Large-N expansion of $(4-\epsilon)$-dimensional oriented manifolds in random media

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(Received 29 March 1993)

The equilibrium statistical mechanics of a d-dimensional "oriented" manifold in an $(N+d)$-dimensional random medium are analyzed in $d=(4-\epsilon)$ dimensions. For $N=1$, this problem describes an interface pinned by impurities. For $d=1$, the model becomes identical to the directed polymer in a random medium. Here, we generalized the functional-renormalization-group method used previously to study the interface problem, and extract the behavior in the double limit ϵ small and N large, finding nonanalytic corrections in $1/N$. For short-range disorder, the interface width scales as $\omega \sim L^{\zeta}$, with $\zeta=[\epsilon/(N+4)]\{1+(1/4e)2^{-[(N+2)/2]}[(N+2)^2/(N+4)][1-4/(N+2)+\cdots]\}$. We also analyze the behavior for disorder with long-range correlations, as is appropriate for interfaces in random-field systems, and study the crossover between the two regimes.

I. INTRODUCTION AND SUMMARY

Oriented elastic manifolds embedded in spaces that contain random impurities that can pin the manifold occur in many physical systems. Both the dimension of the manifold, d, and the dimension of the space in which it is embedded, $D=d+N$, where N is the number of transverse dimensions, play important roles.[1,2] The case $d=1$ corresponds to a directed polymer in a random potential[3,4] which describes the interaction of a single flux line in a type-II superconductor with impurities.[5] Interfaces between two coexisting phases in D-dimensional systems are $(D-1)$-dimensional oriented manifolds whose properties control much of the behavior of such systems in the presence of randomness.[6–8] It has also been argued that flux lattices in superconductors have an intermediate distance regime in which they behave like an oriented manifold with $d=3$ and $N=2$.[9]

From a theoretical point of view, it has become clear in the last few years that elastic manifolds in random media exhibit much of the interesting and subtle phenomena that characterize spin glasses and other complicated disorder dominated phases.[4,10] In addition, the equilibrium statistical mechanics of directed polymers in a random potential can be mapped to the dynamics of an interface growing by random deposition—leading to insights into both problems.[11] In this paper we will analyze the equilibrium behavior of manifolds in random media for d just below the critical dimension of four, focusing on the limit of large N.

By definition, an oriented manifold has no overhangs; therefore, it can be described entirely in terms of a set of N transverse coordinates $\{\phi^i\}$, which are functions of the d internal coordinates $\{x^j\}$ parameterizing the manifold.[12] The Hamiltonian describing such a manifold in the presence of a quenched random potential $V(\phi(\mathbf{x}),\mathbf{x})$ is

$$H=\int d^d x\,\{\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+V(\phi(\mathbf{x}),\mathbf{x})\}\ , \tag{1.1}$$

where $\phi(\mathbf{x})\in\mathbb{R}^N$ is a vector describing the transverse coordinates of the manifold with internal coordinate $x\in\mathbb{R}^d$.

The random potential and/or thermal fluctuations will generally cause the manifold to roughen, resulting in divergent fluctuations in ϕ. It is conventional to parameterize these by a roughness exponent ζ:

$$\langle\{[\phi(\mathbf{x})-\phi(\mathbf{x}')]^2\}_T\rangle\sim|\mathbf{x}-\mathbf{x}'|^{2\zeta}\ , \tag{1.2}$$

where the inner curly brackets with a T subscript denote a thermal average, while the outer angular brackets indicate a statistical average over the random potential.

In the absence of the random potential, the interface will be flat for $d>2$, corresponding to $\zeta=0$, but be thermally rough with $\zeta=\zeta_T=(2-d)/2$ for $d\le 2$ (with logarithmic corrections in two dimensions). For $d<2$, weak randomness is irrelevant when $N>N_T=2d/(2-d)$, and an unpinned phase exists at high temperature.[13] For a strong random potential, low temperature or outside this regime of the $N-d$ plane, the randomness always dominates and the system is controlled by a nontrivial zero-temperature fixed point.

Much is known about the case of one internal dimension ($d=1$), owing both to the mapping to interface growth[11] and to the simplicity of numerical simulations.[14] When both $d=1$ and $N=1$, the roughness exponent is known to be exactly $\zeta=\frac{2}{3}$.[3,15] For $N\ge 2$, in addition to the low-temperature randomness-dominated phase, a high-temperature phase emerges, in which the disorder is irrelevant and $\zeta=\frac{1}{2}$. The value of ζ in the low-temperature phase has been investigated numerically for $N\ge 2$; it decreases with N and appears to approach $\frac{1}{2}$ as $N\to\infty$.[14] The possibility of a finite upper critical dimension, such that $\zeta=\frac{1}{2}$ for $N>N_c$, has been suggested by several authors,[2,16] but no evidence of this has appeared in rather extensive numerical simulations, and others have argued that no such upper critical dimension exists, but rather that ζ decreases continuously to $\frac{1}{2}$ as $N\to\infty$.[4]

For manifolds with $d>4$, a perturbative analysis[8] shows that the interface remains flat ($\zeta=0$). In this regime perturbation theory, or equivalently a simple renormalization-group (RG) treatment (briefly described

0163-1829/93/48(9)/5949(15)/$06.00

in Sec. II) is valid. For $d<2$, as mentioned above, both high-temperature pinned and low-temperature randomness-dominated phases exist for large enough N. Between two and four dimensions, only the disorder-dominated phase exists, characterized by a nontrivial ζ. Attempts to analyze this phase involved perturbative methods about four dimensions, e.g., Parisi and Sourlas,[17] which yield incorrect results [in particular $\zeta=(4-d)/2$] due to the existence of many extrema of the Hamiltonian Eq. (1.1). In order to carry out a proper ϵ expansion, a functional renormalization-group (RG) is needed. This was introduced by Fisher[8] for the case $N=1$, yielding the results $\zeta=\epsilon/3$ and $\zeta=0.2083\epsilon$ for random-field and random-bond disorder, respectively, to lowest order in $\epsilon=4-d$. The first result[7] is believed to be exact for $1<d<4$, while the second is an $O(\epsilon)$ calculation, which required a numerical solution of the RG fixed-point equation.

More recently, Mezard and Parisi[1] (MP) have performed an approximate "replica symmetry-breaking" calculation on the model for general N, and argued that their results are exact in the limit as $N\to\infty$. In the MP method, replica symmetry breaking is introduced as a *variational* ansatz. An interesting question to address is whether the replica symmetry breaking corresponds to some physical aspect of the problem (at least at large N) or is merely a feature of the restricted variational ansatz. This is a particularly interesting issue in light of our $O(\epsilon)$ treatment, which *does not* involve replica symmetry breaking, but is a systematic perturbative RG calculation. It is hoped that a comparison of the two methods may provide insight into whether replica symmetry breaking has any well-defined meaning.

In this paper, we generalize the $(4-\epsilon)$-RG calculation of Ref. 8 to arbitrary N. In the limit of large N, the fixed-point and stability equations become tractable, and we perform an expansion around this limit, working always only to first order in ϵ. For short-range correlated disorder, we find

$$\zeta_{\rm SR}=\frac{\epsilon}{N+4}\left\{1+\frac{1}{4e}2^{-[(N+2)/2]}\frac{(N+2)^2}{N+4}\times\left[1-\frac{4}{N+2}+\cdots\right]\right\}, \tag{1.3}$$

while for a random potential with long-range correlations transverse to the manifold,

$$\langle V(\phi,\mathbf{x})V(\phi',\mathbf{x}')\rangle_C\sim|\phi-\phi'|^{-\gamma}\delta^d(\mathbf{x}-\mathbf{x}')\ , \tag{1.4}$$

there is a continuously variable exponent which is independent of N,

$$\zeta_{\rm LR}=\epsilon/(4+\gamma)\ , \tag{1.5}$$

in agreement with previous results for the particularly interesting case of interfaces in random-field systems which corresponds to $N=1$ and $\gamma=-1$. The long-range fixed point becomes unstable when $\zeta_{\rm LR}<\zeta_{\rm SR}$. As in the $N=1$ case, the long-range result is believed to be exact, while the short-range is true only to $O(\epsilon)$.

Before proceeding with the RG calculation, we briefly outline the remainder of the paper. In Sec. II, the model is described, and an RG procedure is developed to analyze the zero-temperature fixed point in $4-\epsilon$ dimensions. It is shown that a perturbative expansion, which incorrectly deals with the physics of the many metastable states, breaks down. Section III analyzes the correct solution for the fixed point and roughness exponent in the large-N limit, in which analytic results can be obtained. The stability of this fixed point is analyzed in Sec. IV. The behavior for long-range correlated disorder (e.g., random fields) is studied in Sec. V, as well as the stability of the associated fixed points. In Sec. VI we summarize our conclusions and open questions and suggest several possible directions for future work. Appendixes A and B contain various technical details, while Appendix C rederives the RG relations by an iterative minimization of the Hamiltonian and discusses the appearance of many minima. Finally, Appendix D analyzes possible multicritical fixed points.

II. MODEL AND ZERO-TEMPERATURE RENORMALIZATION GROUP

The partition function in terms of the Hamiltonian Eq. (1.1) is

$$Z\{V\}=\int[d\phi]\exp(-H/T)\ . \tag{2.1}$$

We take the random potential V to have a Gaussian distribution, with the two-point correlation function

$$\langle V(\phi,\mathbf{x})V(\phi',x')\rangle=R(\phi-\phi')\delta^{(d)}(\mathbf{x}-\mathbf{x}')\ , \tag{2.2}$$

with a short-distance scale implicitly included in the δ function. To order $\epsilon=4-d$ higher cumulants will be shown to be irrelevant.

To organize a renormalization-group (RG) treatment of the problem, we employ the replica method of averaging over the disorder. Note, however, that we do this only to organize the perturbation expansion. The partition function is now a random variable, and rather than follow the flow of its distribution function directly under the RG, one can follow the full set of moments. These are readily averaged over, yielding terms of the form

$$\overline{Z^p}=\int[d\phi]\exp(-\tilde{H}_p)\ , \tag{2.3}$$

with

$$\tilde{H}_p=\int d^dx\left\{\frac{1}{2T}\sum_\alpha\nabla\phi^\alpha\cdot\nabla\phi^\alpha-\frac{1}{2T^2}\sum_{\alpha\beta}R(\phi^\alpha-\phi^\beta)-\frac{1}{T^3}\sum_{\alpha\beta\gamma}S(\phi^\alpha-\phi^\beta,\phi^\alpha-\phi^\gamma)-\cdots\right\}, \tag{2.4}$$

where the terms with three or more replicas result from non-Gaussian correlations in the disorder; these are generated at higher order in ϵ. We have not included terms with additional gradients, which will be irrelevant. The symmetry under simultaneous shifts of all the replica coordinates corresponds to the statistical symmetry of the disorder under spatial translation normal to the manifold. We have used p instead of the more conventional n for the number of replica indices to avoid confusion with the transverse dimension N of the manifold.

