{G}\cdot\mathbf{r}}=\sum_{\mathbf{G}\in L_R}|\rho_{\mathbf{G}}|e^{-i\Phi_{\mathbf{G}}+i\mathbf{G}\cdot\mathbf{r}}\ , \tag{1}$$

where $|\rho_{\mathbf{G}}|$ and $\Phi_{\mathbf{G}}$ are the mass-density-wave amplitude and phase associated with reciprocal lattice L_R. There exists a set of N basis vectors, $\{\mathbf{G}_n\}$, so that each $\mathbf{G}\in L_R$ can be written as $\Sigma m_n\mathbf{G}_n$ for integers m_n. Moreover, $N=kd$, where k is the number of the mutually incommensurate vectors in the d-dimensional quasicrystal. Generally $k=2$. N phases of the mass density waves describe the elastic behaviors of quasicrystals. A convenient parametrization of these phases is given by

$$\Phi_n=\mathbf{G}_n\cdot\mathbf{u}+\mathbf{G}_n^{\perp}\cdot\mathbf{w}\ . \tag{2}$$

A number of papers give a clear explanation of all quantities on the right-hand side of Eq. (2).[2–7] Here we would like to present a brief review only of the variables **u** and **w**. First, the displacement **u** is analogous to phonon field in crystals. The elementary excitation associated with the phonon mode is propagating. The gradient of **u** describes the change in the shape and volume of the unit cell. Second, the variable **w** is called a phason. The elementary excitation associated with the phason mode is diffusive. The gradient of **w** describes local rearrangements of the unit cells. Finally, the elastic energy density F consists of three terms: the term due to the phonon field F^u, the term due to the phason field F^w, and the possible phonon-phason mixing term F^{uw}, i.e.,

$$F=F^u+F^w+F^{uw}=\tfrac{1}{2}C_{ijkl}E_{ij}E_{kl}+\tfrac{1}{2}K_{ijkl}w_{ij}w_{kl}+R_{ijkl}E_{ij}w_{kl}\ , \tag{3}$$

where

$$E_{ij}=\tfrac{1}{2}(u_{ij}+u_{ji})\ ,\quad u_{ij}\equiv\frac{\partial u_i}{\partial x_j}\ ,\quad w_{ij}\equiv\frac{\partial w_i}{\partial x_j}\ , \tag{4}$$

which are referred to as the phonon and phason strain fields, respectively.

All independent elastic constants and elastic energy have been already given for pentagonal and icosahedral quasicrystals,[3] and for planar quasicrystals with eight-, ten-, and twelvefold symmetries[8] with the group representation theory. In particular, by minimizing the elastic energy De and Pelcovits[9] have derived the following partial differential equations:

$$\begin{aligned}
&\mu\nabla^2u_1+(\lambda+\mu)\frac{\partial}{\partial x_1}\nabla\cdot\mathbf{u}+R\left[\frac{\partial^2w_1}{\partial x_1^2}+2\frac{\partial^2w_2}{\partial x_1\partial x_2}-\frac{\partial^2w_1}{\partial x_2^2}\right]+f_1=0\ ,\\
&\mu\nabla^2u_2+(\lambda+\mu)\frac{\partial}{\partial x_2}\nabla\cdot\mathbf{u}+R\left[\frac{\partial^2w_2}{\partial x_1^2}-2\frac{\partial^2w_1}{\partial x_1\partial x_2}-\frac{\partial^2w_2}{\partial x_2^2}\right]+f_2=0\ ,\\
&K_1\nabla^2w_1+R\left[\frac{\partial^2u_1}{\partial x_1^2}-2\frac{\partial^2u_2}{\partial x_1\partial x_2}-\frac{\partial^2u_1}{\partial x_2^2}\right]+g_1=0\ ,\\
&K_1\nabla^2w_2+R\left[\frac{\partial^2u_2}{\partial x_1^2}+2\frac{\partial^2u_1}{\partial x_1\partial x_2}-\frac{\partial^2u_2}{\partial x_2^2}\right]+g_2=0\ ,
\end{aligned} \tag{5}$$

satisfied by **u** and **w** for planar quasicrystals. Moreover, using Green's functions the authors have solved Eq. (5) and obtained the displacement field corresponding to a dislocation in a planar pentagonal quasicrystal.

The purpose of this paper is to generalize the three-dimensional (3D) elasticity theory to higher-dimensional cases and to establish a general theory of linear elasticity applicable to all quasicrystals. The following section will explain the main contents of this theory including strain and stress tensors, equilibrium equation, generalized

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Hooke's law, elastic energy, and so on. The practical application to quasicrystals is given in Sec. III.

II. GENERALIZED ELASTICITY THEORY

A. Phonon strain field E_{ij} and phason strain field $\partial_j W_i$

According to the cut-and-projection method for the generation of a quasilattice, a 3D quasilattice can be obtained by selected projection of the respective 6D periodical lattice.[10–12] The 6D space E^6 can be decomposed into mutually orthogonal subspaces, $E_\parallel^3$ and $E_\perp^3$:

$$E^6=E_\parallel^3\oplus E_\perp^3 \ . \tag{6}$$

$E_\parallel^3$ and $E_\perp^3$ are called the physical and perpendicular space, respectively. If $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}_4,\mathbf{e}_5,\mathbf{e}_6\}$ are basis vectors in $E_\parallel^3$ and $E_\perp^3$, respectively, two Cartesian coordinate systems are chosen in $E_\parallel^3$ and $E_\perp^3$ so that the coordinate axes are along the above basis vectors.

