Theory of current-voltage instabilities in superlattices

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The theory of the uniform electric-field distribution instability in superlattices is developed. The instability with respect to high-field domain formation was discovered some time ago by Esaki and Chang but no microscopic theory has been proposed so far. To determine the instability condition we consider first the conductivity of a superlattice with a uniform electric-field distribution for arbitrary relations between the width of the miniband, the electric potential drop per period, and the energy uncertainty due to scattering. Such a general case can be described quantitatively with the help of the density matrix if an effective electron temperature is larger than those three characteristic energies. The instability threshold corresponds to an electric field for which the separation between Stark levels in the superlattice becomes larger than the energy uncertainty due to scattering. The physical reason for the instability is that any local increase of the field leads to a larger separation between the Stark levels and to the local decrease of the current. The solution of the instability problems shows that near the instability threshold only long-wavelength fluctuations are unstable. At the initial stage of the development of the instability nonlinear effects accelerate the fluctuation growth.

I. INTRODUCTION

Since the work of Esaki and Chang[1] the instability of the uniform distribution of electric field in superlattices has been observed experimentally many times.[2–4] These experiments show that if an electric field applied to a superlattice becomes greater than some critical value then the uniform distribution of the field along the superlattice breaks down and a high-field domain comes about. Although no direct measurements of high-field domains have been made, they are usually thought about as a large potential drop across one of the barriers forming the superlattice so that the first level in the well at one side of the barrier is in resonance with the second level in the well at the other side.

The physical reason for the formation of high-field domains was shown by Esaki and Chang.[1] They attributed it to a negative differential conductance (NDC) predicted for a superlattice in a high enough electric field.[5–8] For a continuous medium with an N-shape current-voltage characteristic a simple phenomenological theory predicts the formation of high-field domains propagating with some drift velocity.[9,10] Esaki and Chang assumed that the uniform-field distribution in a superlattice is unstable for the same reason as in a continuous medium. This reasonable assumption leaves, however, at least two questions which cannot be answered by such an analogy. The first is that high-field domains in a continuous medium drift while nobody has detected any motion of the domains in superlattices.[11] The second question is related to the fact that the NDC in a superlattice results from a rather small energy bandwidth in the direction perpendicular to layers. A high-field domain destroys the one-band propagation of electrons and so destroys the reason for the NDC. That is, the question arises as to whether the suggested instability would lead to the formation of a high-field domain or to some nonstationary phenomena due to the destruction of the one-band transport.

The first attempt to develop an adequate theory of the instability in superlattices and answer these questions was made by one of the authors.[12] In that work a superlattice was modeled by a series of identical circuits consisting of a nonlinear resistor and a capacitor in parallel. The capacitor and resistor described, respectively, displacement current and tunneling current across a barrier. Although this simple model explained the instability, domain formation, and hysteresis detected in superlattices,[13,3] it also left a few unresolved problems. The main problem is that the modeling of a superlattice by a series of circuits is possible only in the case of noncoherent tunneling across different barriers. In other words, a barrier can be modeled by a resistor if electrons are scattered after each tunneling. In the opposite case minibands are formed and tunneling is coherent across all barriers at the mean free path of an electron.

The present work has been started to study the stability of the uniform electric-field distribution under the condition of coherent tunneling. Actually we considered a general case with any relation between the mean free path and the period of the superlattice.

In Sec. II we describe the physical picture the conductivity and instability in superlattices and show main results of the work.

In Sec. III an equation for the electron density matrix in a superlattice under an electric field is derived. We consider only one miniband (or one level in each well) and neglect tunneling or thermal exciting of electrons in upper minibands. Even in this case the derivation is not simple because the instability takes place in the regime intermediate between the band conductivity and hopping. In general, no quantitative theory exists at this

situation. We prove, however, that in superlattices it is possible to use the equation for the density matrix with the collision operator calculated in the Born approximation.

In Sec. IV the current-voltage characteristic of the superlattice in the case of the uniform electric-field distribution is calculated.

