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Extended irreversible thermodynamics of liquid helium II

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In this work a macroscopic monofluid theory of liquid helium II, which is based on the extended irreversible thermodynamics, is formulated both in the presence and in the absence of dissipative phenomena. The work is a generalization of previous papers, where the extended thermodynamics of an ideal monoatomic fluid was applied to liquid helium II. It is shown that the behavior of helium II can be described by means of an extended thermodynamic theory where four fields, namely density, temperature, velocity, and heat flux are involved as independent fields. In the presence of dissipative phenomena, constitutive relations for the trace and the deviator of the nonequilibrium stress tensor are determined. In these relations, in addition to the normal viscous terms (which take into account the mechanical dissipation), terms proportional to the gradient of heat flux (which take into account the thermal dissipation) are present. The proposed theory is able to explain the propagation of the two sounds that are typical of helium II, and the attenuation calculated for such sounds is in agreement with the experimental results. Finally, the proposed theory is compared with the two-fluid model making apparent the analogies and the differences.

I. INTRODUCTION

Extended thermodynamics (ET) is a macroscopic theory of nonequilibrium processes, which has been formulated in various ways in the last decades.[1–3] The main difference between the ordinary thermodynamics and the ET is that the latter uses dissipative fluxes, besides the traditional variables, as independent fields. As a consequence, the assumption of local equilibrium is abandoned in such theories. In the study of nonequilibrium thermodynamic processes, an extended approach is required when one is interested in sufficiently rapid phenomena, or else when the relaxation times of the fluxes are long; in such cases, a constitutive description of these fluxes in terms of the traditional field variables is impossible, so that they must be treated as independent fields of the thermodynamic process.

The behavior of liquid helium, below the λ *point,* is anomalous with respect to the other substances.[4] In particular, it has an extraordinary ability to flow through a thin capillary, is unable to boil, it is extremely difficult to measure its thermal conductivity, and temperature waves are propagated in it. In order to describe the behavior of this quantum liquid, Landau and other introduce the well-known two-fluid model.[5–8] Other authors have taken into account the inseparability of the superfluid and normal-fluid component and attempted to provide alternatives to the two-fluid model.[9–11]

The anomalous behavior of helium II can also be explained supposing that the quantum phenomena do not allow the introduction of the hypothesis of local equilibrium. From a macroscopic point of view, an extended approach to thermodynamics is required in helium II because the relaxation time of heat flux is comparable with the evolution times of the other variables. This field cannot, therefore, be expressed by means of a constitutive equation as a dependent variable. This point of view is confirmed by the fact that the thermal conductivity of helium II cannot be measured and by the possibility of describing, through the proposed theory, several typical effects of helium II. The two-fluid model by Landau (formulated when a coherent thermodynamic theory for nonequilibrium phenomena was not yet established) shortcuts this problem, because an objective kinematic variable replaces the heat flux, which is the nonequilibrium variable needed to describe the macroscopic behavior of helium II: It is the relative velocity between the two components of the mixture.

An extended thermodynamic theory strongly suggested by the kinetic theory of gases has been formulated in Ref. 2. Moments of various orders of the phase density (which satisfies a Boltzmann equation) are chosen as fundamental fields, while the balance equations for these fields are nothing but the transport equations for the moments. For this reason the theory is valid only for ideal monoatomic fluids. As a limiting case of this theory, the case of a strongly degenerate Bose gas, where Bose-Einstein condensation takes place, has been studied in Ref. 12.

In some previous papers, basing upon such a theory, the behavior of an ideal monoatomic superfluid, that is an ideal monoatomic fluid with extremely low viscosity and extremely high thermal conductivity, has been studied both with and without dissipation. It has been shown that two waves are propagated in it (which become the two sounds characteristic of helium II when thermal expansion is negligible and temperature is low[13]), and that in such a superfluid, thermomechanical phenomena, as the link between the stress and the heat flux and the fountain effect, take place.[13–14] However, owing to the strong constraints imposed by the ET formulated in Ref. 2, the state equation of an ideal monoatomic fluid possesses a very particular functional form; consequently the thermostatic properties of helium II fit into the theory only approximatively and at high pressures and low temperatures, as it may be seen from the data reported in Ref. 15. Finally, the dissipative theory of an ideal monoatomic superfluid developed in Ref. 16, involves a single viscous coefficient, that is the shear viscosity, while it is observed that at low pressures bulk viscosity is not negligible in helium II.[8] All the former reasons require the formulation of a monofluid theory of this liquid, valid in a wider range of temperatures and pressures.

In this work, a monofluid theory of helium II is formulated which is based on extended irreversible thermodynamics formulated by Jou, Casas-Vazquez, and Lebon in Ref. 3. This theory is more general than the one formulated in Ref. 2, because its application is not restricted to ideal monoatomic fluids, and because, besides the fields of ET by Liu and Müller, also the trace of nonequilibrium stress (which is zero in an ideal monoatomic fluid) is introduced.

In Sec. II the extended irreversible thermodynamics is shortly summarized. It is shown, in Sec. III, that the behavior of helium II can be described, in the absence of dissipation, by means of the two scalar fields ρ and T, respectively, density and temperature, and the two vector fields v_i and q_i, respectively, velocity and heat flux. In Sec. IV, constitutive relations for the two fields $m_{\langle ij\rangle}$ and p_V, respectively, deviator and trace of nonequilibrium stress tensor, are determined, when the dissipation is taken into account. Starting from the assumption that the relaxation time of q_i is comparable to the evolution times of the other field variables, while the relaxation times of $m_{\langle ij\rangle}$ and p_V are negligible, one arrives at the constitutive equations for the latter fields, where terms proportional to the gradients of v_i and q_i are present. The propagation of small amplitude waves is studied and it is shown that the attenuation coefficients anticipated by the theory are in agreement with the experimental results. In Sec. V, finally, the proposed theory is compared with the two-fluid model by Landau and Khalatnikov.