Supposing R^6 to be a 6D hypercube, by the cut-and-projection method the projection of all lattice points within some strip embedded in R^6 upon $E_\parallel^3$ gives a quasilattice, which is considered as a continuous medium in the present theory. Meanwhile all points obtained by the projection of the same strip upon $E_\perp^3$ lie on some range, which is also considered as a continuous medium.

Suppose $\tilde{\mathbf{u}}$ to be a 6D displacement vector in E^6, which is projected upon $E_\parallel^3$ and $E_\perp^3$, and becomes

$$\tilde{\mathbf{u}}=\mathbf{u}\oplus\mathbf{w} \ ,$$
$$\mathbf{u}=(u_1,u_2,u_3) \ , \tag{7}$$
$$\mathbf{w}=(w_1,w_2,w_3) \ ,$$

where u_i and w_i $(i=1,2,3)$ are the components of $\mathbf{u}$ and $\mathbf{w}$ in $E_\parallel^3$ and $E_\perp^3$, respectively.

The displacement field $\tilde{\mathbf{u}}$ must fulfill an important requirement in order to be consistent with the basic property of quasicrystals of being structurally independent on the actual location of the physical space $E_\parallel^3$ in the 6D space E^6 (local isomorph classes): the displacement field $\tilde{\mathbf{u}}$ must be invariant by any translation along the perpendicular space $E_\perp^3$, i.e., the displacement field $\tilde{\mathbf{u}}=\mathbf{u}\oplus\mathbf{w}$ in the 6D space E^6 is a function of the physical space position vector $\mathbf{r}_\parallel=\mathbf{r}_\parallel(x_1,x_2,x_3)$ only:[13]

$$\tilde{\mathbf{u}}(\mathbf{r}_\parallel)=\mathbf{u}(\mathbf{r}_\parallel)\oplus\mathbf{w}(\mathbf{r}_\parallel) \ . \tag{8}$$

Let $\nabla=\mathbf{e}_i\nabla_i$ with $\nabla_i=\partial/\partial x_i=\partial_i$ be a differential operator relative to the position vector $\mathbf{r}_\parallel(x_1,x_2,x_3)$, then

$$\nabla\tilde{\mathbf{u}}=\nabla\mathbf{u}\oplus\nabla\mathbf{w} \ , \tag{9}$$

where $\nabla\mathbf{u}$ and $\nabla\mathbf{w}$ are transformed as 3×3 tensors of rank 2, respectively, when the basis of coordinate system is transformed.

Janssen[14] divided symmetry operations for quasiperiodic structure in $E_\parallel^3$ into two types For the operations of the first type, $E_\perp^3$ contains full rational irreducible subspaces V_μ and the corresponding m_μ also occurs in the physical space $E_\parallel^3$. In this case $E_\perp^3$ and $E_\parallel^3$ have the same irreducible representations. The cubic quasiperiodic structure as discussed by Wang *et al.*[15] and by Yang *et al.*[16] belongs to this case. In this case, both $\nabla\mathbf{u}$ and $\nabla\mathbf{w}$ can be decomposed into symmetrical and antisymmetrical components:

$$\nabla\mathbf{u}=(\nabla\mathbf{u})^S+(\nabla\mathbf{u})^A \ , \tag{10}$$

$$\nabla\mathbf{w}=(\nabla\mathbf{w})^S+(\nabla\mathbf{w})^A \ , \tag{11}$$

$$(\nabla\mathbf{u})_{ij}^S=\tfrac{1}{2}(\partial_j u_i+\partial_i u_j)=E_{ij} \ ,$$
$$(\nabla\mathbf{u})_{ij}^A=\tfrac{1}{2}(\partial_j u_i-\partial_i u_j) \ , \tag{12}$$

$$(\nabla\mathbf{w})^S=\tfrac{1}{2}(\partial_j w_i+\partial_i w_j)=F_{ij} \ ,$$
$$(\nabla\mathbf{w})^A=\tfrac{1}{2}(\partial_j w_i-\partial_i w_j) \ . \tag{13}$$

The symmetrical components describe the change in the shape and volume of the unit cell while the antisymmetrical components describe rigid rotations that do not change the elastic energy.

For the operations of the second type, an invariant real irreducible 2D subspace of the rational irreducible space V_μ belongs to $E_\parallel^3$, whereas the remaining real irreducible subspaces of V_μ belong to $E_\perp^3$. In this case the phason displacement $\mathbf{w}$ must be transformed according to an irreducible representation different from that for the phonon displacement $\mathbf{u}$, position vector $\mathbf{r}_\parallel$ and differential operator ∇. Therefore, although $\nabla\mathbf{u}$ can still be decomposed into symmetrical and antisymmetrical components, $\nabla\mathbf{w}$ cannot. Hence all the components of the gradient $\partial_j w_i$ contribute to the elastic energy.