In Sec. V we consider the instability of the uniform-field distribution along the superlattice. In Sec. V A the equation for a density-matrix perturbation $\delta\rho$ is obtained. $\delta\rho$ depends on two Stark numbers, ν and ν'. Because we study the stability of a uniform state the equation for $\delta\rho$ has a simpler form in the Fourier representation with respect to $(\nu+\nu')/2$. The self-consistent electric-field distribution is found in Sec. V B. In Sec. V C the stability of long-wavelength fluctuations is studied. For a long-wavelength perturbation a superlattice can be considered as a continuous medium. The dispersion relation for fluctuations with a short wavelength is obtained in Sec. V D.

In Sec. VI the nonlinear growth of fluctuations in a superlattice with a finite length is studied near the instability threshold.

Section VII contains the discussion of the results and their comparison with available experimental data. In particular we compare our results with the calculations of Tsu and Döhler[23] and discuss the experiments of Sibille *et al.*[15,37–41]

II. PHYSICAL PICTURE AND MAIN RESULTS

The electron transport in a superlattice is characterized by a few energy scales. The main scales are an overlap integral between electron states in adjacent quantum wells Λ, an energy uncertainty induced by electron scattering Γ, and an average potential drop per one period eFd (here e is the electron charge, F is the average electric field, and d is the period of the superlattice).

Without an electric-field quantum-mechanical consideration leads to the formation of a band with the width 4Λ. In a uniform electric-field F, the continuous spectrum in the band is replaced by a system of Stark levels with the energy separation eFd. The spatial size of each Stark state is Λ/eF.[14,8] For different relations between Λ, Γ, and eFd two different transport regimes are possible (Fig. 1).

If $eFd \ll \Gamma \ll \Lambda$ the Stark levels are not resolved and the quantum number characterizing the electron motion perpendicular to the layers is momentum. Electron transport can be described by a distribution function depending on the electron momentum and satisfying the classical Boltzmann equation (region I in Fig. 1). The only difference between this case and usual semiconductor transport is that one cannot use the effective-mass approximation and has to take into account the real electron spectrum in the band. This approach was used by Lebwohl and Tsu[6] and Sibille *et al.*[15] who used the relaxation-time approximation. In a simplified version of this approach the Boltzmann equation is replaced by balance equations[16] or hydrodynamics equations.[5,17–19] The same case was studied also with Monte Carlo calculations.[7,20,21]

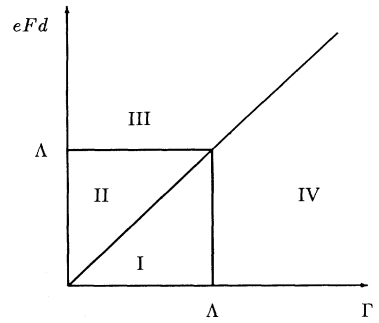

FIG. 1. Different types of conductivity in superlattices. Band conductivity takes place only in region I. In regions II and III electrons are localized at Stark levels. In region III the size of the Stark states is about one period. In region IV electron are localized in wells of the superlattice due to strong scattering. The instability threshold goes along the boundaries between regions I and II and regions III and IV.

There are two possible reasons for a violation of the band spectrum and electron localization. The first is the Stark quantization, when $eFd \gg \Gamma$. The distribution of electrons is characterized by the occupation numbers of the Stark levels and the transport mechanism is hopping between them (regions II and III in Fig. 1). The case of a Stark quantization was studied by Yakovlev,[22] Tsu and Döhler,[23] Döhler, Tsu, and Esati,[24] Movaghar[25] and Calecki, Palmier, and Chomette.[26]

Another reason for electron localization is strong scattering, when $\Gamma \gg \Lambda$. In this case an electron is scattered after each tunneling across a barrier and no miniband exists, no matter what value eFd takes (regions IV and a part of region III in Fig. 1). Note that the localization in separate wells of the superlattice takes place also if the electric field is so strong that $eFd \gg \Lambda$ (region III in Fig. 1). The transport in this case is hopping between adjacent wells of the superlattice. This case can be described with the phenomenological model.[12]

NDC takes place at the boundaries between regions I and II and regions III and IV in Fig. 1. The physical meaning of this result is very clear.[5,27–29,12] For a weak scattering, $\Gamma \ll \Lambda$, and in a small field, $eFd \ll \Gamma$, the Stark levels are not resolved and the electron spectrum is continuous. When the electric field exceeds the value of Γ/ed the continuous electron spectrum is split into Stark levels, which impedes an electron space transfer. Transitions between the Stark levels take place only due to scattering and the current is proportional to the ratio of Γ to the separation between them, eFd. This is the region of a NDC.