II. OUTLINE OF EXTENDED IRREVERSIBLE THERMODYNAMICS OF A FLUID

The fundamental fields of the extended irreversible thermodynamics are the density ρ, the velocity $\mathbf{v}=(v_i)$, the internal energy ϵ, the nonequilibrium stress $\mathbf{m}=(m_{ij})$ and the heat flux $\mathbf{q}=(q_i)$. In the following we suppose the nonequilibrium part of the stress $m_{ij}=p_V\delta_{ij}+m_{\langle ij\rangle}$ decomposed into its trace p_V and its deviator $m_{\langle ij\rangle}$. As we shall consider only thermodynamic processes near equilibrium, it is sufficient to use the following linearized field equations:[3]

$$\frac{\partial\rho}{\partial t}+\rho\frac{\partial v_j}{\partial x_j}=0\ , \tag{2.1a}$$

$$\rho\frac{\partial v_i}{\partial t}+\frac{\partial}{\partial x_j}[(p_E+p_V)\delta_{ij}+m_{\langle ij\rangle}]=0\ , \tag{2.1b}$$

$$\rho\frac{\partial\epsilon}{\partial t}+\frac{\partial q_j}{\partial x_j}+[(p_E+p_V)\delta_{ij}+m_{\langle ij\rangle}]\frac{\partial v_i}{\partial x_j}=0\ , \tag{2.1c}$$

$$\tau_0\frac{\partial p_V}{\partial t}+\lambda_0\frac{\partial v_j}{\partial x_j}-\beta' T\lambda_0\frac{\partial q_j}{\partial x_j}=-p_V\ , \tag{2.1d}$$

$$\tau_2\frac{\partial m_{\langle ik\rangle}}{\partial t}+2\lambda_2\frac{\partial v_{<i}}{\partial x_{k>}}-2\beta T\lambda_2\frac{\partial q_{<i}}{\partial x_{k>}}=-m_{\langle ik\rangle}\ , \tag{2.1e}$$

$$\tau_1\frac{\partial q_i}{\partial t}+\lambda_1\frac{\partial T}{\partial x_i}-\beta' T^2\lambda_1\frac{\partial p_V}{\partial x_j}-\beta T^2\lambda_1\frac{\partial m_{\langle ij\rangle}}{\partial x_j}=-q_i\ . \tag{2.1f}$$

In Eqs. (2.1), p_E is the pressure of thermostatics, T the temperature, τ_0, τ_2, and τ_1 are, respectively, the relaxation times of the nonequilibrium pressure, stress deviator and heat flux; λ_0, λ_2, and λ_1 are the coefficients, which in a normal fluid, can be identified, respectively, with the bulk viscosity, shear viscosity, and heat conductivity. Finally β and β' are coefficients which can be related to the moments of fluctuations.[3,17]

In Ref. 3 the expressions for the entropy η, its production σ^η, and its flux $\mathbf{J}^\eta$ have been determined; up to the second order in nonequilibrium quantities, denoting with $\mathbf{m}^0$ the traceless part of $\mathbf{m}$, they are

$$\eta=\eta_E-\frac{1}{2\rho T^2}\frac{\tau_1}{\lambda_1}\mathbf{q}\cdot\mathbf{q}-\frac{1}{2\rho T}\frac{\tau_0}{\lambda_0}p_V^2-\frac{1}{4\rho T}\frac{\tau_2}{\lambda_2}\mathbf{m}^0{:}\mathbf{m}^0\ , \tag{2.2}$$

$$\mathbf{J}^\eta=\rho\eta\mathbf{v}+\frac{1}{T}\mathbf{q}+\beta' p_v\mathbf{q}+\beta\mathbf{m}^0\cdot\mathbf{q}\ , \tag{2.3}$$

$$\sigma^\eta=\frac{1}{\lambda_1 T^2}\mathbf{q}\cdot\mathbf{q}+\frac{1}{\lambda_0 T}p_V p_V+\frac{1}{2\lambda_2 T}\mathbf{m}^0{:}\mathbf{m}^0\ . \tag{2.4}$$

In the extended thermodynamics, finally, the entropy η satisfies a generalized Gibbs equation; an approximate expression of it is:[3]

$$T d\eta=d\epsilon-\frac{p_E}{\rho^2}d\rho-\frac{1}{2\rho T}\frac{\tau_1}{\lambda_1}\mathbf{q}\cdot d\mathbf{q}-\frac{1}{2\rho}\frac{\tau_0}{\lambda_0}p_V dp_V-\frac{1}{4\rho}\frac{\tau_2}{\lambda_2}\mathbf{m}^0{:}d\mathbf{m}^0\ . \tag{2.5}$$

Finally, the entropy principle, when applied to extended irreversible thermodynamics, leads[3] to the following inequalities for the coefficients appearing in Eqs. (2.1):

$$\lambda_0>0,\ \ \lambda_1>0,\ \ \lambda_2>0;\ \ \tau_0>0,\ \ \tau_1>0,\ \ \tau_2>0\ . \tag{2.6}$$

III. EXTENDED THERMODYNAMICS OF HELIUM II WITHOUT DISSIPATION

Experiments[4] show that phenomena, as heat and matter transport in liquid helium II, take place in almost complete absence of dissipation. Namely, its viscosity is very low and its thermal conductivity is very high. In a first approximation, we will suppose that in helium II the viscosity vanishes and the thermal conductivity is infinite:

$$\lambda_0=0\ , \tag{3.1a}$$

$$\lambda_2=0\ , \tag{3.1b}$$

$$\lambda_1=\infty\ . \tag{3.1c}$$

We analyze expression (2.4) for the entropy production σ^η, in order to verify the compatibility of the conditions (3.1) with entropy conservation. We observe first that, because $1/\lambda_1=0$, the term proportional to the square of heat flux does not appear in entropy production; we can therefore say that, under the hypothesis (3.1c), the heat transport is a reversible process in helium II. Substituting then in (2.4) the conditions (3.1a) and (3.1b), we deduce that, up the first order in nonequilibrum quantities, entropy conservation requires

$$p_V=0; \quad m_{\langle ij\rangle}=0 \; . \tag{3.2}$$

Substituting (3.2) in field equations (2.1d) and (2.1e), we deduce also $\tau_0=0$ and $\tau_2=0$.