Finally, it is easy to verify the compatibility equation:

$$-e_{ijk}e_{lmn}\partial_j\partial_m E_{kn}=0 \ ,$$
$$-e_{ijk}e_{lmn}\partial_j\partial_m\partial_n w_k=0 \ , \tag{14}$$

where e_{ijk} is the alternator symbol.

B. Stress tensors T_{ij}, H_{ij} and equilibrium equation

The phason displacement field $\mathbf{w}$ in the perpendicular space $E_\perp^3$ describes local rearrangement of units cells of the quasicrystals. When the unit cells rearrange, the movement of atoms through barriers needs some forces. Therefore, besides the conventional body force density $\mathbf{f}$ and surface force density $\mathbf{t}$ in the conventional elastic theory, we must introduce a generalized body force density $\mathbf{g}$ and surface force density $\mathbf{h}$ in the elasticity theory of quasicrystals, which will be along the directions in the perpendicular space $E_\perp^3$. Similarly, in addition to the conventional 3×3 stress tensor $\mathbf{T}$ of rank 2, there is another 3×3 stress tensor $\mathbf{H}$ or rank 2, where H_{ij} describes the stress components along the x_i direction in the perpendicular space $E_\perp^3$ applied on the surface orthogonal to the x_j direction in the physical space $E_\parallel^3$.

If $\mathbf{n}$ is a unit vector normal to an element surface as outwards then we have

$$\mathbf{t}=\mathbf{T}\cdot\mathbf{n} \tag{15}$$

and

$$\mathbf{h}=\mathbf{H}\cdot\mathbf{n} \ . \tag{16}$$

Now we extend the Newton's law of motion to the case of quasicrystals. The theorem of momentum possesses the form

$$\frac{d}{dt}\int_V \rho(\dot{\mathbf{u}}\oplus\dot{\mathbf{w}})dV=\int_V(\mathbf{f}\oplus\mathbf{g})dV+\int_S(\mathbf{t}\oplus\mathbf{h})dS \ , \tag{17}$$

where V is an arbitrary volume in $E_{\parallel}^3$ and S is the boundary surface of V. Equation (17) may be decomposed into two equations along $E_{\parallel}^3$

$$\frac{d}{dt}\int_V \rho\dot{\mathbf{u}}\,dV=\int_V \mathbf{f}\,dV+\int_S \mathbf{t}dS \tag{18}$$

and along $E_{\perp}^3$

$$\frac{d}{dt}\int_V \rho\dot{\mathbf{w}}\,dV=\int_V \mathbf{g}\,dV+\int_S \mathbf{h}\,dS \ . \tag{19}$$

Applying Gauss theorem, Eq. (18) yields the equation of motion

$$\partial_j T_{ij}+f_i=\rho\ddot{u}_i \tag{20}$$

and the static equilibrium equation

$$\partial_j T_{ij}+f_i=0 \tag{21}$$

in physical space. Similarly, Eq. (19) yields the equation of motion

$$\partial_j H_{ij}+g_i=\rho\ddot{w}_i \tag{22}$$

and the static equilibrium equation

$$\partial_j H_{ij}+g_i=0 \tag{23}$$

in the perpendicular space.

The theorem of angular momentum for quasicrystals has the form

$$\frac{d}{dt}\int_V \mathbf{r}_{\parallel}\times\rho(\dot{\mathbf{u}}\oplus\dot{\mathbf{w}})dV=\mathbf{M} \ , \tag{24}$$

where

$$\mathbf{M}=\int_V \mathbf{r}_{\parallel}\times(\mathbf{f}\oplus\mathbf{g})dV+\int_S \mathbf{r}_{\parallel}\times(\mathbf{t}\oplus\mathbf{h})dS \tag{25}$$

expresses the torque applied on the body with a volume V and $\mathbf{r}_{\parallel}$ is a position vector in $E_{\parallel}^3$. Because $\mathbf{r}_{\parallel}$ is orthogonal to any vector in $E_{\perp}^3$, we have $\mathbf{r}_{\parallel}\times\mathbf{w}=0$, $\mathbf{r}_{\parallel}\times\mathbf{g}=0$, and $\mathbf{r}_{\parallel}\times\mathbf{h}=0$. Hence Eq. (24) is reduced to the conventional form

$$\frac{d}{dt}\int_V \mathbf{r}_{\parallel}\times\rho\dot{\mathbf{u}}\,dV=\mathbf{M} \ , \tag{26}$$

with

$$\mathbf{M}=\int_V \mathbf{r}_{\parallel}\times\mathbf{f}\,dV+\int_S \mathbf{r}_{\parallel}\times\mathbf{t}\,dS \ . \tag{27}$$

By using Gauss theorem and Eq. (20), we obtain the theorem of stress mutual equivalence from Eq. (26), i.e., the local form of the angular momentum theorem as follows:

$$T_{ij}=T_{ji} \ . \tag{28}$$

Since in $E_{\perp}^3$ there is no similar equation as Eq. (26), it is not possible to obtain the relationship $H_{ij}=H_{ji}$.