The momentum shell RG approach we employ consists of integrating out high-momentum modes in a shell with $\Lambda/b < |\mathbf{p}| < \Lambda$. We take the limit in which the width of this shell is infinitesimal, i.e., $b = e^{dl}$, which simplifies the formulas somewhat. To keep the cutoff fixed, momenta, coordinates, and fields are rescaled according to

$$\begin{aligned} \mathbf{p} &= \mathbf{p}'/b \ , \\ \mathbf{x} &= b\mathbf{x}' \ , \\ \phi(\mathbf{x}) &= b^{\zeta}\phi'(\mathbf{x}') \ . \end{aligned} \tag{2.5}$$

A simple first attempt at an RG analysis consists of expanding the function $R(\phi)$ in a power series, and analyzing the results term by term. Since R is an even function, only even powers of ϕ appear in such an expansion, i.e.,

$$\tilde{H}_{\text{int}} = -\frac{1}{2T^2}\sum_{\alpha,\beta}\sum_{m}\frac{R_m}{m!}(\phi^\alpha-\phi^\beta)^{2m} \ . \tag{2.6}$$

Above four dimensions, the quartic and higher vertices are irrelevant and the theory flows to a simple Gaussian fixed point. Below four dimensions the quartic interaction becomes relevant, and one might hope to make a simple ϵ expansion by going to second order. A simple calculation shows that no perturbative fixed point exists, so that a strong-coupling analysis is necessary.

Physically, the behavior below four dimensions is dominated by the randomness, and should be described by a zero-temperature fixed point. By allowing temperature to renormalize, it is possible to organize an analysis of this fixed point. The RG flows which arise from the scale changes are then

$$\begin{aligned} \left.\frac{dT}{dl}\right|_{\text{SC}} &= (2-d-2\zeta)T \ , \\ \left.\frac{\partial R(\phi)}{\partial l}\right|_{\text{SC}} &= (4-d-4\zeta)R(\phi)+\zeta\phi_i\partial_i R(\phi) \ , \\ \left.\frac{\partial S(\phi_1,\phi_2)}{\partial l}\right|_{\text{SC}} &= (6-2d-2\zeta)S(\phi_1,\phi_2) \\ &\quad +\zeta(\phi_{1i}\partial_{1i}+\phi_{2i}\partial_{2i})S(\phi_1,\phi_2) \ , \\ &\quad \cdots \ . \end{aligned} \tag{2.7}$$

The $\zeta\phi^i\partial_i R(\phi)$ term in the second equation comes from the infinitesimal field rescaling of Eq. (2.5).

The flow equation [Eq. (2.7)] for the temperature is actually exact, due to the "Galilean" invariance of the Hamiltonian Eq. (1.1). This can be seen by considering the change in the free energy corresponding to a uniform tilt of the manifold (or equivalently a change in boundary conditions). If the fields are shifted by a linear function of the coordinates $\phi \to \phi + \mathbf{v}^i x^i$, the probability distribution of the new free energy is identical to that of the old one plus an additive constant

$$\Delta F \equiv \Delta(-T\ln Z) = \tfrac{1}{2}|\mathbf{v}|^2 L^d \ . \tag{2.8}$$

This is an exact statement about the model, and must therefore be true at all stages of the RG; it requires that T (i.e., the coefficient of the stiffness term) only be renormalized by the scale changes.[18] We thus have

$$\frac{dT}{dl} = -\theta T \ , \tag{2.9}$$

with $\theta = 2\zeta + d - 2$ determining the scaling of energies at the zero-temperature fixed point.

One can again attempt to proceed by expanding the function $R(\phi)$ in a power series. The condition fixing the quadratic term R_2 in Eq. (2.6) is then $\zeta = (4-d)/2 = \epsilon/2$. However, since ζ is of order ϵ, all the higher terms in the expansion of $R(\phi)$ also become relevant below four dimensions. It is therefore necessary to keep track of the entire series of $\{R_m\}$, or the entire function $R(\phi)$.

The one-loop RG equations are best derived in their functional form, through the use of Fourier transforms. As an example, consider the *first-order* feedback of the $R(\phi)$ term. As noted above, it will not renormalize itself, but it will contribute to the free energy. The first step in the calculation is to Fourier transform the interaction term,

$$-\tilde{H}_{\text{int}} = \frac{1}{2T^2}\sum_{\alpha,\beta}\int_{\mathbf{x},\kappa} e^{i\kappa\cdot[\phi^\alpha(\mathbf{x})-\phi^\beta(\mathbf{x})]}\tilde{R}(\kappa) \ , \tag{2.10}$$

where we use Greek and Latin letters for Fourier transforms perpendicular (ϕ direction) and parallel ($\mathbf{x}$ direction) to the manifold, respectively, and $\int_{x,\kappa} \equiv \int d^d\mathbf{x}\, d^d\kappa/(2\pi)^d$.

To perform the elimination part of the RG transformation, the fields are split into slowly and rapidly moving parts,

$$\phi(\mathbf{x}) = \phi_<(\mathbf{x}) + \phi_>(\mathbf{x}) \ , \tag{2.11}$$

and a trace is performed over the "fast" fields $\phi_>(\mathbf{x})$. In Eq. (2.10), the slow and fast terms separate into two exponential factors, and it is a simple matter to average over the fast modes. Dropping the label (<) on the slow fields, the traced term becomes

$$\begin{aligned} -\langle \tilde{H}_{\text{int}}\rangle_> = \frac{1}{2T^2}\sum_{\alpha,\beta}\int_{\mathbf{x},\kappa} & e^{i\kappa\cdot[\phi^\alpha(\mathbf{x})-\phi^\beta(\mathbf{x})]}e^{-\kappa^2 T G_>(0)(1-\delta^{\alpha\beta})} \\ & \times \tilde{R}(\kappa) \ , \end{aligned} \tag{2.12}$$

where the free two-point function is

$$G_>(\mathbf{x}) = \int_{\mathbf{q}}^{>} e^{i\mathbf{q}\cdot\mathbf{x}}/q^2 \ , \tag{2.13}$$

where the > on the integral denotes integration over momenta ($d^d\mathbf{q}/(2\pi)^d$) in the shell only. When evaluated at $\mathbf{x}=0$, this function has the well-defined limit

$$G_>(0) = (2\pi)^{-d}S_d\Lambda^{d-2}dl \equiv A_d\, dl \ , \tag{2.14}$$

where S_d is the surface area of a unit sphere in d dimensions. The second exponential in Eq. (2.12) can thus be expanded to yield

$$-\langle \tilde{H}_{\text{int}} \rangle_{>} = \frac{A_d dl}{2T} \int_{\mathbf{x}} \left\{ \sum_{\alpha,\beta} \partial_i \partial_i R(\phi^\alpha - \phi^\beta) - p \partial_i \partial_i R(0) \right\} , \tag{2.15}$$

where the ∂_i act on the internal coordinates of the functions $R(\phi)$ (not on the spatial coordinates $\mathbf{x}$) and repeated Latin indices are summed from 1 to N. The first term is of the appropriate form to feed back into $R(\phi)$, but it is reduced by a factor of T from the term in Eq. (2.4). Physically, this renormalization is due to the averaging of the potential by thermal fluctuations, and is thus negligible at the zero-temperature fixed point of interest. The second term in Eq. (2.15) contributes to the renormalization of the free-energy density

$$\frac{df}{dl} = (d-\theta) f - \frac{A_d}{2} \partial_i \partial_i R(0) + \cdots . \tag{2.16}$$

We now proceed with the analysis to second order in $R(\phi)$; one must multiply two terms and take the connected expectation value over $\phi_>$. Performing the average, one finds the expression

$$\begin{aligned} \frac{1}{8T^4} \sum_{\substack{\alpha_1,\beta_1 \\ \alpha_2,\beta_2}} \int_{\substack{x_1,\kappa_1 \\ x_2,\kappa_2}} & \exp\{ i\kappa_1 \cdot [\phi_1^\alpha(\mathbf{x}_1) - \phi_1^\beta(\mathbf{x}_1)] + i\kappa_2 \cdot [\phi_2^\alpha(\mathbf{x}_2) - \phi_1^\beta(\mathbf{x}_2)] \} \\ & \times \exp\{ -\kappa_1^2 T G_>(0)(1-\delta^{\alpha_1\beta_1}) - \kappa_2^2 T G_>(0)(1-\delta^{\alpha_2\beta_2}) \} \tilde{R}(\kappa_1) \tilde{R}(\kappa_2) \\ & \times \exp\{ -\kappa_1 \cdot \kappa_2 T G_>(\mathbf{x}_1 - \mathbf{x}_2) [\delta^{\alpha_1\alpha_2} - \delta^{\beta_1\beta_2} - \delta^{\alpha_1\beta_2} - \delta^{\beta_1\alpha_2}] \} - \text{disconnected parts} . \end{aligned} \tag{2.17}$$

The terms resulting from expanding the $G_>(0)$ parts are canceled by disconnected pieces, leaving only the expansion of the final term. Expansion of the final exponential gives terms proportional to $1/T^3$ and lower powers of T. The $1/T^3$ term is a three-replica contribution,

$$\frac{1}{2T^3} \sum_{\alpha,\beta,\gamma} \int_{\mathbf{x},\mathbf{x}'} \partial_i R[\phi^\alpha(\mathbf{x}) - \phi^\gamma(\mathbf{x})] \partial_i R[\phi^\alpha(\mathbf{x}') - \phi^\beta(\mathbf{x}')] G_>(\mathbf{x}-\mathbf{x}') , \tag{2.18}$$

since a single Kronecker delta function leaves three free replica indices. Equation (2.18), however, contributes only at large momenta, since $G_>(\mathbf{x})$ contains an integral only over momenta in the shell. Because the generated three-replica term exists only at large momentum, it cannot feed back to generate a zero-momentum term until second order. The resulting contribution to the renormalization of R will then turn out to be higher order in ϵ, since the three-replica piece will be $O(\epsilon^2)$, anticipating that R will be $O(\epsilon)$ at the fixed point. The situation is analogous to the neglect (to lowest order) of the ϕ^6 term in momentum-shell RG for the conventional $\lambda\phi^4$ theory, in which a ϕ^6 interaction with large momentum is generated from two ϕ^4 terms, but does not feed back in a dangerous manner. Figure 1 shows diagrammatically how the three-replica term is generated and feeds back into the two-replica piece.