The same can be described in the classical language. The electric field accelerates an electron and its wave vector k grows proportionally to time, $k \sim eFt/\hbar$. Such a free acceleration can last no longer than the relaxation time $\hbar/\Gamma$. That is, in the case of $eFd \ll \Gamma$ the electron is scattered before its wave vector reaches the value of π/d and it can experience the Bragg reflection from the top of

the band. The current under this condition is limited by scattering and, in general, goes up with the electric field. In the case of $eFd \gg \Gamma$ the electron Bragg reflection limits the electron velocity and leads to a NDC.

In the case of a strong localization, $\Gamma \gg \Lambda$ or $eFd \gg \Lambda$, the transition from region IV to region III in Fig. 1 with an increase of the electric field leads to the violation of the resonance between levels in adjacent wells and reduces the current between them.

It appears that the instability of the uniform electric-field distribution comes about along with the NDC, which supports the original idea of Esaki and Chang.[1] The explanation of this coincidence is the following. For a weak scattering and $eFd \sim \Gamma$ electron scattering competes with the Bragg reflection. In other words, it is possible to think about Stark levels smeared by scattering. Any local increase of the electric field moves apart local Stark levels and impedes the current between them. As a result charges of the opposite sign are accumulated on sides of the electric-field fluctuation. The field created by these charges is of the same sign as the initial fluctuation field, which leads to a further decrease of the current, i.e., to a growth of the fluctuation. In the case of a strong scattering a similar argument is applied to resonant levels in adjacent wells.

In the phenomenological case, $\Gamma \gg \Lambda$ or $eFd \gg \Lambda$, electrons are localized in different wells and the density matrix is nearly diagonal in the number of the well. Transitions between wells can be described in the Born approximation. The opposite case is more complicated. The problem is that the threshold of the NDC and the instability corresponds to the transition between band transport and hopping. The parameter of the perturbation theory for band transport is the ratio of Γ to the characteristic energy, which in the case of narrow bands is the width of the band or eFd, whichever is larger. And, in general, near the boundary between regions I and II in Fig. 1 the perturbation theory fails. If we come from the side of the hopping transport, the condition justifying the perturbation theory is opposite, $eFd/\Gamma \gg 1$, and it is also violated near the boundary between regions I and II.

The situation in superlattices is dramatically simplified because of the in-plane electron motion. In this case there is another energy which characterizes the in-plane electron motion, e.g., temperature T. The condition $T \gg \Gamma$ is enough for the justification of the perturbation theory. To make this fact clear, in Sec. III we derive the equation for the density matrix with the help of the Keldysh technique.[30] The derivation is similar to that for the Boltzmann equation.[30]

We consider a strong elastic relaxation mechanism and a weak inelastic relaxation mechanism. Although an expression for the elastic collision operator is written down for impurity scattering the explicit form of the scattering matrix element is not used anywhere and all results can be applied for the cases of the acoustic- or optical-phonon scattering as well. Acoustic-phonon scattering is quasielastic and leads to a momentum relaxation much faster than the energy relaxation. Optical-phonon scattering at temperatures below the phonon energy in the first approximation also leads only to the momentum relaxation. Indeed, because of a small phonon dispersion an absorption followed by the emission of a phonon does not change the electron energy.[31]

It is important to note that the energy uncertainty Γ results from elastic scattering. Apparently perturbation theory is justified for a weak inelastic scattering. For this reason we do not make a specific derivation for the inelastic part of the collision operator, which is available in textbooks.

The resulting equation for the density matrix is rather complicated and in the following sections it is simplified for the case when the width of an electron energy distribution is much larger than eFd, Γ, and Λ. This is not a very strong limitation. At high temperatures T, the width of the distribution is T (we measure temperature in energy units) and the required condition is satisfied. At low temperatures the width of the electron energy distribution is determined by an energy relaxation and larger than T due to a heating of the electron gas.