C. Elastic energy density and elastic constant

The elastic energy of quasicrystals is a function of the phonon strain field E_{ij} and the phason strain field $\partial_j w_i$.[2–8] The elastic energy density F can be expanded in terms of the Tayler series in the vicinity of $E_{ij}=0$ and $\partial_j w_i=0$ to the second order

$$\begin{aligned}F(E_{mn},\partial_n w_m)&=\frac{1}{2}\left[\frac{\partial^2 F}{\partial E_{ij}\partial E_{kl}}\right]_0 E_{ij}E_{kl}+\frac{1}{2}\left[\frac{\partial^2 F}{\partial(\partial_j w_i)\partial(\partial_l w_k)}\right]_0 \partial_j w_i \partial_l w_k\\&\quad+\frac{1}{2}\left[\frac{\partial^2 F}{\partial E_{ij}\partial(\partial_l w_k)}\right]_0 E_{ij}\partial_l w_k+\frac{1}{2}\left[\frac{\partial^2 F}{\partial(\partial_j w_i)\partial E_{kl}}\right]_0 \partial_j w_i E_{kl}\\&=\tfrac{1}{2}C_{ijkl}E_{ij}E_{kl}+\tfrac{1}{2}K_{ijkl}\partial_j w_i\partial_l w_k+\tfrac{1}{2}R_{ijkl}E_{ij}\partial_l w_k+\tfrac{1}{2}R'_{ijkl}\partial_j w_i E_{kl} \ .\end{aligned} \tag{29}$$

In Eq. (29)

$$C_{ijkl}=\frac{1}{2}\left[\frac{\partial^2 F}{\partial E_{ij}\partial E_{kl}}\right]_0 \tag{30}$$

are quadratic elastic constants in the classical elasticity theory with

$$C_{ijkl}=C_{klij}=C_{jikl}=C_{ijlk} \ . \tag{31}$$

We can denote all C_{ijkl} by a 9×9 symmetric matrix $[C]$;

$$K_{ijkl}=\left[\frac{\partial^2 F}{\partial(\partial_j W_i)\partial(\partial_l W_k)}\right]_0 \tag{32}$$

are elastic constants of the phason field in $E_{\perp}^3$ with

$$K_{ijkl}=K_{klij} \ , \tag{33}$$

which can be also denoted by a 9×9 symmetric matrix $[K]$;

$$R_{ijkl}=\left[\frac{\partial^2 F}{\partial E_{ij}\partial(\partial_l w_k)}\right]_0 ,$$

$$R'_{ijkl}=\left[\frac{\partial^2 F}{\partial(\partial_j w_i)\partial E_{kl}}\right]_0 \quad (34)$$

are the elastic constants associated with the phonon-phason coupling. Obviously,

$$R_{ijkl}=R_{jikl} , \quad R'_{ijkl}=R'_{ijlk} , \quad R'_{klij}=R_{ijkl} , \quad (35)$$

but

$$R_{ijkl}\neq R_{klij} , \quad R'_{ijkl}\neq R'_{klij} . \quad (36)$$

We denote them by 9×9 matrices $[R]$ and $[R']$, respectively, and

$$[R']=[R]^T . \quad (37)$$

Four matrices $[C]$, $[K]$, $[R]$, and $[R']$ compose a 18×18 matrix $[C \;\; K \;\; R]$:

$$[C \;\; K \;\; R]=\begin{bmatrix}[C] & [R]\\ [R'] & [K]\end{bmatrix}=\begin{bmatrix}[C] & [R]\\ [R]^T & [K]\end{bmatrix} . \quad (38)$$

If the row matrix

$$[E \;\; w]=(E_{11},E_{22},E_{33},E_{23},E_{31},E_{12},E_{32},E_{13} , E_{21},\partial_1 w_1,\partial_2 w_2,\partial_3 w_3,\partial_3 w_2,\partial_1 w_3,\partial_2 w_1, \partial_2 w_3,\partial_3 w_1,\partial_1 w_2) , \quad (39)$$

then the elastic energy density F can be written as

$$F=\frac{1}{2}[E \;\; w]\begin{bmatrix}[C] & [R]\\ [R]^T & [K]\end{bmatrix}\begin{bmatrix}E\\ w\end{bmatrix} , \quad (40)$$

which coincides with Eq. (3).

D. Generalized Hooke's law

By an argument like that given in the classical elasticity theory we obtain

$$T_{mn}=\frac{\partial F}{\partial E_{mn}} , \quad H_{mn}=\frac{\partial F}{\partial(\partial_n w_m)} . \quad (41)$$

Substituting Eq. (3) into the above equation gives generalized Hooke's law

$$T_{ij}=C_{ijkl}E_{kl}+R_{ijkl}\partial_l w_k ,$$
$$H_{ij}=K_{ijkl}\partial_l w_k+R_{klij}E_{kl} , \quad (42)$$

which can be expressed in the matrix form

$$\begin{bmatrix}T\\ H\end{bmatrix}=\begin{bmatrix}[C] & [R]\\ [R]^T & [K]\end{bmatrix}\begin{bmatrix}E\\ w\end{bmatrix} , \quad (43)$$

where the order of T_{ij},H_{ij} in $[{}^T_H]$ is the same one as the order of E_{ij},w_{ij} in $[{}^E_w]$.