The $1/T^2$ parts generate both two- and three-replica terms. The three-replica term is down by a factor of T, and can be neglected at the zero-temperature fixed point. Keeping track of the factors, one finds a contribution

$$\begin{aligned} \left. \frac{\partial R(\phi)}{\partial l} \right|_{O(R^2)} = \frac{1}{2T^2} \sum_{\alpha,\beta} \int_{\mathbf{x},\mathbf{x}'} & G_>(\mathbf{x}-\mathbf{x}')^2 \{ \partial_i \partial_j R[\phi^\alpha(\mathbf{x}) - \phi^\beta(\mathbf{x})] \partial_i \partial_j R[\phi^\alpha(\mathbf{x}') - \phi^\beta(\mathbf{x}')] \\ & - 2\partial_i \partial_j R[\phi^\alpha(\mathbf{x}) - \phi^\beta(\mathbf{x})] \partial_i \partial_j R(0) \} . \end{aligned} \tag{2.19}$$

Since we are interested in the renormalization of the long-wavelength portion, the kernel $K(\mathbf{x}) \equiv G_>(\mathbf{x})^2$ must be evaluated at zero momentum, i.e.,

$$\tilde{K}(\mathbf{p}=0) = \int_{\mathbf{x}} G_>(\mathbf{x})^2 = \int_{\mathbf{x},\mathbf{p},\mathbf{p}'}^{>} \frac{\exp[i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}]}{\mathbf{p}^2 \mathbf{p}'^2} = \int_{\mathbf{p}}^{>} \frac{1}{\mathbf{p}^4} = S_d (2\pi)^{-3} \Lambda^{d-4} dl . \tag{2.20}$$

By rescaling $R(\phi)$ by a constant multiple, this factor can be removed. The full RG equation for R then becomes

$$\frac{\partial R(\phi)}{\partial l} = (4-d-4\zeta) R(\phi) + \zeta \phi^i \partial_i R(\phi) + [\tfrac{1}{2} \partial_i \partial_j R(\phi) \partial_i \partial_j R(\phi) - \partial_i \partial_j R(\phi) \partial_i \partial_j R(0)] + O(R^3) . \tag{2.21}$$

This equation can also be formally derived by expanding $R(\phi)$ in $\phi_>$ and keeping terms up to $O(\phi_>^4)$, although such a treatment does not properly treat the case of nonanalytic $R(\phi)$. The flow equation (2.21) is exactly equivalent to the infinite series of RG equations obtained from all one-loop diagrams in an ordinary diagrammatic approach. It was used previously in Refs. 2 and 19. In Appendix C, we derive the RG equation schematically by directly minimizing H over the fast degrees of freedom $\phi_>$ without the use of replicas or field-theoretic techniques.

At this point, one may try to directly analyze the flows and fixed points of Eq. (2.21): If a fixed point $R \sim \epsilon$ is found, then the other terms in $\tilde{H}_p$ will not play a role to

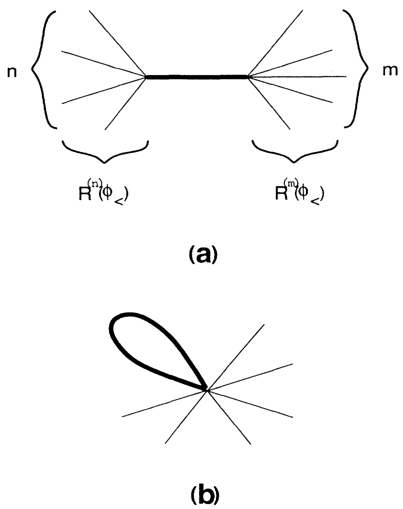

FIG. 1. Diagrams representing the generation and feedback of the three-replica term S [see the third line of Eq. (2.7)]. (a) shows how such an operator is generated at second order in R by terms with one contraction. Since the momentum of the internal line is within the momentum shell, the diagram only contributes at large momentum. Summing up all such terms resulting from the expansion of R [Eq. (2.22)] gives Eq. (2.18). Since the fixed-point value of S is $O(\epsilon^2)$, the only potentially dangerous contribution for our analysis is first order in S. This feedback comes from diagrams such as that of (b), with a single loop. Since the original S vertices were generated only at higher momentum, such terms do not renormalize R.

$O(\epsilon)$. We will take this approach in Sec. III, but first it is instructive to investigate the manner in which the standard polynomial RG breaks down. A conventional RG approach is equivalent to expanding $R(\phi)$ in a power series

$$R(\phi)=\sum_m \frac{R_{2m}}{(2m)!}\phi^{2m} , \tag{2.22}$$

and following the flow equations for the coefficients. The first two equations are

$$\begin{aligned}\frac{\partial R_2}{\partial l}&=(\epsilon-2\zeta)R_2 ,\\ \frac{\partial R_4}{\partial l}&=\epsilon R_4+\frac{N+8}{3}R_4^2 .\end{aligned} \tag{2.23}$$

While the quadratic term could be fixed by requiring $\zeta=\epsilon/2$ (the naive perturbative result), R_4 then flows off and is not stabilized at second order. Examination of the flow equation for R_4 shows that it becomes infinite after a finite amount of renormalization. This is an artifact of the truncation to second order; nevertheless, R_4 will rapidly become $O(1)$.

We thus see that there is no fixed point of the perturbative RG for which $R(\phi)$ is analytic. It is the assumption of analyticity which leads to this conclusion, and we shall see that nonanalytic fixed points of Eq. (2.21) do exist. The correct behavior for small ϕ can be found by directly examining Eq. (2.21) with $\partial R/\partial l=0$. This will be done in Sec. III.

Before proceeding, it is useful to recall how a normal perturbative analysis leads to drastically wrong results. Various methods have been used to show that, in a field-theoretic expansion, formally R_2 has no nontrivial renormalizations.[17] An apparent fixed point can then be found by setting $\zeta=\epsilon/2$ to all orders in ϵ. This results in the so-called "dimensional reduction" result $\theta=2$. In the supersymmetric formulation of Parisi and Sourlas which averages over *all* of the extrema of H with positive and negative weights, the only operator which appears corresponds to R_2. Thus this analysis completely misses the flow of the other operators, such as R_4, out of the regime in which a perturbative analysis might have been valid. As we discuss in the conclusion, it is likely that there is an exact upper bound for ζ which is violated by the naive perturbative result.

III. FIXED-POINT ANALYSIS IN THE LARGE-N LIMIT

In this section, we analyze the behavior of the fixed points in $(4-\epsilon)$ dimensions in the limit of large N. We look for solutions where the R is $O(1/N)$ by rescaling $R\to R/N$. The flow equation [Eq. (2.21)] then takes the form

$$\begin{aligned}\frac{\partial R(\phi)}{\partial l}=&(4-d-4\zeta)R(\phi)+\zeta\phi^i\partial_i R(\phi)\\ &+\frac{1}{N}[\tfrac{1}{2}\partial_i\partial_j R(\phi)\partial_i\partial_j R(\phi)\\ &-\partial_i\partial_j R(\phi)\partial_i\partial_j R(0)] .\end{aligned} \tag{3.1}$$

For rotationally invariant solutions, the ansatz $R(\phi)=Q(\phi^2/2)$ yields

$$\begin{aligned}\frac{\partial Q}{\partial l}=&(4-d-4\zeta)Q+2\zeta yQ'+\frac{1}{2}(Q')^2-Q'Q_0'\\ &+\frac{1}{N}[2yQ'Q''+2y^2(Q'')^2-2yQ''Q_0'] ,\end{aligned} \tag{3.2}$$

where primes denote differentiation with respect to y, defined

$$y\equiv\phi^2/2 , \tag{3.3}$$

and

$$Q_0'\equiv Q'(y=0) . \tag{3.4}$$

Differentiating once and pulling out the $O(\epsilon)$ factor by defining

$$Q'=-(4-d-2\zeta)\frac{N}{N+2}u ,$$

and (3.5)

$$t=(4-d-2\zeta)l ,$$

gives the final form of the fixed-point equation:

$$\begin{aligned}\frac{\partial u}{\partial t}=&u+\beta yu'+u'[u(0)-u]\\ &-\mu\{3y(u')^2+2y^2u'u''+yu''[u-u(0)]\}=0 , \end{aligned} \tag{3.6}$$

with

$$\beta \equiv 2\zeta/(4-d-2\zeta) ,$$
$$\mu \equiv 2/(N+2) . \tag{3.7}$$

Any physical solution of this equation will have a finite value of u at the origin, so that by a choice of scale we can set

$$u(0)=1 . \tag{3.8}$$

The behavior at large y will fix the value of β in a way which is somewhat analogous to more conventional eigenvalue problems. If we start with potential correlations which are short range, it is natural to expect the fixed points also to correspond to short-range correlations. We thus look for fixed points with $R(\phi)$ decaying rapidly for large ϕ, i.e., $u(y)$ decaying rapidly for large y.

Naively, in the large-N limit the terms proportional to μ in Eq. (3.6) can be simply dropped, and the resulting problem has a much simpler form:

$$u_\infty + \beta y u'_\infty + u'_\infty(1-u_\infty)=0 . \tag{3.9}$$

We will see that this approximation is not valid globally, but this equation determines the primary solution, called the outer solution in the language of boundary-layer theory.[20] This is in a sense the solution for $N=\infty$. The boundary layer, or region in which this solution is not valid, occurs for large y. The solution in this tail region is conventionally denoted the inner solution (as it is valid inside the boundary layer). The primary limit [Eq. (3.9)] has only power-law solutions at infinity unless $\beta=0$. We thus tentatively choose this value (anticipating later corrections for large but finite N). In this case the solution to Eq. (3.9) is given implicitly by

$$u_\infty - \ln u_\infty = y+1 , \tag{3.10}$$

using the boundary condition $u(0)=1$. For small y, this equation yields two possible behaviors for $u(y)$,

$$u(y) \approx 1 \pm \sqrt{2y} . \tag{3.11}$$

We choose the minus solution to obtain a solution decaying as $\exp(-y)$ for large y. The value $\beta=0$ corresponds to $\zeta=0$, so that the interface remains flat to $O(\epsilon)$ at $N=\infty$. A more careful analysis yields the detailed form of the corrections.

To expand towards finite N, the natural first step is to examine perturbatively the effect of the terms dropped in Eq. (3.9). Since the zeroth-order solution is exponential for large y, it is immediately clear that perturbation theory breaks down in this regime, due to the presence of the term

$$\mu y u'' \gg u \quad \text{for } y \gg 1/\mu ,$$
$$\text{with} \tag{3.12}$$
$$u \sim \exp(-y) .$$

The solution Eq. (3.10) is therefore valid (even approximately) only for $y \ll 1/\mu$, and we thus have a boundary layer for $y \gg 1/\mu$.