For a uniform electron distribution the density matrix depends on two Stark numbers, not separately but only on their difference. The equations for the matrix elements resemble the Boltzmann equations for even and odd parts of the electron distribution function in the hot-electron theory for wide band semiconductors.[32,33] We consider acoustic-phonon energy relaxation. For this mechanism the energy relaxation operator is reduced to a differential operator and the Boltzmann equation can be solved analytically without any further approximation. The electron energy distribution and current-voltage characteristic is expressed in terms of an integral containing $\Gamma(E)$, where E is the electron energy. For typical elastic scattering mechanisms (impurities, acoustic or optical phonons) the dependence of Γ on the energy is not very strong and all qualitative features can be obtained without an exact form of that dependence.

The study of self-consistent nonuniform electric-field distribution in Sec. V B leads to a more complicated equivalent circuit than in a simple model suggested in Ref. 12. The main difference is that a well is not a metallic plate. It has a finite polarizability and an electric field penetrates through it. Each well can be characterized by an effective dielectric constant and an effective capacitance. An equivalent electric circuit for a superlattice is more complicated than that suggested in Ref. 12 (see Fig. 2).

The dispersion relation connecting the frequency and

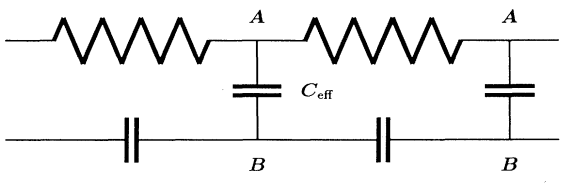

FIG. 2. Equivalent circuit for a superlattice. Electrons are confined in points A while the potentials which affect them correspond to potentials in points B. The electron transport can be described by the resistors only in the case of sequential tunneling.

the wavelength of fluctuations is rather simple for a long wavelength. The most important parameter in this case is $\omega\tau_E$, where ω is the frequency (or just the inverse time scale) of a fluctuation and τ_E is the energy relaxation time. If $\omega\tau_E \ll 1$ the fluctuation can be described by phenomenological equations (see, e.g., Ref. 9). The evolution of the perturbation is characterized by three phenomenological coefficients, a differential conductivity, a drift velocity, and a diffusion coefficient. The opposite case is not described by phenomenological coefficients because fluctuations also perturb the electron energy distribution. The main difference of this case from the first one is that the drift can be neglected compared to the diffusion. These results show another disadvantage of the model used in Ref. 12 where a diffusion current was neglected. The diffusion is not inherent exclusively to long-wavelength fluctuations. It just reveals that for a nonuniform electron distribution the current between adjacent wells depends not only on the potential difference between them but also on the difference of electron concentrations.

The dispersion relation for any wavelength can be studied only numerically. The most important result is that for any set of parameters which we checked the instability came about first for long-wavelength fluctuations (see Fig. 3). Then the question arises how such an instability leads to a high-field domain with the length of one period. A possible answer comes from the study of the nonlinear evolution of fluctuations.

In general, two situations are conceivable. Nonlinear effects can limit the growth and lead to a nonuniform stationary state. The other possibility is that nonlinear effects intensify the growth of fluctuations until other physical phenomena become important or a stationary

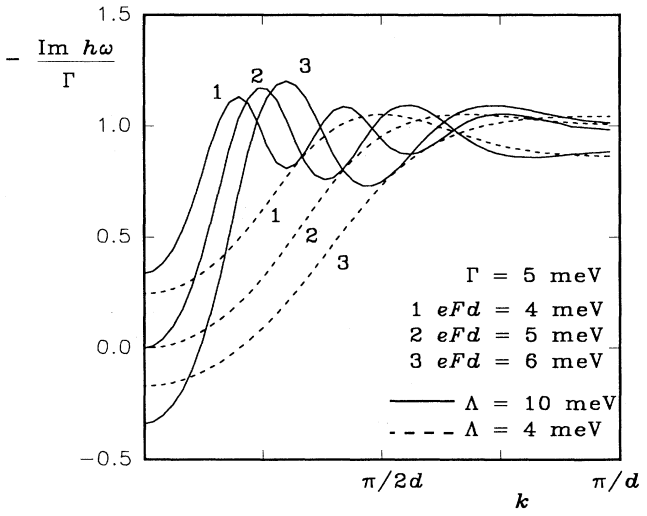

FIG. 3. The dimensionless attenuation coefficient, $-\mathrm{Im}\omega/\Gamma$, as a function of the fluctuation wave vector, k, for typical superlattice parameters. Positive values of $\mathrm{Im}\omega$ correspond to unstable fluctuations. When the electric field exceeds the threshold value the instability appears first at small k.