E. Equilibrium equation

Substituting Eq. (42) into Eqs. (20)–(23) gives equilibrium equations

$$C_{ijkl}\partial_j\partial_l u_k+R_{ijkl}\partial_j\partial_l w_k+f_i=0(=\rho\ddot{u}_i) ,$$
$$K_{ijkl}\partial_j\partial_l w_k+R_{klij}\partial_j\partial_l u_k+g_i=0(=\rho\ddot{w}_i) , \quad (44)$$

which are nonhomogeneous partial differential equations satisfied by **u** and **w**. Using the boundary condition and solving the above equations, we will obtain **u** and **w** in different cases; for example, the elastic displacement field of dislocation and declination.

III. APPLICATION TO QUASICRYSTALS

A. Planar quasicrystal with fivefold symmetry

With group theory all quadratic invariants and independent elastic constants have been derived for the planar quasicrystal with fivefold symmetry.[3,7]

For the phonon field C_{ijkl} have the same expression as isotropic media

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \quad (i,j,k,l=1,2) . \quad (45)$$

For the phason field quadratic invariants are

$$(\partial_1 w_1+\partial_2 w_2)^2+(\partial_1 w_2-\partial_2 w_1)^2 ,$$
$$(\partial_1 w_2+\partial_2 w_1)^2+(\partial_1 w_1-\partial_2 w_2)^2 . \quad (46)$$

It follows that

$$K_{1111}=K_{2222}=K_{2121}=K_{1212}=K_1 ,$$
$$K_{1122}=K_{2211}=-K_{2112}=-K_{1221}=K_2 , \quad (47)$$

other $K_{ijkl}=0$. Equation (47) can be written as

$$K_{ijkl}=K_1\delta_{ik}\delta_{jl}+K_2(\delta_{ij}\delta_{kl}-\delta_{il}\delta_{jk}) . \quad (48)$$

There is an invariant

$$(E_{11}-E_{22})(\partial_1 w_1+\partial_2 w_2)+2E_{12}(\partial_1 w_2-\partial_2 w_1) \quad (49)$$

associated with the phonon-phason mixing term. It follows that

$$R_{1111}=R_{1122}=-R_{2211}=-R_{2222}=R_{1221}=R_{2121} =-R_{1212}=-R_{2112}\equiv R \quad (50)$$

other $R_{ijkl}=0$. Equation (50) can be written as

$$R_{ijkl}=R(\delta_{i1}-\delta_{i2})(\delta_{ij}\delta_{kl}-\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) . \quad (51)$$

The matrix $[C \;\; K \;\; R]$ is

$$[C\ \ K\ \ R]=\begin{bmatrix} \lambda+2\mu & \lambda & 0 & 0 & R & R & 0 & 0 \\ \lambda & \lambda+2\mu & 0 & 0 & -R & -R & 0 & 0 \\ 0 & 0 & \mu & \mu & 0 & 0 & -R & R \\ 0 & 0 & \mu & \mu & 0 & 0 & -R & R \\ R & -R & 0 & 0 & K_1 & K_2 & 0 & 0 \\ R & -R & 0 & 0 & K_2 & K_1 & 0 & 0 \\ 0 & 0 & -R & -R & 0 & 0 & K_1 & -K_2 \\ 0 & 0 & R & R & 0 & 0 & -K_2 & k_1 \end{bmatrix} . \tag{52}$$

Substituting Eq. (52) into Eq. (43) gives the generalized Hooke's law with the form

$$\begin{aligned} T_{11}&=\lambda(E_{11}+E_{22})+2\mu E_{11}+R(\partial_1 w_1+\partial_2 w_2)\ , \\ T_{22}&=\lambda(E_{11}+E_{22})+2\mu E_{22}-R(\partial_1 w_1+\partial_2 w_2)\ , \\ T_{12}&=2\mu E_{12}+R(\partial_1 w_2-\partial_2 w_1)=T_{21}\ , \\ H_{11}&=R(E_{11}-E_{22})+K_1\partial_1 w_1+K_2\partial_2 w_2\ , \\ H_{22}&=R(E_{11}-E_{22})+K_1\partial_2 w_2+K_2\partial_1 w_1\ , \\ H_{12}&=-2RE_{12}+K_1\partial_2 w_1-K_2\partial_1 w_2\ , \\ H_{21}&=2RE_{12}+K_1\partial_1 w_2-K_2\partial_2 w_1\ . \end{aligned} \tag{53}$$

Substituting Eq. (52) into Eq. (40) gives the expression for the elastic energy density, which coincides with that of Ref. 3. Substituting Eqs. (45), (47), and (51) into Eq. (44) gives the partial differential equation, which coincides with Eq. (5) obtained in Ref. 9.