Fortunately, we can proceed with the analysis by noting that for $y \gg 1$, the nonlinear terms in Eq. (3.6) become negligible, and the equation can be reduced to the form

$$u + \beta y u' + u' + \mu y u'' = 0 , \tag{3.13}$$

where we have allowed for $\beta \neq 0$ [but $O(\mu)$], which will turn out to be the case for N large. Equation (3.13) is valid within the boundary layer. It is important to note that both this equation for the tail and the primary equation [Eq. (3.9)] are valid in the (asymptotically infinite) region $1 \ll y \ll 1/\mu$, which makes it possible to match the solutions of the two equations in this domain (see Fig. 2).

The linear tail equation [Eq. (3.13)] is second order, and therefore has two independent solutions. For $\mu y, \beta y \ll 1$, the solutions are

$$u_a(y) \sim Ce^{-y} ,$$
$$u_b(y) \sim Cu^{-1/u} . \tag{3.14}$$

From the behavior of the primary solution, Eq. (3.10), in the matching region $y \gg 1$, we see that we must choose the solution $u_a(y)$ (with possibly a small admixture of u_b vanishing in the $N \to \infty$ limit). For a given value of β, both the primary and tail solutions are thus completely determined, giving a uniformly valid solution to the full equation [Eq. (3.6)] for large N.

For short-range correlated disorder, the bare unrenormalized function $u(y)$ decays exponentially (or more rapidly) for large y. It is straightforward to see that this exponential decay is preserved by the flows, from Eq. (3.6), or directly from Eq. (3.1). Higher-order terms from higher loops can generate at most power-law corrections to the initial exponential behavior (from terms with all but one of the R's evaluated at $\phi=0$). The nonrenormalization of the exponential behavior can be seen in a schematic way directly from the Hamiltonian. When a particular fast mode is integrated out, as in the previous section, the integral performed is of the form

$$\int [d\phi_>] \exp[-\phi_>^2/2 + R(\phi_< + \phi_>)] . \tag{3.15}$$

For large $\phi_<$, the contributions from small and large $\phi_>$ can be easily estimated. For small $\phi_>$, the argument of R is large, so that it can be approximated by an exponential

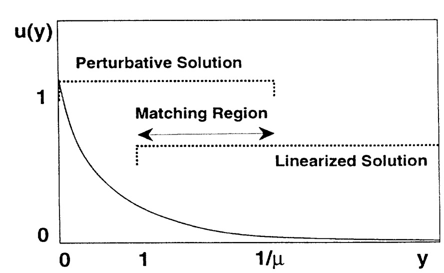

FIG. 2. Regions of validity of the primary (perturbative) solution, Eq. (3.9), and the tail (linearized) solution, Eq. (3.13). In the large-N (small-μ) limit, the size of the matching region ($1 \ll y \ll 1/\mu$) grows without bound.

decay $R\sim\exp(-\phi_<^2/2)$, and the integral yields an exponentially decaying function of $\phi_<$. For large $\phi_>$, the integral is dominated by $\phi_>\approx-\phi_<$, so that now the quadratic term dominates and again yields an exponential function of $\phi_<$, of the exact form that corresponds to $u(y)\sim\exp(-y)$.

To be a valid fixed-point function for short-range correlated disorder, therefore, $u(y)$ must have an exponential tail at large y. By analyzing the tail equation [Eq. (3.13)] for $y\gg 1/\beta, 1/\mu$, we again find two possible behaviors:

$$u_c(y)\sim Cy^{1/\beta-1/\mu}e^{-(\beta/\mu)y}\,,$$
$$u_d(y)\sim Cy^{-1/\beta}\,. \tag{3.16}$$

For most values of β, the required solution $u_a(y)$ in the intermediate region $1\ll y\ll 1/\mu$, $1/\beta$ will be a linear combination of $u_c(y)$ and $u_d(y)$, and thus will have a power-law tail at large y. For some special values of β, however, $u_a(y)$ will correspond exactly to $u_c(y)$, so that the power-law tail of $u_d(y)$ does not contribute. This will be the eigenvalue-like condition that determines β, and hence the short-range roughness exponent $\zeta_{\rm SR}$.

It is, in fact, simple to guess one such value of β. From the fact that β vanishes for $N=\infty$ ($\mu=0$), one expects that $\beta=O(\mu)$. It is easy to check that for $\beta=\mu$, an exact solution of Eq. (3.13) satisfying the matching conditions is $u=e^{-1-y}$. This value of β corresponds to a roughening exponent of

$$\zeta\approx\epsilon/(4+N)\,, \tag{3.17}$$

which we anticipate will be valid for large N.[21] Note that this agrees with a recent replica symmetry-breaking calculation, which was claimed to be valid in the large-N limit.[1]

To compute the next-order corrections to the result [Eq. (3.17)], we need to consider in detail the effects of the neglected terms. For the primary solution ($y\ll 1/\mu$), these can be computed perturbatively in μ. However, they will not affect β unless the nonlinearities for $y\gg 1$ are also taken into account, since the solution found above can be scaled to match the corrections to the primary solution. We therefore first consider the effects of nonlinearities for $y\gg 1$.

To look for solutions close to the original one [u_∞ in Eq. (3.10)], we make the change of variables

$$u=\exp\left[-y-\int_0^y dy'\sigma(y')\right]\,,$$
$$\beta=\mu(1+b)\,, \tag{3.18}$$
$$\mu y=\eta\,,$$

where the final change of variables was made to concentrate on the change of character of the solution for $y\sim 1/\mu$. With some algebra, Eq. (3.6) is transformed to

$$\left[-b+\frac{\eta-1}{\eta}\right]\sigma-b+\sigma^2-\mu\frac{d\sigma}{d\eta}+g(\eta)=0\,, \tag{3.19}$$

where

$$g\equiv\frac{\exp(-y-\int_0^y\sigma)}{\eta}\left[(1+\sigma)-4\eta(1+\sigma)^2+2\frac{\eta^2}{\mu}(1+\sigma)^3+[2\eta^2(1+\sigma)+\mu n]\frac{d\sigma}{d\eta}\right]\,. \tag{3.20}$$

We anticipate that $\sigma(\eta)$ will be exponentially small (in $1/\mu$) for $\eta\sim 1$. Then we can ignore terms of order σ^2 and also terms of order $\sigma(\eta)\exp(-\eta/\mu)$. We see, however, that $g(\eta)$ has a part which depends on the behavior of $\sigma(\eta)$ for η of order $1/\mu$ (i.e., y of order one) via u, Eq. (3.18). We can obtain this by perturbing in β and μ about the solution u_∞ which is good in this region. Anticipating that $b\ll 1$, a perturbative calculation performed in Appendix A yields

$$u\approx u_\infty+\mu u_1\approx e^{-1-y}[1-\mu+O(\mu^2)]\quad\text{for } y\gg 1\,. \tag{3.21}$$

This expression is valid for $\eta\ll 1$ but we will see that the corrections embodied in Eq. (3.18) will be small for the desired full solution. From Eq. (3.21), the g term becomes

$$g=\frac{e^{-1-\eta/\mu}}{\eta}\left[\frac{2\eta^2}{\mu}+1-4\eta-2\eta^2+O(\mu)\right]\,. \tag{3.22}$$

Neglecting the σ^2 [and $\sigma\exp(-\eta/\mu)$] terms in Eq. (3.19), we obtain a linear homogeneous equation for σ. This can be solved straightforwardly by introducing the integrating factor

$$F(\eta)\equiv\exp\left[-\frac{1}{\mu}\int^\eta d\tilde\eta\left[\frac{\tilde\eta-1}{\tilde\eta}-b\right]\right]$$
$$=\exp\left\{\frac{1}{\mu}[(b-1)\eta+\ln\eta]\right\}\,, \tag{3.23}$$

whence

$$\sigma(\eta)\approx\frac{1}{\mu F(\eta)}\left\{C+\int_0^\eta d\tilde\eta F(\tilde\eta)[g(\tilde\eta)-b]\right\}\,. \tag{3.24}$$

For $\eta\to 0$, this must match onto the solution Eq. (3.21) implying that, since $F(\eta)$ vanishes for small η, the integration constant C must be zero. At the other end, since $F(\eta)$ also vanishes for large η, σ will diverge unless the integral in Eq. (3.24) is zero. We thus obtain the integral condition

$$\int_0^\infty d\eta[b-g(\eta)]e^{-1/\mu[\eta-\ln\eta]}=0\,. \tag{3.25}$$

With the condition of Eq. (3.25), Eq. (3.24) yields

$$\sigma(\eta)\approx\frac{b}{\mu F(\eta)}\int_\eta^\infty d\tilde\eta\,F(\tilde\eta)$$
$$\approx\frac{b}{1-b}+\frac{b}{\eta(1-b)^2}+O(\eta^{-2})\,, \tag{3.26}$$

for $\eta\gg 1$ and hence

$$u(y)\sim y^{-b/[\mu(1-b)^2]}\exp\left[-y\left[1+\frac{b}{1-b}\right]\right]\,, \tag{3.27}$$

for $y \gg 1/\mu$. Up to corrections of order b^2 in σ [which arise from the neglected terms in Eq. (3.19)] this agrees with the behavior of $u_c(y)$ from Eq. (3.16). We have thus found the desired exponentially decaying full solution valid for the full range of y.[22]

As $\mu \to 0$, both terms in Eq. (3.25) can be evaluated by steepest descents. Expansion of each term around the saddle points $\eta^* \in \{1, \frac{1}{2}\}$ yields

$$b = \frac{1}{2e} 2^{-1/\mu} \frac{1}{\mu} [1 - 2\mu + O(\mu^2)] , \tag{3.28}$$

which is exponentially small for small μ, justifying our approximations. We are now in a position to obtain the roughness exponent from Eq. (3.28). For large N, we have

$$\begin{aligned} \zeta &= \frac{\epsilon}{N+4} \left[1 + \frac{b}{1+\mu} \right] \\ &= \frac{\epsilon}{N+4} \left\{ 1 + \frac{1}{4e} 2^{-[(N+2)/2]} \frac{(N+2)^2}{N+4} \right. \\ &\qquad \left. \times \left[1 - \frac{4}{N+2} + \cdots \right] \right\} . \end{aligned} \tag{3.29}$$

Note that an *approximate* analysis of the RG flow [Eq. (3.1)] by Natterman and Leschhorn[19] also gave the same prefactor, but a different exponentially small correction. If we naively truncate the series after the "1" in the last bracket, we obtain for $N=1$,

$$\zeta(N=1) = \frac{\epsilon}{5}(1 + 0.0585) \approx 0.2117\epsilon . \tag{3.30}$$

It is clear from the increasing nature of the next term that the series is asymptotic; nevertheless, the magnitude of the correction to $\epsilon/5$ compares fairly well with the direct numerical solution of Eq. (3.1) for $N=1$, which yielded $\zeta = 0.2083\epsilon$.