We consider now the balance equation (2.1f) of the heat flux. The experimental results show that in helium II the thermal conductivity cannot be measured. The researchers who tried to measure it found it very high and observed that its value seemed to depend on the device used for the measurement. These facts led Mendelsohn[4] to state that "the concept of 'heat conductivity' in the accepted sense as a constant ratio of heat current density to the temperature gradient has thus lost its usefulness when dealing with liquid helium II." This behavior can be explained in the frame of ET observing that q_i is an independent variable. As already noted experiments show that the coefficient λ_1 is very high. Equation (2.1f) implies that τ_1 too must be very high. In what follows, we assume that the ratio between λ_1 and τ_1 is finite and nonzero. We put therefore,

$$\frac{\lambda_1}{\tau_1}=\zeta \; . \tag{3.3}$$

From (2.6) we deduce that ζ is strictly positive.

From the previous considerations, we conclude that, in order to describe the behavior of liquid helium II, the two scalar fields ρ and T, and the two vector fields v_i and q_i must be introduced as independent variables. The linearized equations for these fields can be obtained by substituting in Eqs. (2.1a), (2.1b), (2.1c), and (2.1f) the relations (3.1), (3.2), and (3.3):

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+\rho\frac{\partial v_j}{\partial x_j}=0 \; , \\
&\frac{\partial v_i}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}=0 \; , \\
&\frac{\partial T}{\partial t}+\frac{Tp_T}{\rho c_V}\frac{\partial v_j}{\partial x_j}+\frac{1}{\rho c_V}\frac{\partial q_j}{\partial x_j}=0 \; , \\
&\frac{\partial q_i}{\partial t}+\zeta\frac{\partial T}{\partial x_i}=0 \; .
\end{aligned} \tag{3.4}$$

In Eqs. (3.4), p denotes the pressure of thermostatics, (previously denoted p_E), the subscript T near p indicates partial differentiation with respect to this variable, and c_V is the constant volume specific heat.

We observe that Eqs. (3.4) are identical with the linearized ones, obtained in Ref. 13, by imposing entropy conservation in an ideal monoatomic inviscid fluid, with high thermal conductivity. The case considered here is a generalization of the one analyzed in Ref. 13, because the present theory does not fix *a priori* the relation between pressure and internal energy, which characterizes the ideal monoatomic fluids.

As it was shown in Ref. 13, Eqs. (3.4) describe the propagation in liquid helium II of two waves, whose speeds u are the solutions of the characteristic equation

$$(u^2-V_1^2)(u^2-V_2^2)-W_1W_2u^2=0 \; , \tag{3.5}$$

where

$$V_1^2=p_\rho \; ; \tag{3.6a}$$

$$V_2^2=\frac{\zeta}{\rho c_V} \; ; \tag{3.6b}$$

$$W_1=\frac{p_T}{\rho} \; ; \tag{3.6c}$$

$$W_2=\frac{Tp_T}{\rho c_V} \; . \tag{3.6d}$$

We recall now that in helium II thermal expansion is very low. In this hypothesis, W_1 and W_2 vanish and Eq. (3.5) admit the solutions $u_{1,2}=\pm V_1$ and $u_{3,4}=\pm V_2$, corresponding to the two sounds typical of helium II: To the wave whose speed is $u=\pm V_1$, only vibrations of density and velocity are associated, while in the wave whose speed is $u=\pm V_2$, only temperature and heat flux vibrate. This agrees with experimental observations. The coefficient ζ defined in (3.3), can be determined by (3.6b), once the expression of the second sound is known.

Substituting (3.2) in Eq. (2.2) the expression for the entropy η of helium II, in the absence of dissipation ($\sigma^\eta=0$), is obtained:

$$\eta=\eta_E-\frac{1}{2\rho\zeta T^2}\mathbf{q}\cdot\mathbf{q} \; . \tag{3.7}$$

Jou and co-workers[18,19] have shown that the coefficient ζ can be related to second-order moments of the fluctuations of heat flux. They assume that the classical Einstein formula

$$Pr=A\exp\left[\frac{1}{2k_B}(\delta^2\eta)_E\right] , \tag{3.8}$$

where k_B is the Boltzmann constant, A is a normalization constant and $\delta^2\eta$ is the second differential of entropy, holds also in ET. Taking into account (3.7), the latter equation leads to the following expression for the second-order moments of the fluctuations of heat flux:

$$\langle\delta q_i\delta q_j\rangle=k_B\zeta\frac{T^2}{V}\delta_{ij} \; , \tag{3.9}$$

where V is the volume occupied by the fluid.

Finally, we observe that Gibbs equation for helium II can be written as

$$T\,d\eta=d\epsilon-\frac{p}{\rho^2}d\rho-\frac{1}{\rho\zeta T}\mathbf{q}\cdot d\mathbf{q} \; . \tag{3.10}$$

Using this latter relation, reasoning as in Ref. 14, we can explain, also in this theory, the static fountain effect. In order to explain other thermomechanical phenomena, as the link between the stress and the heat flux, it is necessary to use a nonlinear formulation of this theory. This investigation will be the object of further study.