B. Planar quasicrystal with eightfold symmetry

According to the study by Socolar on planar quasicrystal with eightfold symmetry,[8] we have the following results: the phonon field is isotropic; C_{ijkl} take the same form as Eq. (45); the phason field is anisotropic. Its quadratic invariants are

$$\begin{aligned} &(\partial_1 w_1-\partial_2 w_2)^2\ ,\quad (\partial_2 w_1+\partial_1 w_2)^2\ , \\ &(\partial_1 w_1+\partial_2 w_2)^2+(\partial_1 w_2-\partial_2 w_1)^2\ . \end{aligned} \tag{54}$$

In comparison with the standard form $K_{ijkl}\partial_j w_i \partial_l w_k/2$, we have

$$\begin{aligned} &K_{1111}=K_{2222}=K_1\ ,\quad K_{1122}=K_{2211}=K_2\ , \\ &K_{1221}=K_{2112}=K_3\ ,\quad K_{2121}=K_{1212}=K_1+K_2+K_3\ , \end{aligned} \tag{55}$$

other $K_{ijkl}=0$. Equation (55) can be written as

$$\begin{aligned} K_{ijkl}=&(K_1-K_2-K_3)\delta_{ik}\delta_{jl}+K_2\delta_{ij}\delta_{kl}+K_3\delta_{il}\delta_{jk} \\ &+2(K_2+K_3)(\delta_{i1}\delta_{j2}\delta_{k1}\delta_{l2}+\delta_{i2}\delta_{j1}\delta_{k2}\delta_{l1})\ . \end{aligned} \tag{56}$$

There is an invariant

$$(E_{11}-E_{22})(\partial_1 w_1+\partial_2 w_2)+2E_{12}(\partial_1 w_2-\partial_2 w_1) \tag{57}$$

coupling **u** and **w**. It follows that all R_{ijkl} have the same form as Eq. (51). The matrix $[C\ \ K\ \ R]$ is

$$[C\ \ K\ \ R]=\begin{bmatrix} \lambda+2\mu & \lambda & 0 & 0 & R & R & 0 & 0 \\ \lambda & \lambda+2\mu & 0 & 0 & -R & -R & 0 & 0 \\ 0 & 0 & \mu & \mu & 0 & 0 & -R & R \\ 0 & 0 & \mu & \mu & 0 & 0 & -R & R \\ R & -R & 0 & 0 & K_1 & K_2 & 0 & 0 \\ R & -R & 0 & 0 & K_2 & K_1 & 0 & 0 \\ 0 & 0 & -R & -R & 0 & 0 & K_1+K_2+K_3 & K_3 \\ 0 & 0 & R & R & 0 & 0 & K_3 & K_1+K_2+K_3 \end{bmatrix} . \tag{58}$$

By Eq. (40) the elastic energy density F is

$$\begin{aligned} F=&\tfrac{1}{2}\lambda(\nabla\cdot\mathbf{u})^2+\mu E_{ij}E_{ij}+\tfrac{1}{2}K_1\partial_j w_i\partial_j w_i+\tfrac{1}{2}K_2[(\partial_2 w_1)^2+(\partial_1 w_2)^2+2\,\partial_1 w_1\partial_2 w_2]+\tfrac{1}{2}K_3(\partial_2 w_1+\partial_1 w_2)^2 \\ &+R[(E_{11}-E_{22})(\partial_1 w_1+\partial_2 w_2)+2E_{12}(\partial_1 w_2-\partial_2 w_1)]\ . \end{aligned} \tag{59}$$

Using Eqs. (42) or (43), we have the generalized Hooke's law as follows:

$$T_{11}=\lambda(E_{11}+E_{22})+2\mu E_{11}+R(\partial_1 w_1+\partial_2 w_2)\ ,\quad T_{22}=\lambda(E_{11}+E_{22})+2\mu E_{22}-R(\partial_1 w_1+\partial_2 w_2)\ ,$$
$$T_{12}=2\mu E_{12}+R(\partial_1 w_2-\partial_2 w_1)=T_{21}\ ,$$
$$H_{11}=R(E_{11}-E_{22})+K_1\partial_1 w_1+K_2\partial_2 w_2\ ,\quad H_{22}=R(E_{11}-E_{22})+K_1\partial_2 w_2+K_2\partial_1 w_1\ , \tag{60}$$
$$H_{12}=-2RE_{12}+(K_1+K_2+K_3)\partial_2 w_1+K_3\partial_1 w_2\ ,\quad H_{21}=2RE_{12}+(K_1+K_2+K_3)\partial_1 w_2+K_3\partial_2 w_1\ .$$

u and **w** satisfy the dynamical equation

$$\mu\nabla^2 u_1+(\lambda+\mu)\partial_1(\nabla\cdot\mathbf{u})+R(\partial_1\partial_1 w_1+2\,\partial_1\partial_2 w_2-\partial_2\partial_2 w_1)+f_1=0\ ,$$
$$\mu\nabla^2 u_2+(\lambda+\mu)\partial_2(\nabla\cdot\mathbf{u})+R(\partial_1\partial_1 w_2-2\partial_1\partial_2 w_1-\partial_2\partial_2 w_2)+f_2=0\ ,$$
$$K_1\nabla^2 w_1+(K_2+K_3)(\partial_2\partial_2 w_1+\partial_1\partial_2 w_2)+R(\partial_1\partial_1 u_1-2\partial_1\partial_2 u_2-\partial_2\partial_2 u_1)+g_1=0\ , \tag{61}$$
$$K_1\nabla^2 w_2+(K_2+K_3)(\partial_1\partial_2 w_1+\partial_1\partial_1 w_2)+R(\partial_1\partial_1 u_2+2\partial_1\partial_2 u_1-\partial_2\partial_2 u_2)+g_2=0\ .$$