In Appendix D, we show that in addition to the fixed point found in this section, there is a discrete infinite series of fixed points with $u(y)$ changing sign but still decaying rapidly. At this point, whether these are physically meaningful is unclear.

IV. STABILITY ANALYSIS

In this section, we analyze the stability of the short-range fixed point found in the previous section to perturbations with both short- and long-range correlations. Since this fixed point represents a *phase* of the system, rather than a critical point, there should be no short-range perturbations which are relevant. From the scale invariance of the fixed-point equation, it is clear, however, that there is at least a marginal operator connecting the line of short-range fixed points with different $u(0)$. This does not alter the physics, since it just corresponds to redefinitions of dimensional quantities and is thus an uninteresting *redundant* operator. Intuitively, one expects perturbations with sufficiently long-range correlations to be relevant, causing the system to flow to an appropriate long-range fixed point; we will see that this is indeed the case.

We look for the eigenoperators in the usual way, by considering a function $u(y)$ initially very close to a fixed point solution $u^*(y)$, i.e.,

$$u(y) = u^*(y) + v(y) , \tag{4.1}$$

where $v(y)$ is a small perturbation. Inserting this into the flow equation, Eq. (3.6), and keeping only terms first order in $v(y)$ gives

$$\begin{aligned} \frac{\partial v}{\partial t} &= v + v'(1-u^*) + u^{*\prime}[v(0) - v] + \beta y v' \\ &\quad + \mu y [v'' + u^{*\prime\prime} v(0)] \\ &\quad + \mu\{6yu^{*\prime}v' + 2y^2(u^{*\prime}v'' + u^{*\prime\prime}v') \\ &\qquad + y[u^{*\prime\prime}(v - v(0)) + (u^* - 1)v'']\} , \end{aligned} \tag{4.2}$$

where we have chosen to perturb around the fixed point with $u^*(0)=1$. Note that it is not permitted at this point to choose $v(0)$, since it may not be a constant. The right-hand side of Eq. (4.1) can be thought of as a (nonlocal) linear operator acting on $v(y)$. Just as in finite-dimensional RG's, solutions with simple exponential t dependence can be found (the question of completeness is discussed in Ref. 23) if the spatially dependent part obeys the eigenvalue equation

$$\begin{aligned} v + v'(1-u) + u'(1-v) + \beta y v' + \mu y (v'' + u'') \\ + \mu O(uv) = \lambda v , \end{aligned} \tag{4.3}$$

where we have dropped the asterisk on $u(y)$ and chosen $v(0)=1$. Since eigenvectors are defined only up to a constant, we have the freedom to choose a scale for $v(y)$. It is straightforward to show that there are no eigenfunctions with $v(0)=0$, the only choice not equivalent by a choice of scale to $v(0)=1$. This is shown in Appendix B. The terms in the curly brackets in Eq. (4.2) are of order μuv. For small μ, they are small for all y, and will be neglected in what follows.

As for the fixed-point equation, the solutions to the eigenvalue equation can be found in two regions and the pieces matched asymptotically. For $y \ll 1/\mu, 1/\beta$, the equation can be rewritten in the standard form for first-order linear differential equations by dropping the $O(\mu uv)$ terms,

$$v'(y) + a(y)v(y) = b(y) , \tag{4.4}$$

where

$$\begin{aligned} a(y) &= \left[\frac{1-\lambda-u'(y)}{1-u(y)} \right] , \\ b(y) &= \frac{-u'(y)}{1-u(y)} . \end{aligned} \tag{4.5}$$

Using the fact that $u(y)$ satisfies Eq. (3.9) (with $\beta=0$), and taking some care due to the singularity in $u'(y)$ for small y, one finds the solution

$$v = \frac{1}{\lambda} \frac{u^{1-\lambda} - u}{1-u} , \tag{4.6}$$

for $y \ll 1/\mu, 1/\beta$. In the matching regime, $y \gg 1$, this solution behaves like

$$v \sim e^{-1-y}[e^{\lambda y}-1]/\lambda \; . \tag{4.7}$$

For $y \gg 1$, Eq. (4.3) reduces to a linear equation very similar to the fixed-point equation [Eq. (3.13)] in this regime,

$$\mu y v'' + (1+\beta y)v' + (1-\lambda)v = -u' - \mu y u'' \; . \tag{4.8}$$

As in the preceding section (for the behavior as a function of β), one expects a discrete set of exponential solutions which match onto Eq. (4.7). A natural substitution is thus $v = e^{-y}w$, which yields

$$\mu y w'' + (1-\mu y)w' - \lambda w = (1-\mu y)e^{-1} \; , \tag{4.9}$$

using $\beta = \mu$ and $u = e^{-1-y}$, which are valid for small μ. Expanding the solution in a power series, $w(y) = \sum_m w_m y^m$, results in the following recursion relations for the set $\{w_m\}$:

$$\begin{aligned} w_1 &= e^{-1} + \lambda w_0 \; , \\ w_2 &= \frac{-\mu e^{-1}}{2(1+\mu)} + \frac{\lambda+\mu}{2(1+\mu)} w_1 \; , \\ w_{m+1} &= \frac{\lambda + \mu m}{(m+1)(1+\mu m)} w_m , \quad m \ge 2 \; . \end{aligned} \tag{4.10}$$

If the series does not terminate, then the recursion relation at high order simplifies to $w_{m+1} \approx w_m/m$, so that $w_m \sim 1/m!$. This implies that for large y, $w(y) \sim e^y$ (times a power law of y arising from the corrections to the recurrence relation), so that the corresponding eigenfunction $v(y)$ has power-law decay. This implies that the desired short-range eigenfunctions form a discrete set, corresponding to the condition that the series terminate at the mth order, for $m = 1, 2, \ldots$.

For $m \ge 2$, these conditions yield the eigenvalues $\lambda = -m\mu$. For $m = 1$, the matching conditions fix w_0. We guess that $|\lambda| \ll 1$, and check for self-consistency. Under this condition Eq. (4.7) becomes $w \sim e^{-1}y$, so that $w_0 = 0$ and $w_1 = e^{-1}$, which satisfies the first recursion relation, as it must for the solutions to match, and yields the condition $\lambda = 0$ in order for w_2 to vanish. This is thus just the redundant marginal operator resulting from the choice of normalization of the fixed-point solution mentioned earlier. The full set of physical short-range eigenvalues is therefore

$$\lambda = -2\mu, -3\mu, -4\mu, \ldots \tag{4.11}$$

for small μ. Note that there is no eigenfunction with $\lambda = -\mu$. It is absent because of the inhomogeneous terms in Eq. (4.9), as discussed above.

V. LONG-RANGE CORRELATED DISORDER

We now analyze the behavior for random potentials with long-range power-law correlations in ϕ. We first consider the stability of the short-range fixed point analyzed above to such long-range correlated perturbations.

The behavior of the power-law eigenfunctions is much simpler than the exponentially decaying solutions found above. For any value of λ not in the discrete set of Eq. (4.11), the solution which is well behaved at the origin has power-law decay at infinity. There is thus a continuum of power-law eigenfunctions with $v \sim y^{-\Gamma}$, corresponding to $R(\phi) \sim \phi^{-\gamma}$, with $\Gamma = 1 + \gamma/2$. The associated eigenvalues are

$$\lambda = 1 - \beta\Gamma \; , \tag{5.1}$$

for all Γ except those corresponding to the short-range solutions [Eq. (4.11)], i.e., for

$$\Gamma \ne \frac{1}{\beta}, \frac{(1-2\mu)}{\beta}, \frac{(1-3\mu)}{\beta}, \frac{(1-4\mu)}{\beta}, \ldots \; . \tag{5.2}$$

From Eq. (5.1), we see that perturbations are irrelevant for $\Gamma > 1/\beta$.[23]

We now show that the behavior for distributions with long-range correlations is, in fact, rather simple and general. As seen in Sec. III, most solutions of the fixed-point equation have power-law tails. It is only for certain special values of ζ, corresponding to the short-range eigenvalues, that the well-behaved solution at $y = 0$ connects to an exponentially decaying solution at infinity. In all other cases, the solution has a power-law decay for large y, which is dictated entirely by the value of ζ and ϵ.

A simple large-y analysis of Eq. (3.6) gives the value of ζ quoted in the introduction:

$$\zeta_{\mathrm{LR}} = \epsilon/(4+\gamma) \; , \tag{5.3}$$

for

$$\langle \mathbf{V}(\phi,\mathbf{x})\mathbf{V}(\phi',\mathbf{x}')\rangle_C \sim |\phi-\phi'|^{-\gamma}\delta^d(\mathbf{x}-\mathbf{x}') \; . \tag{5.4}$$

It is easy to see that this result will be unaffected by the higher-order terms in an RG expansion, since higher powers of $R(\phi)$ always appear with two derivatives, and multiplying negative power laws results in a more negative power law. (We restrict our attention to $\gamma > -2$, which is needed to make the problem well defined.) Thus, at least within the perturbative RG, only the scale change terms in the RG flows [i.e., those in Eq. (3.1) that are multiplied by ϵ and ζ] are needed to fix ζ. The exponent γ thus fixes the roughness exponent at the long-range fixed point exactly. We believe that this result should be strictly true in all dimensions, but an actual nonperturbative proof would clearly be desirable.

For Eq. (5.1), we see that the condition that the long-range correlations dominate and that Eq. (5.3) applies is that $\gamma < 1/\beta_{\mathrm{SR}}$, implying $\gamma < \gamma_c \equiv \epsilon/\zeta_{\mathrm{SR}} - 4$. Since this also arises from just the rescaling part of the RG flows, we expect it to be true in all dimensions less than four. As γ decreases, the short-range fixed point will become unstable when $\gamma = \gamma_c$; and for $\gamma < \gamma_c$, ζ will vary continuously away from ζ_{SR} according to Eq. (5.3).