but time-dependent regime (e.g., periodical) is set up. We considered the nonlinear growth of fluctuations when the electric field is very close to the instability threshold and the instable region includes only one degree of freedom of the superlattice. It appears that at the initial stage nonlinear effects accelerate the growth of fluctuations in superlattices with parameters usually used in experiments. This means that high harmonics excited as a result of nonlinear interaction can substantially reduce the characteristic length scale of growing fluctuations.

III. DERIVATION OF THE KINETIC EQUATION FOR A SUPERLATTICE

It is convenient to start calculations in the Wannier representation (i.e., the representation diagonal with respect to electron states in different wells) because in this representation it is easy to write down the Hamiltonian of a superlattice. Electron wave functions in the Wannier representation are

$$\psi_{n,\alpha,\mathbf{p}}(x,\mathbf{r}) = \frac{1}{\sqrt{S}} e^{i\mathbf{p}\cdot\mathbf{r}} w_n(x-\alpha d), \tag{1}$$

where $w_n(x-\alpha d)$ is the wave function of the level with the energy E_n in the αth well, d is the period of the superlattice, $\mathbf{p}$ and $\mathbf{r}$ are the in-plane wave vector and coordinate, and S is the normalization area. The electron Hamiltonian of a perfect superlattice in a uniform electric field F is

$$\mathcal{H}_0 = (\mathcal{E}_n + E_p - eFd\alpha)\delta_{\alpha\alpha'} + \Lambda_n(\delta_{\alpha\alpha'+1} + \delta_{\alpha\alpha'-1}). \tag{2}$$

Here e and m are the electron charge and mass, respectively, $E_p = \hbar^2 p^2/2m$, and Λ_n is the tunneling matrix element between nth levels in nearest wells. We neglect tunneling on the distance more than one period. Also we consider the electric field F so small that eFd is much smaller than the distance between energy levels in a well. The mixing of the levels due to the field and the tunneling between different levels in the nearest wells are neglected.

Eigenvalues of the Hamiltonian Eq. (2) are $E_{n\nu\mathbf{p}} = \mathcal{E}_n + E_p - eFd\nu$, and eigenfunctions are expressed in terms of Bessel functions

$$\langle\alpha|\nu\rangle = J_{\nu-\alpha}\left(\frac{2\Lambda_n}{eFd}\right). \tag{3}$$

We will use subscripts α and β for the Wannier representation, and μ and ν for the Stark representation in which the Hamiltonian Eq. (2) is diagonal.

Without a scattering the Stark quantization takes place in any small electric field. Only in zero electric field the Hamiltonian Eq. (2) gives a band spectrum with the width of the bands $4\Lambda_n$.

We assume that the Fermi level is below the second subband. Then below the stability threshold and on the first stage of the development of the instability the second and higher subbands are empty. So hereafter we assume that only the first subband is occupied and omit the energy $\mathcal{E}_1$ and the subscript showing the number of a subband.

The total Hamiltonian contains $\mathcal{H}_0$, a self-consistent electron potential energy induced by a nonuniform distribution of electrons among the wells $e\hat{\Phi}$, and a scattering potential $\hat{U}$. We consider that the main scattering mechanism is impurity scattering. The overlap of electron wave functions in different wells is small and only diagonal elements with respect to the wells $\Phi_{\alpha\alpha}$ and $U_{\alpha\alpha}$ will be taken into account. Matrix elements of the operators $e\hat{\Phi}$ and $\hat{U}$ in the Stark representation have the form

$$\Phi_{\nu\nu'} = \sum_{\alpha} J_{\nu-\alpha} J_{\nu'-\alpha} \Phi_{\alpha\alpha} , \tag{4}$$

$$U^{\nu\nu'}_{\mathbf{pp'}} = \sum_{\alpha} J_{\nu-\alpha} J_{\nu'-\alpha} U^{\alpha\alpha}_{\mathbf{pp'}} . \tag{5}$$

The argument $2\Lambda/eFd$ in Bessel functions will be omitted.