C. Planar quasicrystal with twelvefold symmetry

By group theory, planar quasicrystal with twelvefold symmetry has the same elastic properties as planar quasicrystal with eightfold symmetry both for the phonon field and for the phason field. However, there is no phonon-phason mixing term in the elastic energy.[8] Thus $R_{ijkl}=0$. The matrix $[C\ \ K\ \ R]$ is

$$[C\ \ K\ \ R]=\begin{bmatrix} \lambda+2\mu & \lambda & 0 & 0 & 0 & 0 & 0 & 0\\ \lambda & \lambda+2\mu & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0\\ 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & K_1 & K_2 & 0 & 0\\ 0 & 0 & 0 & 0 & K_2 & K_1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & K_1+K_2+K_3 & K_3\\ 0 & 0 & 0 & 0 & 0 & 0 & K_3 & K_1+K_2+K_3 \end{bmatrix}. \tag{62}$$

The elastic energy density

$$F=\tfrac{1}{2}\lambda(\nabla\cdot\mathbf{u})^2+\mu E_{ij}E_{ij}+\tfrac{1}{2}K_1\partial_j w_i\partial_j w_i+\tfrac{1}{2}K_2[(\partial_1 w_2)^2+(\partial_2 w_1)^2+2\,\partial_1 w_1\partial_2 w_2]+\tfrac{1}{2}K_3(\partial_1 w_2+\partial_2 w_1)^2\ . \tag{63}$$

The generalized Hooke's law has the form

$$T_{ij}=[\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})]E_{kl}\quad(i,j,k,l=1,2)\ ,$$
$$H_{11}=K_1\partial_1 w_1+K_2\partial_2 w_2\ ,\quad H_{22}=K_1\partial_2 w_2+K_2\partial_1 w_1\ , \tag{64}$$
$$H_{12}=(K_1+K_2+K_3)\partial_2 w_1+K_3\partial_1 w_2\ ,\quad H_{21}=(K_1+K_2+K_3)\partial_1 w_2+K_3\partial_2 w_1\ .$$

u and **w** satisfy

$$\mu\nabla^2 u_1+(\lambda+\mu)\partial_1(\nabla\cdot\mathbf{u})+f_1=0\ ,\quad \mu\nabla^2 u_2+(\lambda+\mu)\partial_2(\nabla\cdot\mathbf{u})+f_2=0\ ,$$
$$K_1\nabla^2 w_1+(K_2+K_3)(\partial_2\partial_2 w_1+\partial_1\partial_2 w_2)+g_1=0\ ,\quad K_1\nabla^2 w_2+(K_2+K_3)(\partial_1\partial_2 w_1+\partial_1\partial_1 w_2)+g_2=0\ . \tag{65}$$

D. 3D icosahedral quasicrystal

All quadratic invariants and the elastic energy have been already derived for 3D icosahedral quasicrystal with the group theory.[3,7] From these results we know C_{ijkl} take the same form as Eq. (45) except that in this case $i,j,k,l=1,2,3$. The nonvanishing K_{ijkl} are

$$K_{1111}=K_{2222}=K_{1212}=K_{2121}\equiv K_1\ ,$$
$$K_{1131}=K_{3111}=K_{1113}=K_{1311}=K_{2213}=K_{1322}=K_{2312}=K_{1223}=-K_{2231}$$
$$=-K_{3122}=-K_{2321}=-K_{2123}=-K_{1232}=-K_{3212}=-K_{3221}=-K_{2132}\equiv K_2\ , \tag{66}$$
$$K_{3333}=K_1+K_2\ ,$$
$$K_{2323}=K_{3131}=K_{3232}=K_{1313}=K_1-K_2\ .$$

Therefore,

$$[K]=\begin{bmatrix} K_1 & 0 & 0 & 0 & K_2 & 0 & 0 & K_2 & 0 \\ 0 & K_1 & 0 & 0 & -K_2 & 0 & 0 & K_2 & 0 \\ 0 & 0 & K_1+K_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_1-K_2 & 0 & K_2 & 0 & 0 & -K_2 \\ K_2 & -K_2 & 0 & 0 & K_1-K_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_2 & 0 & K_1 & -K_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -K_2 & K_1-K_2 & 0 & -K_2 \\ K_2 & K_2 & 0 & 0 & 0 & 0 & 0 & K_1-K_2 & 0 \\ 0 & 0 & 0 & -K_2 & 0 & 0 & -K_2 & 0 & K_1 \end{bmatrix}, \tag{67}$$

$$[R]=R\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}. \tag{68}$$

Using Eq. (40) we have the elastic energy density F with the same form as in Ref. 3. By Eq. (43) the generalized Hooke's law is given as follows