A special case of long-range correlated randomness corresponds to intefaces in random field systems which have $N = 1$ and $\gamma = -1$. The general result of Eq. (5.3) implies that $\zeta_{\mathrm{RF}} = (4-d)/3$ as obtained by many authors.[6–8]

Since the short-range fixed point becomes linearly unstable for γ below γ_c, it should be possible to observe the instability of the long-range fixed point in the opposite regime, as γ increases to γ_c. This is actually a subtle problem. If one proceeds with a naive calculation of the eigenfunctions around the long-range fixed point, one arrives again at Eq. (4.3). In this case, however, a simple argument demonstrates that *all* the solutions have power-law form. For large y, Eq. (4.3) becomes

$$\mu y v''+(1+\beta y)v'+(1-\lambda)v \approx A\Gamma[1-\mu(\Gamma+1)]y^{-\Gamma-1} , \tag{5.5}$$

using $u^*(y)\sim Ay^{-\Gamma}$ for large y. Since the power law dictated by the fixed-point function appears on the right-hand side, the homogeneous terms must balance this, and the only way they can do so is to develop power-law tails themselves. Therefore, all the eigenfunctions have power-law decay at large y. This means that the growth of short-range correlations does *not* manifest itself in the usual manner as a relevant eigenvalue. Although we have not calculated this explicitly within the framework of the large-N RG, we expect the following behavior: As γ increases towards γ_c, the fixed-point correlation function of the random potential will become more and more like the short-range fixed point with the regime in which the power-law tail appears moving out to larger and larger y, eventually disappearing for $\gamma>\gamma_c$.[24]

VI. CONCLUSIONS

In this paper we have studied the problem of an oriented manifold with d internal and N transverse dimensions. For d near 4, it became possible to treat the zero-temperature fixed point in an expansion in $\epsilon=4-d$, in terms of a second-order nonlinear differential equation. At $N=\infty$, this equation could be solved exactly for the most interesting case of short-range disorder, yielding a roughness exponent $\zeta_{\rm SR}=\epsilon/(4+N)$. For large but finite N, boundary-layer techniques were employed to estimate the leading corrections, which were found to be nonanalytic, vanishing as 2^{-N}. The magnitude of this nonanalytic correction for $N=1$ is comparable to the correction to $\epsilon/5$ found numerically: $\zeta=0.2083\epsilon$. The behavior in large N near four dimensions saturates a lower bound $\zeta\geq(4-d)/(4+N)$ that is believed to be exact.[25] The leading correction to this that we have found is indeed positive as it should be. Formally, preliminary analysis of our RG flow equations in the opposite limit yields $\zeta(N=0)=\epsilon/4$. This is believed to an exact upper bound for general N.[25] All known numerical and analytical results do indeed lie in the range

$$\frac{4-d}{4+N}\leq\zeta\leq\frac{4-d}{4} . \tag{6.1}$$

It is interesting that our large-N ϵ-expansion result agrees with the large-N results of Mezard and Parisi, claimed to be valid for general dimension $(2<d<4)$ by replica symmetry-breaking techniques. A preliminary large-N analysis of the zero-temperature minimization problem corresponding to our RG procedure does *not* suggest that the $O(\epsilon)$ result should be exact for $\epsilon<2$ in this limit, although further study may change this conclusion. The analysis does, however, suggest that the next corrections to ζ, which are probably of $O(\epsilon^{3/2})$, may be calculable without taking into account the effects of multiple local minima.

Both extensions of the ϵ expansion and investigation of the large-N limit beyond the ϵ expansion are worthwhile future endeavors. An important question is whether or not the replica symmetry breaking used in the vibrational ansatz by Mezard and Parisi has any well-defined physical interpretation beyond that of the general scaling picture of manifolds in random media (discussed in detail in the directed polymer context by Fisher and Huse[4]). Answering this might bear fruit for understanding other random systems, such as spin and vortex glasses. Finally, it is possible that some of the techniques used here might be applicable for other problems such as periodic (e.g., charge-density wave or flux lattices) or nonperiodic (e.g., polymerized membranes) elastic manifolds which exist in random media of the same dimension.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation, through Harvard University's Materials Research Lab, Grant No. DMR9106237. D.S.F. was also support by the A.P. Sloan Foundation.

APPENDIX A: PERTURBATIVE CALCULATION IN PRIMARY REGION

For the primary region, $y\ll 1/\mu$, simple perturbation theory can be used to find the effects of the $O(\mu)$ terms of Eq. (3.6). The solution is expanded in a power series in μ,

$$u=u_\infty+\mu u_1+\mu^2u_2+\cdots , \tag{A1}$$

and terms are grouped order by order, assuming (as confirmed in Sec. III) that $\beta=O(\mu)$. The zeroth-order equation is just Eq. (3.9), while the first-order terms give

$$u_1'(1-u_\infty)+u_1(1-u_\infty')=3y(u_\infty')^2+2y^2u_\infty'u_\infty'' +yu_\infty''(u_\infty-1)-(\beta/\mu)yu_\infty' , \tag{A2}$$

where $u_\infty(y)$ is the solution of Eq. (3.9). The right-hand side of this equation can be rewritten completely in terms of u_∞ by using Eq. (3.10) to eliminate the y dependence, and Eq. (3.9) to eliminate derivative terms. After some lengthy algebra, one finds

$$u_1'(1-u_\infty)+u_1(1-u_\infty')=H(u_\infty) , \tag{A3}$$

where

$$H(x)\equiv H_0(x)+H_\beta(x) , \tag{A4}$$

with

$$H_0(x+1)=[-2/x^2-2/x+3x+3] +\ln(x+1)[4/x^3+6/x^2-1/x-3] +\ln^2(x+1)[-2/x^4-4/x^3-2/x^2] , \tag{A5}$$

and

$$H_\beta(x)=-\frac{\beta}{\mu}[x-\ln(x)-\ln(x)/(x-1)] .$$

Since Eq. (A3) now has no explicit y dependence, we switch to the dependent variable u_∞, using

$$u_1'\equiv\frac{du_1}{dy}=\frac{du_1}{du_\infty}u_\infty'=\frac{du_1}{du_\infty}\frac{u_\infty}{u_\infty-1} . \tag{A6}$$

Equation (A3) becomes

$$\frac{du_1}{du_\infty}+\frac{u_1}{u_\infty(u_\infty-1)}=-H(u_\infty)/u_\infty , \tag{A7}$$

which has the solution

$$u_1(u_\infty)=\frac{u_\infty}{1-u_\infty}\int_{u_\infty}^1\frac{1-x}{x^2}H(x)dx , \tag{A8}$$

where the boundary condition $u_1(y=0)=0$ has been imposed. The important limit for the matching carried out in Sec. III is $y\gg1$, corresponding to $u_\infty\sim\exp(-1-y)\to0$. In this limit, the integral yields

$$\mu u_1(y)\approx e^{-1-y}[\tfrac{1}{2}(\beta-\mu)(u^2-1)-\mu] . \tag{A9}$$

In Sec. III, it was found that $|\beta-\mu|\ll\mu$, so that the first term in the brackets can be neglected. The full solution is then still a pure exponential in the matching region, but with a different coefficient:

$$u(y)\approx e^{-1-y}(1-\mu)\quad\text{for } 1\ll y\ll1/\mu . \tag{A10}$$

APPENDIX B: EIGENFUNCTIONS WITH $v(0)=0$

By choosing $v(0)=0$, we arrive at an equation for this case analogous to Eq. (4.3),

$$v+(1-u)v'-u'v+\beta yv'+\mu y(v''+u'')=\lambda v , \tag{B1}$$

where we have already neglected the $\mu O(uv)$ terms. In the perturbative region ($y\ll1/\mu$), the equation analogous to Eq. (4.4) is

$$v'(1-u)+(1-\lambda-u')v=0 , \tag{B2}$$

which in this case is homogeneous, and correspondingly simpler to solve. Using Eq. (3.9) and Eq. (3.10), one finds the general solution

$$v=Cu^{1-\lambda}(1-u) . \tag{B3}$$

As $y\to0$, $u\to1-\sqrt{2y}$, so that

$$v(y)\to\frac{C}{\sqrt{2y}} . \tag{B4}$$

For v to be nonsingular for small y, the only choice is $C=0$, so that no such eigenfunctions exist.

This is in some respects a surprising result, since for any perturbation of $u^*(y)$, one should be able to rescale the resulting function to leave $u(0)$ unchanged. Therefore one might expect the stability analysis to be phrased precisely in terms of those perturbations which *do not* change $u(0)$. The rescaling, however, is equivalent to adding some amount of the marginal eigenfunction which moves along the fixed line. The function representing the combined effects of perturbation and rescaling [with $v(0)=0$], is, in this case, not an eigenfunction.

APPENDIX C: SCHEMATIC MINIMIZATION

To understand the effects of multiple minima in the random potential on the validity of the RG used here, it is instructive to consider a simple model problem representing the iterative minimization at a particular length scale. This provides a physical derivation of the renormalization-group flow equation [Eq. (2.21)]. Schematically, we imagine integrating out a single Fourier mode $\phi_>$ of momentum $\mathbf{p}$. At zero temperature, this reduces to the problem of minimization over this N-dimensional vector. The renormalized potential will then be given by

$$V_R(\phi_<)=\min_{\phi_>}[\tfrac{1}{2}\phi_>^2+V(\phi_<+\phi_>)] , \tag{C1}$$

where the magnitude of the momentum cutoff $p=\Lambda$ has been set to one for convenience, and, more importantly, we have ignored the spatial dependence of the potential $V(\phi,\mathbf{x})$. A similar formulation of the RG would appear in the treatment of elastic manifolds on a hierarchical lattice. Although Eq. (C1) may appear to be an unreasonable approximation, we will find that the lowest RG order flow equations that we have used in this paper are exactly reproduced. Indeed, the treatment of a suitably modified version of Eq. (C1) which takes into account the spatial dependence does not differ substantially from the approximate version described here (though the required notation makes it rather more cumbersome).

For some purposes of simplicity, we will concentrate on the case of $N=1$, which, for small ϵ, does not differ much from the case of general N. For the remaining part of this appendix, we introduce the notation $x\equiv\phi_<$ and $y\equiv\phi_>$. The $N=1$ problem is then

$$\epsilon^{1/2}V_R(x)=\min_y U(x,y)\equiv\min_y\{\tfrac{1}{2}y^2+\epsilon^{1/2}V(x+y)\} , \tag{C2}$$

where a factor of $\epsilon^{1/2}$ has been extracted to make V of order 1. To estimate quantities, we use the fixed-point values for the correlation function of $V(y)$,

$$\begin{aligned}&\langle V(x)V(x')\rangle=\tilde{R}(x-x')\equiv R(x-x')/\epsilon ,\\&\langle V(x)\rangle=0 .\end{aligned} \tag{C3}$$

Because the minima in Eq. (C2) will be at small y for small ϵ, it is the small-distance behavior of $\tilde{R}(x)$ which will determine the behavior of the model. Based on the behavior of the fixed-point function $R(x)$ in the RG of Sec. II, we assume that $\tilde{R}(x)$ can have a discontinuity in its third derivative at $x=0$.