We will assume an arbitrary relation between the width of the subband 4Λ, the electric potential drop per period eFd, and the energy uncertainty due to scattering. All these energies, however, will be considered much smaller than the width of the electron energy distribution in each layer of the superlattice. The last assumption allows us to derive a kinetic equation for the electron density matrix. We will make use of the Keldysh technique and derive this equation in the same way as the Boltzmann equation is usually derived.[30]

In the Keldysh technique the kinetic equation results from the Dyson equation which can be written in two equivalent forms,

$$\hat{G}^{-1}_{01} G_{12} = \sigma_z + \sigma_z \Sigma_{13} G_{32} , \tag{6}$$

$$\hat{G}^{-1*}_{02} G_{12} = \sigma_z + \Sigma_{13} G_{32} \sigma_z . \tag{7}$$

The matrix Keldysh Green function G_{12} depends on two sets of variables, $\{\mu, \mathbf{p}, t\}$, and a sum and integral with respect to the variables with the subscript 3 is implied in Eqs. (6) and (7). The operators in the left-hand sides (lhs),

$$\hat{G}^{-1}_0 = i\hbar \frac{\partial}{\partial t} - \mathcal{H}_1 , \tag{8}$$

contain the Hamiltonian

$$\mathcal{H}_1 = (E_p - eFd\nu)\delta_{\nu\nu'} + e\Phi_{\nu\nu'} . \tag{9}$$

σ_z is the Pauli matrix.

Green functions depend on two times, t_1 and t_2. Instead of them the difference, $t_1 - t_2$, and the sum, $t = (t_1 + t_2)/2$, times can be introduced. The characteristic values of the difference time is of the order of $\hbar$ divided by the characteristic energy, i.e., the maximum of the Fermi energy and temperature. The time t characterizes much more slow variation of occupation numbers. So, in the right-hand sides (rhs) of Eqs. (6) and (7) all functions can be considered dependent on the same time t and the integration with respect to t_3 is reduced to the integration with respect to difference times. Then the Fourier transform with respect to the difference time leads to the Green functions depending on t and a frequency ω which is a spectral variable.

The density matrix can be expressed in terms of G^{-+}. The difference and half of the sum of $(-+)$ matrix elements of Eqs. (6) and (7) has the form

$$i\hbar \frac{\partial}{\partial t} G^{-+}_{12} - [\mathcal{H}_1, G^{-+}]_{12} = \Sigma^r_{13} G^{-+}_{32} - \Sigma^{-+}_{13} G^a_{32} + G^r_{13} \Sigma^{-+}_{32} - G^{-+}_{13} \Sigma^a_{32} , \tag{10}$$

$$\hbar\omega G^{-+}_{12} - \tfrac{1}{2}\{\mathcal{H}_1, G^{-+}\}_{12} = \tfrac{1}{2}(\Sigma^r_{13} G^{-+}_{32} - \Sigma^{-+}_{13} G^a_{32} - G^r_{13} \Sigma^{-+}_{32} + G^{-+}_{13} \Sigma^a_{32}) . \tag{11}$$

For the calculation of self-energies we need also equations for advanced and retarded Green functions. Equations for them are obtained from Eqs. (6) and (7),

$$\hbar\omega G^r_{12} - \tfrac{1}{2}\{\mathcal{H}_1, G^r\}_{12} = 1 + \tfrac{1}{2}\{\Sigma^r, G^r\}_{12} , \tag{12}$$

$$G^a_{12} = G^{r*}_{21} . \tag{13}$$

Here $[a, b] = ab - ba$ and $\{a, b\} = ab + ba$.

The superlattice is uniform in the y, z plane and Green functions as well as self-energies are diagonal with respect to $\mathbf{p}$. We consider the case of a weak scattering when the energy uncertainty due to the scattering is much smaller than the width of the in-plane electron energy distribution, the electron temperature, or the Fermi energy. The important point is that the self-energies can be calculated in the first approximation. The correction to this approximation (diagrams with crossings of impurity lines) contain the ratio of the energy uncertainty to the Fermi energy or electron temperature, whichever is larger. In the case of a superlattice with narrow minibands such a characteristic energy can be the energy of in-plane motion, i.e., electron temperature. This fact makes possible the development of the quantitative transport theory for superlattices in the regime intermediate between band transport and hopping. It is worth noting that this is not the case in three-dimensional (3D) narrow-band semiconductors where the characteristic energy (for one-band transport) does not exceed the width of the band.