$$\begin{aligned}
&T_{11}=\lambda\theta+2\mu E_{11}+R(\partial_1 w_1+\partial_2 w_2+\partial_3 w_3+\partial_3 w_1)\,,\quad T_{22}=\lambda\theta+2\mu E_{22}-R(\partial_1 w_1+\partial_2 w_2-\partial_3 w_3+\partial_3 w_1)\,,\\
&T_{33}=\lambda\theta+2\mu E_{33}-2R\partial_3 w_3\,,\quad T_{23}=2\mu E_{23}+R(\partial_2 w_3-\partial_2 w_1-\partial_1 w_2)=T_{32}\,,\\
&T_{31}=2\mu E_{31}+R(\partial_1 w_1-\partial_2 w_2+\partial_1 w_3)=T_{13}\,,\quad T_{12}=2\mu E_{12}+R(\partial_1 w_2-\partial_3 w_2-\partial_2 w_1)=T_{21}\,,\\
&H_{11}=R(E_{11}-E_{22}+2E_{31})+K_1\partial_1 w_1+K_2(\partial_1 w_3+\partial_3 w_1)\,,\quad H_{22}=R(E_{11}-E_{22}-2E_{31})+K_1\partial_2 w_2+K_2(\partial_3 w_1-\partial_1 w_3)\,,\\
&H_{33}=R(E_{11}+E_{22}-2E_{33})+(K_1+K_2)\partial_3 w_3\,,\quad H_{23}=-2RE_{12}+(K_1-K_2)\partial_3 w_2+K_2(\partial_2 w_1-\partial_1 w_2)\,,\\
&H_{31}=2RE_{31}+K_2(\partial_1 w_1-\partial_2 w_2)+(K_1-K_2)\partial_1 w_3\,,\quad H_{12}=-2R(E_{23}+E_{12})+K_1\partial_2 w_1+K_2(\partial_3 w_2-\partial_2 w_3)\,,\\
&H_{32}=2RE_{23}+(K_1-K_2)\partial_2 w_3-K_2(\partial_2 w_1+\partial_1 w_2)\,,\quad H_{13}=R(E_{11}-E_{22})+K_2(\partial_1 w_1+\partial_2 w_2)+(K_1-K_2)\partial_3 w_1\,,\\
&H_{21}=2R(E_{12}-E_{23})-K_2(\partial_3 w_2+\partial_2 w_3)+K_1\partial_1 w_2\,,
\end{aligned} \tag{69}$$

where $\theta=E_{11}+E_{22}+E_{33}$. Substituting Eq. (69) into Eqs. (21) and (23) gives the equilibrium equation

$$\begin{aligned}
&\mu\nabla^2 u_1+(\lambda+\mu)\partial_1(\nabla\cdot\mathbf{u})+R(\partial_1\partial_1 w_1+2\,\partial_1\partial_3 w_1-\partial_2\partial_2 w_1+2\,\partial_1\partial_2 w_2-2\partial_2\partial_3 w_2+2\partial_1\partial_3 w_3)+f_1=0\,,\\
&\mu\nabla^2 u_2+(\lambda+\mu)\partial_2(\nabla\cdot\mathbf{u})+R(-2\,\partial_1\partial_2 w_1-2\,\partial_2\partial_3 w_1+\partial_1\partial_1 w_2-2\,\partial_1\partial_3 w_2-\partial_2\partial_2 w_2+2\,\partial_2\partial_3 w_3)+f_2=0\,,\\
&\mu\nabla^2 u_3+(\lambda+\mu)\partial_3(\nabla\cdot\mathbf{u})+R(\partial_1\partial_1 w_1-\partial_2\partial_2 w_1-2\,\partial_1\partial_2 w_2+\partial_1\partial_1 w_3+\partial_2\partial_2 w_3-2\,\partial_3\partial_3 w_3)+f_3=0\,,\\
&K_1\nabla^2 w_1+K_2(2\,\partial_1\partial_3 w_1-\partial_3\partial_3 w_1+2\,\partial_2\partial_3 w_2+\partial_1\partial_1 w_3-\partial_2\partial_2 w_3)\\
&\qquad+R(\partial_1\partial_1 u_1-\partial_2\partial_2 u_1+2\,\partial_1\partial_3 u_1-2\,\partial_1\partial_2 u_2-2\,\partial_2\partial_3 u_2+\partial_1\partial_1 u_3-\partial_2\partial_2 u_3)+g_1=0\,,\\
&K_1\nabla^2 w_2+K_2(2\,\partial_2\partial_3 w_1-2\,\partial_1\partial_3 w_2-2\,\partial_1\partial_2 w_3-\partial_3\partial_3 w_2)\\
&\qquad+R(2\partial_1\partial_2 u_1-2\partial_2\partial_3 u_1+\partial_1\partial_1 u_2-\partial_2\partial_2 u_2-2\,\partial_1\partial_3 u_2-2\,\partial_1\partial_2 u_3)+g_2=0\,,\\
&(K_1-K_2)\nabla^2 w_3+K_2(\partial_1\partial_1 w_1-\partial_2\partial_2 w_1-2\,\partial_1\partial_2 w_2+2\partial_3\partial_3 w_3)\\
&\qquad+R(2\partial_1\partial_3 u_1+2\,\partial_2\partial_3 u_2+\partial_1\partial_1 u_3+\partial_2\partial_2 u_3-2\,\partial_3\partial_3 u_3)+g_3=0\,.
\end{aligned} \tag{70}$$

Finally, we would like to point out that the present version is a generalization of the classical elasticity theory describing conventional crystals, so the fundamental results obtained here should be applicable to both crystals and quasicrystals. As we know, the phason displacement field **w** in Eq. (2) is absent for the usual crystals. Consequently, $w_i=0$, $H_{ij}=0$, $g_i=0$, $K_{ijkl}=0$, and $R_{ijkl}=0$. In this case all the formulas given in Sec. II automatically reduce to the classical ones.[17,18]

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