In Ref. 14, a boundary condition, able to explain, in a monofluid theory of superfluid helium, the reversible flow through a very thin capillary (superleak), has been formulated. This condition, which we shall impose also in this more general theory of helium II, can be stated as follows: *The component of the entropy flux density tangent to the walls of the vessel vanishes;* $J_t^\eta=0$. In a superleak, indeed, this boundary condition implies the vanishing of entropy flux density through the whole section of the capillary. This boundary condition, to the lowest order, using (2.3), can be written

$$q_t+\rho T\eta v_t=0 \ . \tag{3.11}$$

IV. THE ATTENUATION OF THE TWO SOUNDS IN LIQUID HELIUM II

We show now that helium II can be described through the two scalar fields ρ and T and the two vector fields v_i and q_i, even when entropy is not conserved.

In order to deduce a dissipative system of field equations for helium II, we determine constitutive relations for the trace p_V and for the deviator $m_{\langle ij\rangle}$ of the nonequilibrium stress, depending on the derivatives of the fundamental fields. We consider Eqs. (2.1d) and (2.1e). As we have seen, under the hypothesis of entropy conservation, their solutions are zero. We suppose now that entropy is not conserved. Experimentally the relaxation times τ_0 and τ_2 are very small. Under the approximation of entropy conservation, we have been forced to put them zero. In a first approximation, they can be supposed to be zero even when entropy is not conserved. Such hypotheses correspond to retain the evolution time of the stress extremely small, in comparison with the evolution times of the other variables. Substituting, therefore, $\tau_0=0$ and $\tau_2=0$, in (2.1d) and (2.1e), one gets

$$p_V=-\lambda_0\frac{\partial v_j}{\partial x_j}+\beta' T\lambda_0\frac{\partial q_j}{\partial x_j} \ , \tag{4.1}$$

$$m_{\langle ik\rangle}=-2\lambda_2\frac{\partial v_{<i}}{\partial x_{k>}}+2\beta T\lambda_2\frac{\partial q_{<i}}{\partial x_{k>}} \ . \tag{4.2}$$

These are the constitutive equations for the trace and the deviator of the nonequilibrium stress in liquid helium II we searched for. These equations can be identified with the ones obtained in Ref. 16, by using the ET of an ideal monoatomic superfluid, if we put

$$\beta=-\frac{2}{5Tp}, \quad \lambda_0=0 \ . \tag{4.3}$$

Equations (4.1) and (4.2) contain, in addition to terms proportional to the gradient of velocity, terms depending on the gradient of the heat flux. We will show that the first terms allow us to explain the attenuation of the first sound in helium II, the latter terms explain the attenuation of the second sound.

We write then the field equations of helium II in the presence of dissipation. We shall use as field variables p, T, v_i, and q_i. Substituting relations (4.1) and (4.2) into the field equations (2.1a), (2.1b), (2.1c), and (2.1f), the following linearized system is obtained:

$$\frac{\partial p}{\partial t}-p_T\frac{\partial T}{\partial t}+\rho p_\rho\frac{\partial v_j}{\partial x_j}=0 \ , \tag{4.4a}$$

$$\frac{\partial v_i}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}-\frac{\lambda_0}{\rho}\frac{\partial}{\partial x_i}\left[\frac{\partial v_j}{\partial x_j}-\beta' T\frac{\partial q_j}{\partial x_j}\right] -\frac{\lambda_2}{\rho}\frac{\partial}{\partial x_j}\left[2\frac{\partial v_{<j}}{\partial x_{i>}}-2\beta T\frac{\partial q_{<j}}{\partial x_{i>}}\right]=0 \ , \tag{4.4b}$$

$$\frac{\partial T}{\partial t}+\frac{Tp_T}{\rho c_V}\frac{\partial v_j}{\partial x_j}+\frac{1}{\rho c_V}\frac{\partial q_j}{\partial x_j}=0 \ , \tag{4.4c}$$

$$\frac{\partial q_i}{\partial t}+\zeta\frac{\partial T}{\partial x_i}+\lambda_0\beta' T^2\zeta\frac{\partial}{\partial x_i}\left[\frac{\partial v_j}{\partial x_j}-\beta' T\frac{\partial q_j}{\partial x_j}\right] -\lambda_2\beta T^2\zeta\frac{\partial}{\partial x_j}\left[2\frac{\partial v_{<j}}{\partial x_{i>}}-2\beta T\frac{\partial q_{<j}}{\partial x_{i>}}\right]=-\frac{1}{\tau_1}q_i \ . \tag{4.4d}$$

We consider the propagation of harmonic plane waves. Putting $\mathcal{U}=(p,T,v_i,q_i)$, we look for a solution of the linearized equations (4.4) having the form:

$$\mathcal{U}=\mathcal{U}_0+\overline{\mathcal{U}}e^{i(Kn_jx_j-\omega t)} \ , \tag{4.5}$$

where $\mathcal{U}_0=(p_0,T_0,0,0)$ and where $K=k_r+ik_s$ is the complex wave number. We suppose further that the oversigned quantities denote small amplitudes whose products can be neglected. Inserting (4.5) in linearized field equations (4.4), putting $\bar{v}_n=\bar{v}_jn_j$, $\bar{q}_n=\bar{q}_jn_j$, and multiplying finally the last two equations by the unit vector n_i, orthogonal to the wave front, the following algebraic system for the small amplitudes is obtained:

$$\begin{aligned}
&-\omega\bar{p}-\omega[p_T]_0\bar{T}+K[\rho p_\rho]_0\bar{v}_n=0 \ ,\\
&\left[-\omega+iK^2\left[\frac{1}{\rho}(\lambda_0+\tfrac{4}{3}\lambda_2)\right]_0\right]\bar{v}_n+K\left[\frac{1}{\rho}\right]_0\bar{p}+iK^2\left[\frac{T}{\rho}(\lambda_0\beta'+\tfrac{4}{3}\lambda_2\beta)\right]_0\bar{q}_n=0 \ ,\\
&-\omega\bar{T}+K\left[\frac{Tp_T}{\rho c_V}\right]_0\bar{v}_n+K\left[\frac{1}{\rho c_V}\right]_0\bar{q}_n=0 \ ,\\
&\left[-\omega-i\left[\frac{1}{\tau_1}\right]_0-iK^2[T^3\zeta(\lambda_0\beta'^2+\tfrac{4}{3}\lambda_2\beta^2)]_0\right]\bar{q}_n+K[\zeta]_0\bar{T}+iK^2[T^2\zeta(\lambda_0\beta'+\tfrac{4}{3}\lambda_2\beta)]_0\bar{v}_n=0 \ .
\end{aligned} \tag{4.6}$$