The extremal condition for Eq. (C2) is

$$y=\epsilon^{1/2}F(x+y) , \tag{C4}$$

where

$$F(y)\equiv-V'(y) \tag{C5}$$

is the force at the "position" y, and primes have been introduced to denote derivatives. From Eq. (C3) the correlations of the force at short distances are

$$\langle F(x)^2 \rangle = 1 \,,$$
$$\langle [F(x)-F(0)]^2 \rangle \sim |x| \quad \text{for } |x| \ll 1 \,. \tag{C6}$$

The perturbative RG performed in Sec. II is equivalent to assuming a perturbation series for y in $\epsilon^{1/2}$ in Eq. (C4) and expanding $F(x+y)$ to obtain solutions order by order in $\epsilon^{1/2}$. There are two possible ways in which such an expansion may break down. Most obviously, the linear behavior in Eq. (C6) implies that derivatives of $F(y)$ are typically infinite, so that the analyticity assumed by a power series in $\epsilon^{1/2}$ may break down. Secondly, the extremal condition [Eq. (C4)] may have multiple solutions, and an iterative solution may converge to any of these, including both maxima and nonglobal minima.

Note that if $\tilde{R}(x)$ were smooth, then $\langle [V''(x)]^2 \rangle < \infty$ and the full potential U in Eq. (C2) would have strictly positive curvature with high probability for small ϵ and hence a unique minimum. The apparent assumption of analyticity of $\tilde{R}(x)$ may be circumvented by the use of a different iterative procedure to find the minimum of the potential U. The simplest such method, which reproduces the results of the perturbation series for the analytic case, is a version of gradient descent (see Fig. 3). One defines a sequence of approximants $\{y_0, y_1, y_2, \ldots\}$ to Eq. (C4) by

$$y_{n+1} = \epsilon^{1/2} F(x+y_n) \,, \tag{C7}$$

for $n = 0, 1, 2 \ldots$.

It is interesting to note that the stability properties of such a mapping discriminate between minima and maxima. Letting $y_n = y^* + \delta y$ and linearizing, one finds

$$\delta y_{n+1} = -V''(y^*)\delta y_n \,, \tag{C8}$$

which is stable for $|V''(y^*)| < 1$. Since the curvature of the potential U in Eq. (C2) is just $1+V''(y)$, this condition excludes all maxima (as well as minima which are sufficiently narrow). Note, however, that the iterative scheme does not guarantee convergence, and one cannot rule out limit cycles or other more complicated behavior for particular realizations of the disorder. Furthermore, even if it does converge, it may not be to the desired global minimum. As we shall see, however, we will obtain an estimate of the global minimum with sufficient accuracy for our present purposes.

Iterating Eq. (C7) yields the first few approximants,

$$y_0 = 0 \,,$$
$$y_1 = \epsilon^{1/2} F(x) \,, \tag{C9}$$
$$y_2 = \epsilon^{1/2} F[x + \epsilon^{1/2} F(x)], \text{ etc .}$$

Using Eq. (C6) the corrections at each level of approximation may be estimated. From Eq. (C7),

$$y_{n+1} - y_n = \epsilon^{1/2}[F(x+y_n) - F(x+y_{n-1})]$$
$$\sim \epsilon^{1/2} |y_n - y_{n-1}|^{1/2} \,, \tag{C10}$$

with a random coefficient of $O(1)$. Iterating this, an infinite series of nontrivial powers of ϵ appear. For the nth approximant, we thus have

$$y_n \approx A\epsilon^{1/2} + B\epsilon^{3/4} + C\epsilon^{7/8} + \cdots + Z\epsilon^{(2^n-1)/2^n} \,. \tag{C11}$$

The difference from the perturbation series result for the analytic case appears first in y_2, via the appearance of the $\epsilon^{3/4}$ term in Eq. (C11).

From Eq. (C4), we may find an upper bound for the size of the region in which minima are likely to exist. (In fact, we only calculate this bound for the region within which there are *extrema*. For $N=1$, the furthest out extrema are in fact always minima, but for large N this difference may be important.) Suppose one extrema is located at the point y^*, satisfying Eq. (C4). Then for another extremum to be located within a distance δy, the condition must again be satisfied at the point $y+\delta y$. For large δy, this is clearly extremely unlikely, since the linear term grows, while the random force remains bounded and of $O(\epsilon^{1/2})$ with high probability. For small δy, the variations of the force grow like $(\delta y)^{1/2}$ from Eq. (C6), i.e., *faster* than the linear term in Eq. (C4). Thus there will be a length scale below which the variations of the force dominate, and other extrema are possible. Equating the two terms in Eq. (C4) gives

$$\delta y \sim \epsilon^{1/2} |\delta y|^{1/2} \Longrightarrow \delta y < O(\epsilon) \,, \tag{C12}$$

which is the desired upper bound on the separation of extrema. We see that the separation between extrema is smaller than *any* of the correction terms obtained in (C11). This is illustrated in Fig. 3. This suggests that the corrections due to multiple minima appear at higher order in ϵ than the iterative corrections which arise from the nonanalyticity of $V(y)$. To check this, we must analyze how the corrections to y affect the renormalized potential and its correlations.

To estimate the corrections from the terms in Eq.

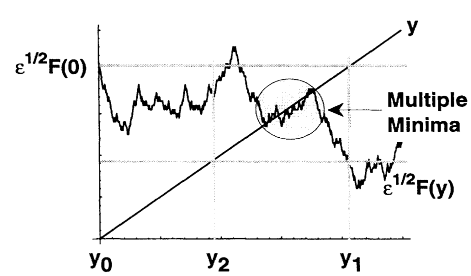

FIG. 3. Graphical illustration of the iterative minimization of Eq. (C7). Given a guess y_i for the location of the minima, the next approximation is found by following a vertical line at this value of y until it intersects the random force curve. Extending a horizontal line to the 45° line through the origin (representing the uniform restoring force of the harmonic potential) gives the value of y for the next iteration. The second two approximants, y_1 and y_2, resulting from the initial $y_0 = 0$ are shown here. The many intersections between the random force curve and the 45° line in the figure represent multiple extrema which occur on smaller scales [see Eq. (C20)].

(C11), it is useful to write the random potential in the form

$$V(x+y)=V(x)+yV'(x)+W(x;y)\ , \tag{C13}$$

where

$$W(x;y)\equiv\int_0^y [V'(x+z)-V'(x)]dz\ . \tag{C14}$$

The renormalized potential can then be evaluated by inserting the iterative solution Eq. (C11) into Eq. (C2), yielding

$$\epsilon^{1/2}V_R(x)=\epsilon^{1/2}V(x)+\frac{A^2(x)}{2}\epsilon+\frac{B^2(x)}{2}\epsilon^{3/2}+\frac{C^2(x)}{2}\epsilon^{7/4}+\cdots+A(x)B(x)\epsilon^{5/4}+A(x)C(x)\epsilon^{11/8}$$
$$+\cdots+\epsilon^{1/2}V'(x)[A(x)\epsilon^{1/2}+B(x)\epsilon^{3/4}+\cdots]+\epsilon^{1/2}W[x;A(x)\epsilon^{1/2}+B(x)\epsilon^{3/4}+\cdots]\ . \tag{C15}$$

This expression simplifies somewhat when the function $A(x)=F(x)=-V'(x)$ from Eq. (C9) is inserted,

$$\epsilon^{1/2}V_R(x)=\epsilon^{1/2}V(x)-\frac{A^2(x)}{2}\epsilon+\frac{B^2(x)}{2}\epsilon^{3/2}+\cdots+B(x)C(x)\epsilon^{13/8}+\cdots+\epsilon^{1/2}W[A(x)\epsilon^{1/2}+B(x)\epsilon^{3/4}+\cdots]\ . \tag{C16}$$

The fact that the second term is negative reflects the approach to the minimum. Note that the cross terms AC, AD, etc., have now canceled. The RG flow requires renormalization of the correlation function of the disorder, Eq. (C3). The renormalized potential in Eq. (C16) will have a nonzero expectation value, since the minimization procedure decreases the energy for all realizations of $V(y)$. To find the renormalized correlation function, therefore, it is necessary to take the truncated (cumulant) expectation value,

$$\begin{aligned}\epsilon R_R(x)&=\epsilon\langle V_R(x)V_R(0)\rangle_C\\&=\epsilon\langle V(x)V(0)\rangle-\tfrac{1}{2}\epsilon^{3/2}\langle A^2(x)V(0)+A^2(0)V(x)\rangle_C+\tfrac{1}{2}\epsilon^2\langle B^2(x)V(0)+B^2(0)V(x)\rangle_C\\&\quad+\tfrac{1}{4}\epsilon^2\langle A^2(x)A^2(0)\rangle_C+2\epsilon\langle V(x)\hat{W}[0]\rangle_C\epsilon+\epsilon\langle\hat{W}[x]\hat{W}[0]\rangle_C\\&\quad-\tfrac{1}{2}\epsilon^{3/2}\langle A^2(x)\hat{W}[0]+A^2(0)\hat{W}[x]\rangle_C+\tfrac{1}{2}\epsilon^2\langle B^2(x)\hat{W}[0]+B^2(0)\hat{W}[x]\rangle_C+O(\epsilon^{17/8})\ ,\end{aligned} \tag{C17}$$

where in the expression $\hat{W}[x]$ means the full expression from the last term of Eq. (C16), evaluating all internal coefficients [i.e., $A(x)$, $B(x)$, etc.] at the point x, and we have dropped all terms of explicitly higher order than ϵ^2. The function $\hat{W}[x]$ is, however, itself small, so that some further terms can be dropped. From using Eq. (C14) and the fact that $F(x)-F(x+z)\sim\sqrt{z}$ [Eq. (C6)], we see that $W(x;y)\sim|y|^{3/2}$. Since $y\sim\epsilon^{1/2}$, $\hat{W}[x]\sim\epsilon^{3/4}+O(\epsilon)$, where the $O(\epsilon)$ term arises from the $B(x)$ in Eq. (C11) and higher-order terms in $A(x)$; this $O(\epsilon)$ term is needed to obtain the V_R correlations to $O(\epsilon^2)$ but higher-order terms are not. In addition we see that of all the terms involving $\hat{W}$ in Eq. (C17), only the $V\hat{W}$ term will contribute at this order. We thus see that to $O(\epsilon^2)$, we may drop all terms in y beyond the $B(x)$ term. To this order, we may thus use the second-order iterative solution

$$\begin{aligned}\epsilon^{1/2}V_R(x)&\approx\epsilon^{1/2}V\{x+\epsilon^{1/2}F[x+\epsilon^{1/2}F(x)]\}\\&\quad+\tfrac{1}{2}\epsilon\{F[x+\epsilon^{1/2}F(x)]\}^2\ .\end{aligned} \tag{C18}$$

The correlations of $V_R(x)$ and $V_R(0)$ can be calculated directly from this form of V_R yielding

$$\begin{aligned}R_R(x)&=\epsilon\langle V_R(x)V_R(0)\rangle_C\\&=R(x)+\tfrac{1}{2}[R''(x)]^2-R''(x)R''(x)+O(\epsilon^{5/2})\ .\end{aligned} \tag{C19}$$

To this order, the correct answer can be obtained from Eq. (C17) by expanding $W(x;y)$ formally as $W(x;y)\approx(y^2/2)V''(x)$ and averaging directly the $\hat{W}V$ term in Eq. (C17). This is valid because only one V'' appears here. To analyze the $\hat{W}\hat{W}$ term, an expansion in y *fails* and $\langle\hat{W}\hat{W}\rangle\sim\epsilon^{3/2}$ rather than the naive ϵ^2. We thus expect that the effects of the nonanalyticity of R will affect R_R at order $\epsilon^{5/2}$. These terms need to be balanced by adjustments to ζ, suggesting $O(\epsilon^{3/2})$ corrections to our $O(\epsilon)$ result for ζ.