In (4.6) the subscript 0 denotes quantities referring to the unperturbed state $\mathcal{U}_0$. In what follows, for the sake of simplicity, this subscript will be neglected. The system (4.6) possesses nontrivial solutions if and only if its determinant vanishes:

$$\left[\omega^2-K^2(V_1^2-W_1W_2)+iK^2\frac{\omega}{\rho}(\lambda_0-\tfrac{4}{3}\lambda_2)\right]\cdot\left[\omega^2+i\frac{1}{\tau_1}\omega-K^2V_2^2+iK^2\omega T^3\zeta(\lambda_0\beta'^2+\tfrac{4}{3}\lambda_2\beta^2)\right]$$
$$+\left[K^2\frac{1}{\rho T}W_2-iK^2\omega\frac{T}{\rho}(\lambda_0\beta'+\tfrac{4}{3}\lambda_2\beta)\right]\cdot\left[K^2\zeta W_2+iK^2\omega T^2\zeta(\lambda_0\beta'+\tfrac{4}{3}\lambda_2\beta)\right]=0\ . \quad (4.7)$$

We suppose now that the thermal expansion is zero ($W_1=W_2=0$) and the dissipation is small ($k_r \gg k_s$); we suppose further that the frequency ω is not too small. With these assumptions, the solutions of (4.7) are

$$u_1^2=V_1^2,\quad k_s^{(1)}=\frac{\omega^2}{2\rho u_1^3}(\lambda_0+\tfrac{4}{3}\lambda_2)\ , \quad (4.8)$$

$$u_2^2=V_2^2\ , \quad (4.9a)$$

$$k_s^{(2)}=\frac{1}{2u_2}\frac{1}{\tau_1}+\frac{\omega^2T^3\zeta}{2u_2^3}(\lambda_0\beta'^2+\tfrac{4}{3}\lambda_2\beta^2)\ . \quad (4.9b)$$

We analyze now the attenuation coefficients: First we observe that the expression of the attenuation coefficient $k_s^{(1)}$ of the first sound is identical to the one deduced by Landau and Khalatnikov, using the two-fluid model.[8] The attenuation coefficient of the second sound appears different from the one obtained in Ref. 8. However, in accord with the experimental data, it contains a term proportional to the square of the frequency ω. A comparison between the coefficient $k_s^{(2)}$ deduced here and the one of the two-fluid model of Landau and Khalatnikov will be done in Sec. V.

V. COMPARISON WITH THE TWO-FLUID MODEL

The two-fluid model by Landau regards helium II as a two-component mixture: the normal component with normal entropy, viscosity and thermal conductivity, and the superfluid component, with zero entropy.

As it is shown in Ref. 13, if we perform in Eqs. (3.4) the transformation of variables

$$v_i=\frac{\rho_s}{\rho}v_i^s+\frac{\rho_n}{\rho}v_i^n\ , \quad (5.1a)$$

$$q_i=-\rho_sT\eta_EV_i\ , \quad (5.1b)$$

where $V_i=(v_i^s-v_i^n)$ is the relative velocity between the two components of the mixture, we obtain a system of linearized field equations which can be identified with the linearized equations of Landau, if we put

$$\zeta=\rho\frac{\rho_s}{\rho_n}T\eta^2\ . \quad (5.2)$$

The Gibbs equation, in the two-fluid model description of helium II, takes the form

$$d(\rho\epsilon)=Td(\rho\eta)+\mu_Ed\rho+\frac{\rho_s}{\rho}V_id(\rho_nV_i) \quad (5.3)$$

in the center of mass frame, and retaining only terms up to the first order in V_i.[20] Performing in the latter relation the change of variable (5.1b), (5.2), and remembering that $\mu_E=\epsilon-T\eta+(p/\rho)$, we obtain, to the first order in q_i

$$T\,d\eta=d\epsilon-\frac{p}{\rho^2}d\rho-\frac{1}{\rho\zeta T}q_idq_i\ , \quad (5.4)$$

which is identical to the Gibbs Eq. (3.10) of the monofluid extended model here formulated.

In the presence of dissipative phenomena, the two-phase model postulates that the heat flux is tied to equilibrium entropy η_E by the relation

$$q_i=-\rho_sT\eta_EV_i+\kappa\frac{\partial T}{\partial x_i}\ , \quad (5.5)$$

which generalizes (5.1b). The parameter κ is the thermal conductivity of the mixture. Incidentally, we observe that the terms heat flux and thermal conductivity have different meaning in the two-fluid theory and in the extended one. In fact in the former, the heat flux represents the correction term, of the Fourier type, appearing in the expression (5.5) of the energy flux, and the small parameter κ here appearing is the thermal conductivity of the mixture, which, in the two-fluid model of helium II, is supposed similar to that of a normal fluid. In the extended theory, which is a monofluid one, the energy flux is instead identified with the heat flux. The parameter λ_1, appearing in (2.1f), can be identified with the thermal conductivity only in a normal fluid, where, the relaxation time τ_1 being negligible, one arrives at a constitutive equation for q_i of the Fourier type. On the contrary, in our extended model of helium II, the heat flux q_i is an independent variable, and the parameter λ_1, which is extremely high, and which we call still (extended) thermal conductivity, has nothing to bear with the small parameter κ of the two-fluid model.

In what follows, we compare the field equations (4.4) with the dissipative linearized equations of the two-fluid model, as formulated by Khalatnikov,[8] that are reported here for an easier comparison:

$$\frac{\partial\rho}{\partial t}+\rho\frac{\partial v_j}{\partial x_j}=0\ , \quad (5.6a)$$

$$\frac{\partial v_i}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_i}\left[-\rho_s\xi_1^{(LK)}\frac{\partial V_j}{\partial x_j}-\xi_2^{(LK)}\frac{\partial v_j^n}{\partial x_j}\right]$$
$$+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[-2\eta_2^{(\mathrm{LK})}\frac{\partial v^n_{<j}}{\partial x_{i>}}\right]=0\ , \quad (5.6b)$$

$$\frac{\partial T}{\partial t}+\frac{Tp_T}{\rho c_V}\frac{\partial v_j}{\partial x_j}+\frac{1}{\rho c_V}\frac{\partial}{\partial x_j}\left[-\rho_s T\eta V_j+\kappa\frac{\partial T}{\partial x_j}\right]=0\ , \tag{5.6c}$$

$$\frac{\partial v_i^s}{\partial t}+\frac{\partial \mu_E}{\partial x_i}+\frac{\partial}{\partial x_i}\left[-\rho_s\xi_3^{(LK)}\frac{\partial V_j}{\partial x_j}-\xi_4^{(LK)}\frac{\partial v_j^n}{\partial x_j}\right]=0\ . \tag{5.6d}$$

We carry out in field equations (4.4) the formal change of variables (5.1a) and (5.5), neglecting the terms of the second order in the dissipative coefficients λ_0, λ_2, $1/\lambda_1$, and κ.

The balance equations of mass and energy are (4.4a) and (4.4c), the first one is obviously identical to (5.6a), while substituting (5.5) in the second one, we obtain (5.6c).

We determine now the expressions of the deviator and the trace of the viscous stress tensor in the conventional variables of the two-fluid model. Performing in relations (4.1) and (4.2) the change of variables (5.1a) and (5.5), we get

$$p_V=-\lambda_0\frac{\partial v_j^n}{\partial x_j}-\lambda_0\rho_s\left[\frac{1}{\rho}+\beta' T^2\eta\right]\frac{\partial V_j}{\partial x_j}\ , \tag{5.7}$$

$$m_{\langle ik\rangle}=-2\lambda_2\frac{\partial v^n_{<i}}{\partial x_{k>}}-2\lambda_2\rho_s\left[\frac{1}{\rho}+\beta T^2\eta\right]\frac{\partial V_{<i}}{\partial x_{k>}}\ . \tag{5.8}$$

In (5.7) and (5.8) the terms depending on κ are absent, because they are of the second order in the dissipation coefficients.

The field equation for the momentum of helium II in the new variables is obtained by substituting (5.7) and (5.8), in Eq. (4.4b). One obtains

$$\frac{\partial v_i}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{1}{\rho}\frac{\partial}{\partial x_i}\left[-\rho_s\xi_1^{(E)}\frac{\partial V_j}{\partial x_j}-\xi_2^{(E)}\frac{\partial v_j^n}{\partial x_j}\right]+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[-2\rho_s\eta_1^{(E)}\frac{\partial V_{<j}}{\partial x_{i>}}-2\eta_2^{(E)}\frac{\partial v^n_{<j}}{\partial x_{i>}}\right]=0\ , \tag{5.9}$$

where

$$\xi_1^{(E)}=\lambda_0\left[\frac{1}{\rho}+\beta' T^2\eta\right],\quad \xi_2^{(E)}=\lambda_0\ , \tag{5.10}$$

$$\eta_1^{(E)}=\lambda_2\left[\frac{1}{\rho}+\beta T^2\eta\right],\quad \eta_2^{(E)}=\lambda_2\ . \tag{5.11}$$

As we see, Eq. (5.9) is identical with (5.6b) if we put

$$\lambda_0\left[\frac{1}{\rho}+\beta'\eta T^2\right]=\xi_1^{(LK)},\quad \lambda_0=\xi_2^{(LK)}\ , \tag{5.12a}$$

$$\lambda_2\left[\frac{1}{\rho}+\beta\eta T^2\right]=0,\quad \lambda_2=\eta_2^{(LK)}\ . \tag{5.12b}$$

In particular $\eta_1^{(E)}$ must vanish. This condition is satisfied, supposing $\lambda_2\neq 0$, if the coefficient β takes the value

$$\beta=-\frac{1}{\rho\eta T^2}\ . \tag{5.13}$$

In order to compare the last extended field equation (the one for the heat flux) with the fourth two-fluid field equation (the one for the superfluid velocity), we perform in Eq. (4.4d) the change of variables (5.1a), (5.5), and (5.2). One gets

$$\frac{\partial V_i}{\partial t}\frac{\rho}{\rho_n}\eta\frac{\partial T}{\partial x_i}+\frac{\partial}{\partial x_i}\left[-\frac{1}{\rho_n}\eta T^2\beta'\lambda_0\frac{\partial v_j^n}{\partial x_j}\frac{\rho\rho_s}{\rho_n}\eta T^2\beta'\lambda_0\left[\frac{1}{\rho}+\beta' T^2\eta\right]\frac{\partial V_j}{\partial x_j}\right]+\frac{\partial}{\partial x_j}\left[-2\frac{\rho}{\rho_n}\eta T^2\beta'\lambda_2\frac{1}{\rho}\frac{\partial v^n_{<j}}{\partial x_{i>}}-2\frac{\rho\rho_s}{\rho_n}\eta T^2\beta'\lambda_2\left[\frac{1}{\rho}+\beta T^2\eta\right]\frac{\partial V_{<i}}{\partial x_{k>}}\right]-\frac{1}{\rho_s T\eta}\frac{\partial}{\partial t}\left[\kappa\frac{\partial T}{\partial x_i}\right]-\frac{1}{\tau_1}V_i=0\ . \tag{5.14}$$