So far, the effects of multiple extrema have not been included. Their effects can be estimated by including a further correction term in y,

$$y=A\epsilon^{1/2}+B\epsilon^{3/4}+\cdots+\eta\epsilon+\cdots\ . \tag{C20}$$

By repeating arguments along the lines of those above, it is a simple matter to estimate the leading corrections due to a nonzero η. One finds that the first contribution to $R(x)$ occurs at $O(\epsilon^3)$. This is higher order than all the leading corrections from the nonanalyticities arising in the iterative procedure.

Thus, although the singularity in $R''(x)$ at $x=0$ is associated with the existence of many extrema [since $V''(0)$ has infinite variance], the direct effects of these multiple extrema only show up at higher order in ϵ; to the order needed here, choosing *any* of the minima provides enough accuracy.

The simple approximation to the renormalization-group flows analyzed in this appendix suggests that there will be corrections to ζ, starting at $O(\epsilon^{3/2})$, with an apparently infinite sequence of higher-order corrections appearing before $O(\epsilon^2)$, at which order the effects of multiple minima begin to appear. Although the picture is quite appealing, the results should not be taken as definitive predictions of the powers involved, since a complete analysis should involve a self-consistency condition to determine the small-x behavior of the correlations. Such an analysis may well involve a boundary layer for small x with smoothing of the fixed-point function on scales smaller than some (>1) power of ϵ.

It is straightforward to extend the analysis of this appendix to general (fixed) N in the limit of small ϵ. Since in the limit of large N, ζ is formally small, even if ϵ is not small, one might hope to be able to justify truncation of the RG flows for all ϵ (or at least $\epsilon<2$) for N large. We have not been able to do this, and, indeed, preliminary indications suggest the opposite conclusion: that even for large-N, higher-order terms in ϵ are needed. A more detailed study of this limit would clearly be instructive.

APPENDIX D: MULTICRITICAL SHORT-RANGE FIXED POINTS

As remarked in Sec. III, Eq. (3.13) possesses a discrete family of solutions which are well behaved at the origin and decay exponentially at infinity. These can be matched onto the primary solution to yield additional fixed points of the RG flows. To find these, we perform a power-series expansion similar to the one used for the stability analysis [Eq. (4.10)]. Defining

$$u \equiv e^{-(\beta/\mu)y} w \, , \qquad w=\sum_m w_m y^m \, , \tag{D1}$$

Eq. (3.13) yields a simple recursion relation for the set $\{w_m\}$:

$$w_{m+1}\left[\frac{\beta/\mu-1+\beta m}{(m+1)(1+\mu m)}\right]w_m \; . \tag{D2}$$

If the series does not terminate, the large-m behavior of the coefficients is

$$w_m \sim \frac{1}{m!}\left[\frac{\beta}{\mu}\right]^m , \tag{D3}$$

so that $u(y)$ decays more slowly than an exponential. A short-range $u(y)$ is obtained whenever the series terminates, which yields the condition

$$\beta=\frac{\mu}{1+m\mu}, \quad m=0,1,2,\ldots \; . \tag{D4}$$

In terms of the roughening exponent,

$$\zeta=\frac{\epsilon}{4+N+2m} \; . \tag{D5}$$

The case $m=0$ corresponds to the simple exponential found in Sec. III, while for higher m the solutions have some oscillations and correspond to smaller values of ζ.

It is a simple matter to extend the results of Sec. IV to calculate the stability around the new fixed points. One finds that for the mth fixed point, there are m relevant eigenvalues corresponding to short-range correlated perturbations, so that the solution found in Sec. III is stable, while the remaining solutions represent a hierarchy of multicritical solutions.

Although such solutions exist formally, we have not fully investigated the criteria under which these solutions represent physically meaningful fixed points. At least initially, the function $R(\phi)$ is highly constrained by the positivity condition for the probability distribution of $V(\phi)$. In particular, taking the Fourier transform of the potential-potential correlation function, we must have

$$\langle \tilde{V}(\kappa)\tilde{V}(-\kappa)\rangle=\tilde{R}(\kappa)\geq 0 \; . \tag{D6}$$

Although the interpretation as a correlation function suggests that this positivity property is preserved by the RG, the nonlocality (in κ) of the terms in the RG flows generated by fluctuations has prevented us from finding a simple proof. Nevertheless, it seems likely that the positivity is preserved. It is straightforward to check the multicritical fixed points obtained above for this criterion. If they do not satisfy Eq. (D6) they cannot be physical. A simple computation for the first multicritical solution,

$$R_1(\phi)=\left[1-\mu^2-\frac{\mu\phi^2}{2}\right]\exp\left[-\frac{\phi^2}{2(1+\mu)}\right] , \tag{D7}$$

yields the Fourier transform

$$\tilde{R}_1(\kappa)=\left[\frac{1}{2}+\frac{\mu}{2}(1+\mu)^2\kappa^2\right]\exp\left[-\frac{\kappa^2}{2}(1+\mu)\right] , \tag{D8}$$

in the large-N limit, which satisfies the positivity criterion [Eq. (D6)] and thus might be physically attainable.

If we restrict consideration to distributions in which the function $u(y)$ has no zeros [or equivalently $R(|\phi|)$ has no nontrivial extrema], it is possible to show, however, that the multicritical solutions, which have at least

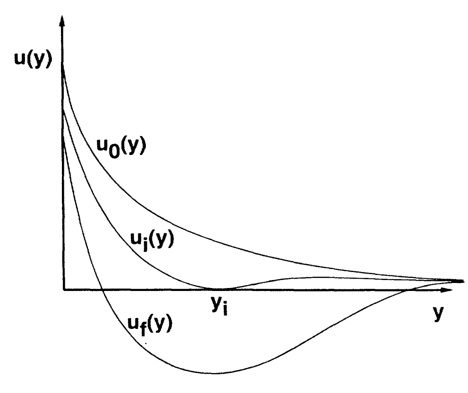

FIG. 4. Illustration of the preservation of the lack of zeros by the RG flows [Eq. (3.6)] for the function $u(y)$. For the initial function $u_0(y)$ to develop into the final function $u_f(y)$, with internal zeros, it must pass through an intermediate state $u_i(y)$, at which it is tangent at some point y_i with the y axis. Equation (D10) shows that such a point of tangency is repelled, so that the putative crossing does not occur.

one zero are inaccessible. A simple argument proceeds as follows (see Fig. 4): Consider the evolution of an initial function $u_0(y)$ which has no zero crossings. For the function $u_0(y)$ to evolve into one of these multicritical solutions, it must at some intermediate stage when it first has a zero be tangent with the y axis at some point y_i [like the function $u_i(y)$ in Fig. 4]. (Note that, since the behavior at ∞ is preserved by the flows, the zero cannot come in from ∞ and avoid the tangency condition.) At this intermediate point, the function must obey

$$u_i(y_i;t_i)=0 \ ,$$
$$u_i'(y_i;t_i)=0 \ , \tag{D9}$$
and
$$u_i''(y_i,t_i)>0 \ .$$

From Eq. (3.6), one then finds

$$\left.\frac{\partial u(y_i;t)}{\partial t}\right|_{t=t_i}=\mu u_0(t_i)y_i u''(y_i,t_i)>0 \ , \tag{D10}$$

so that the putative point disappears. Therefore such a zero cannot occur, and any function $u(y)$ with zeros is inaccessible from an initial u without zeros.

At this point it is unclear whether the formal multicritical fixed points found here are accessible for less restrictive initial correlation functions, and, if so, what their physical significance is. In particular, one might expect the $m=1$ critical point to separate two phases with different behavior. If one of these is the rough phase analyzed in this paper, what is the nature of the other phase? We leave these as intriguing open questions.

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[21]Actually, Eq. (3.13) has a discrete family of eigenfunctions satisfying the appropriate boundary conditions for a short-range fixed-point function and matching to the primary solution. In the linearized approximation to which we have worked so far, these can all be found explicitly. This calculation is performed in Appendix D, where it is shown that these may describe multicritical points for special correlations of the disorder.
[22]Note that for extremely large $y \gg 1/b^2$, this solution is not uniformly good; however, to obtain this regime we can easily match onto the known simple behavior of $u_c(y)$ for $1/\mu \ll y \ll 1/b^2$.
[23]Note that the completeness of these eigenfunctions for the two classes of fixed points remains to some extent an open question. Since the linearized operator in Eq. (4.3) is not self-adjoint, standard completeness theorems do not apply. In general, one may look for different sets of left and right eigenfunctions, and proceed in this way to examine completeness and decompose arbitrary perturbations. Unfortunately, the nonlocality of the operator implies that the associated adjoint operator contains δ functions, and the analysis becomes complicated by convergence issues. For the short-range fixed points, however, completeness should hold, since the eigenfunctions are simply exponentials multiplying polynomials, which should be equivalent to a more standard basis by a simple orthogonalization. For long-range perturbations, much less can be stated. It is unclear how to decompose a particular perturbation in this basis, and also to what extent short-range perturbations are representable in terms of such functions. We *do* expect, however, that the behavior of the eigenfunctions as a function of γ does dictate the relevance of perturbations with power-law tails.
[24]It is the nonuniformity of this limit which led to the incorrect results of Ref. 2. There, it was assumed that the growth of short-range correlations occurs via the appearance of an exponential damping of the long-range power law, implying that the power-law prefactor appearing in Eq. (3.16) is equal to $y^{-\gamma_c}$. A comparison of the large-N results of Eqs. (3.29) and (3.16) demonstrates explicitly the incorrectness of this assumption.
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