Keeping in mind that $v_i^s=(\rho_n/\rho)V_i+v_i$, we multiply (5.12) by ρ_n/ρ and we sum (5.9) to the equation so obtained. Recalling, finally, the relation $d\mu_E=(1/\rho)dp-\eta dT$, which ties the equilibrium chemical potential μ_E to the equilibrium variables, we can write the field equations of the superfluid velocity in the following way:

$$\frac{\partial v_i^s}{\partial t}+\frac{\partial \mu_E}{\partial x_i}+\frac{\partial}{\partial x_i}\left[-\rho_s\xi_3^{(E)}\frac{\partial V_j}{\partial x_j}-\xi_4^{(E)}\frac{\partial v_j^n}{\partial x_j}\right]+\frac{1}{\rho}\frac{\partial}{\partial x_j}\left[-2\rho_s\eta_3^{(E)}\frac{\partial V_{<j}}{\partial x_{i>}}2\eta_4^{(E)}\frac{\partial v^n_{<j}}{\partial x_{i>}}\right]-\frac{\zeta}{\eta}\frac{\partial}{\partial t}\left[\kappa\frac{\partial T}{\partial x_i}\right]=\frac{\rho_n}{\rho}\frac{1}{\tau_1}V_i\ , \tag{5.15}$$

where

$$\xi_3^{(E)}=\lambda_0\left[\frac{1}{\rho}+\beta' T^2\eta\right]^2,\quad \xi_4^{(E)}=\lambda_0\left[\frac{1}{\rho}+\beta' T^2\eta\right]\ , \tag{5.16}$$

$$\eta_3^{(E)}=\lambda_2\left[\frac{1}{\rho}+\beta T^2\eta\right]^2,\quad \eta_4^{(E)}=\lambda_2\left[\frac{1}{\rho}+\beta T^2\eta\right]\ . \tag{5.17}$$

From a comparison of (5.15) with (5.6d), we conclude that these equations can be identified, if the following conditions are satisfied:

$$\lambda_0\left[\frac{1}{\rho}+\beta'\eta T^2\right]^2=\xi_3^{(LK)},\quad \lambda_0\left[\frac{1}{\rho}+\beta'\eta T^2\right]=\xi_4^{(LK)}\ , \tag{5.18a}$$

$$\lambda_2 \left[\frac{1}{\rho} + \beta \eta T^2 \right]^2 = 0, \quad \lambda_2 \left[\frac{1}{\rho} + \beta \eta T^2 \right] = 0 \ . \tag{5.18b}$$

and if the thermal conductivity of the two-fluid model as well as the heat flux production of the extended model vanish:

$$\kappa = 0, \quad \frac{1}{\tau_1} = 0 \ . \tag{5.19}$$

Conditions (5.19) imply that, in both models, the heat transport is a reversible process. We conclude therefore that, while in the two theories the mechanical dissipation can be identified, it is not so for the thermal dissipation.

We analyze further the coefficients $\xi_\alpha^{(E)}$ and $\eta_\alpha^{(E)}$ of the extended model. From (5.10) and (5.16) we deduce that the coefficients $\xi_\alpha^{(E)}$ are not all independent, but they are related by the conditions

$$\xi_1^{(E)} = \xi_4^{(E)}, \quad \frac{\xi_1^{(E)}}{\xi_2^{(E)}} = \frac{\xi_3^{(E)}}{\xi_4^{(E)}} = \frac{1}{\rho} + \beta' \eta T^2 \ . \tag{5.20}$$

Consequently, only two of the four coefficients $\xi_\alpha^{(E)}$ are independent.

Both the conditions above hold for the corresponding coefficients of the Landau model. Equation (5.20a) is, in fact, postulated in the two-fluid model to satisfy the Onsager reciprocity principle; Eq. (5.20b) is not explicitly postulated in that model. In fact in Khalatnikov dissipative hydrodynamic equations the three kinetic coefficients $\xi_1^{(LK)}$, $\xi_2^{(LK)}$, and $\xi_3^{(LK)}$ need only satisfy the inequality $\xi_2^{(LK)}\xi_3^{(LK)} \geq [\xi_1^{(LK)}]^2$ (Ref. 8, Chap. 9). However, taking into account explicit expressions given by Khalatnikov (Ref. 8, Chap. 21) for these coefficients, one realizes that they satisfy the equality $\xi_2^{(LK)}\xi_3^{(LK)} = [\xi_1^{(LK)}]^2$.

We can therefore conclude that in the two-fluid model, as well as in the extended one, only two of the coefficients ξ_α are independent.

In the dissipative model of helium II by Landau and Khalatnikov, the coefficients η_1, η_3, and η_4 are zero. This circumstance is a consequence of the fact that the velocity of the superfluid component of helium II is supposed curl free. This assumption, to which Landau was led by considerations of quantum mechanics, is however, not confirmed experimentally.[21] Feynman[22] solved this paradox by supposing that the superfluid component, although curl free at a microscopic level, creates quantized vortices at an intermediate level; these vortices, when averaged over a macroscopic volume element, yield a nonzero value for the curl of v_i^s. The above considerations lead us to think that, also in the monofluid macroscopic theory of helium II here developed, no constraints must be imposed on the curl of the vector fields v_i and q_i. The coefficients η_1, η_3, and η_4 of the extended model must be then nonzero, so that (5.13) is not satisfied. We show finally that these coefficients are not arbitrary, but must satisfy relations analogous to the ones valid for the coefficients of bulk viscosity; from (5.11) and (5.17) we deduce indeed that

$$\eta_1^{(E)} = \eta_4^{(E)}, \quad \frac{\eta_1^{(E)}}{\eta_2^{(E)}} = \frac{\eta_3^{(E)}}{\eta_4^{(E)}} = \frac{1}{\rho} + \beta \eta T^2 \ . \tag{5.21}$$

Consequently, only two of the four coefficients $\eta_\alpha^{(E)}$ are independent.

Let us now consider again the attenuation coefficient of the second sound $k_s^{(2)}$ expressed by (4.9b). In this section, this coefficient will be denoted $[k_s^{(2)}]^{(E)}$, in order to distinguish it from the corresponding one of the two-fluid model,[8] whose expression is reported here:

$$[k_s^{(2)}]^{(LK)} = \frac{\omega^2}{2\rho u_2^3} \frac{\rho_s}{\rho_n} \cdot \left[\xi_2^{(LK)} + \rho^2 \xi_3^{(LK)} - 2\rho \xi_1^{(LK)} + \frac{4}{3}\eta_2^{(LK)} + \left[\frac{\rho_n}{\rho_s} \frac{k}{T} \frac{\partial T}{\partial \eta} \right] \right] \ . \tag{5.22}$$

For the purpose of comparing (5.22) with (4.9b), we observe that the latter, using the expressions for the coefficients $\xi_\alpha^{(E)}$ and $\eta_\alpha^{(E)}$ [defined in (5.10), (5.11), (5.16), and (5.17)] and Eq. (5.2), can be written in the following way:

$$[k_s^{(2)}]^{(E)} = \frac{1}{2u_2^2} \frac{1}{\tau_1} + \frac{\omega^2}{2\rho u_2^3} \frac{\rho_s}{\rho_n} \cdot \left[\xi_2^{(E)} + \rho^2 \xi_3^{(E)} - 2\rho \xi_1^{(E)} + \frac{4}{3}\eta_2^{(E)} + \left[\frac{8}{3}\rho \eta_1^{(E)} - \frac{4}{3}\rho^2 \frac{[\eta_1^{(E)}]^2}{\eta_2^{(E)}} \right] \right] \ . \tag{5.23}$$

Supposing then $\xi_\alpha^{(E)} = \xi_\alpha^{(LK)}$ and $\eta_2^{(E)} = \eta_2^{(LK)}$, we conclude that the attenuation of the second sound is the same in the two models if the coefficient $1/\tau_1$, which takes into account the production of the heat flux in the extended model, is supposed zero and further if the terms inside the inner brackets in (5.22) and (5.23) are equal; this equality can also be written as

$$\kappa \frac{\rho_n}{\rho_s} \frac{1}{T} \frac{\partial T}{\partial \eta} = \frac{4}{3}\lambda_2 \left[\frac{1}{\rho^2 \eta^2 T^4} - \beta^2 \right] \ . \tag{5.24}$$

According to Khalatnikov the left-hand side of (5.24) is the term giving the main contribution to the attenuation of the second sound. It comes from the terms proportional to the temperature gradient in the energy balance equation (5.6c). Consequently in the two-fluid model, the second sound attenuation is due essentially to the presence of a Fourier-type dissipative term in the field equations. In our extended model, the attenuation of the second sound depends, on the contrary, on the coefficients β and β', that is on the contribution of the nonequilibrium stress coming from the heat flux gradient (trace and deviator).

We analyze finally the physical meaning of the term containing the coefficient $1/\tau_1$, which in our extended model, appears in the production of heat flux, while it is absent in the dissipative model of Khalatnikov. In our theory, this coefficient is very small and it is tied to the extended heat conductivity λ_1 by the relation (3.4). If, in a first approximation, we put $1/\tau_1=0$, we deduce from (4.9b) [or (5.23)] that the attenuation coefficient $k_s^{(2)}$ is proportional to the square of the frequency ω. The theory here developed implies on the contrary, if $1/\tau_1$ is supposed nonzero, an additional attenuation for the second sound, which is proportional to $1/\tau_1$ and independent of the frequency ω. Attenuation measurements of the second sound may allow the determination of τ_1 (and λ_1).

We observe further that, in the variables of the two-fluid model, $1/\tau_1$ appears in the right side of (5.15) as a coefficient in a term of mutual friction. This term is absent in the dissipative model of helium II of Khalatnikov.[8] However, to the purpose of explaining the dissipation evidenced in the flow of helium II through channels when the counterflow takes place, a term proportional to the cube of heat flux has been introduced *ad hoc,* and it has been attributed to the presence of vortices of superfluid.[23–25] Such a term emerges automatically in a nonlinear extended theory as a term of third order in the production of heat flux. This term takes into account, at the macroscopic level, the generation of vortices at the microscopic level.

VI. CONCLUSIONS

In this work, restricting the study to the linear phenomena, it is shown that the main properties of superfluid helium, both in the absence and in the presence of dissipation, can be explained using a monofluid model based on the extended irreversible thermodynamics formulated in Ref. 3.

The theory explains the propagation of two sounds of helium II with their typical properties. The attenuation for these sounds is in agreement with the experimental results.

The main difference between the monofluid theory presented here and the two-fluid model is that, while in the latter the thermal dissipation (needed to explain the attenuation of the second sound) is due to a dissipative term of a Fourier type, in the extended model it is a consequence of terms dependent on the gradient of the heat flux q_i, which are present in the expressions of the trace and the deviator of nonequilibrium stress, besides the traditional viscous terms. These quantities arise naturally in the evolution equations of this stress (trace and deviator) once the relaxation times of such dissipative fluxes are supposed zero.

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