For simplicity we assume that electrons in different wells are scattered by different impurities. It implies that the screening radius is smaller than the period of the superlattice. Then

$$\Sigma^r_{\mu\mu'}(\omega, \mathbf{p}) = N_I C_{\mu\mu'\nu\nu'} \int \frac{d\mathbf{p'}}{(2\pi)^2} |V_{\mathbf{pp'}}|^2 G^r_{\nu\nu'}(\omega, \mathbf{p'}) , \tag{14}$$

$$\Sigma^a_{\mu\mu'}(\omega, \mathbf{p}) = \Sigma^{r*}_{\mu'\mu}(\omega, \mathbf{p}) , \tag{15}$$

$$\Sigma^{-+}_{\mu\mu'}(\omega,\mathbf{p}) = -N_I C_{\mu\mu'\nu\nu'} \int \frac{d\mathbf{p}'}{(2\pi)^2} |V_{\mathbf{pp}'}|^2 G^{-+}_{\nu\nu'}(\omega,\mathbf{p}') \ , \tag{16}$$

where

$$C_{\mu\mu'\nu\nu'} = \sum_{\alpha} J_{\mu-\alpha} J_{\mu'-\alpha} J_{\nu-\alpha} J_{\nu'-\alpha} \ , \tag{17}$$

N_I is the two-dimensional (2D) density of impurities, and $V_{\mathbf{pp}'}$ is the matrix element of one impurity potential.

The retarded Green function can be found from Eqs. (12) and (14). Equations (11) and (16) are linear and uniform with respect to $G^{-+}_{\nu\nu'}(\mathbf{p})$ and this function cannot be found from them. Actually, Eqs. (11) and (16) can be used to find only the dependence of G^{-+} on ω. The integral of G^{-+} with respect to ω , the density matrix

$$\rho_{\nu\nu'}(\mathbf{p},t) = \hbar \int \frac{d\omega}{2\pi i} G^{-+}_{\nu\nu'}(\omega,\mathbf{p},t) \ , \tag{18}$$

has to be determined with the help of Eq. (10). The equation for the density matrix can be obtained by the integration of Eq. (10) with respect to ω,

$$\frac{\partial \rho}{\partial t} - \frac{1}{i\hbar}[\mathcal{H}_1,\rho] = -\int \frac{d\omega}{2\pi} (\Sigma^r G^{-+} - \Sigma^{-+} G^a + G^r \Sigma^{-+} - G^{-+}\Sigma^a) \ . \tag{19}$$

In the rhs of this equation it is necessary to substitute the solution to Eqs. (11) and (16).

The operator of the electron velocity in the x direction in the Wannier representation has the form $v_{\alpha\beta} = (i\Lambda d/\hbar)(\delta_{\alpha,\beta+1} - \delta_{\alpha,\beta-1})$. So, the current density between the αth and $(\alpha+1)$th wells is

$$j_{\alpha,\alpha+1}(F) = \frac{2ie\Lambda}{\hbar} \sum_{\mu\mu'} \int \frac{d\mathbf{p}}{(2\pi)^2} (J_{\alpha-\mu} J_{\alpha+1-\mu'} - J_{\alpha+1-\mu} J_{\alpha-\mu'}) \rho_{\mu\mu'}(\mathbf{p},t) \ . \tag{20}$$

Equation (20) is reduced to the expression used by Tsu and Döhler[23] in the case of hopping conductivity and a uniform electric-field distribution (see Calecki, Palmier, and Chomette[26]).

IV. CONDUCTIVITY OF SUPERLATTICE WITH UNIFORMLY DISTRIBUTED ELECTRIC FIELD

If the electric field is uniformly distributed along the superlattice $\hat{\Phi} = 0$. Then, for a weak scattering, terms containing self-energies in Eqs. (11) and (12) can be neglected compared to $\hbar\omega$ and solutions to them are

$$G^r_{\nu\nu'}(\omega,\mathbf{p}) = \frac{\delta_{\nu\nu'}}{\hbar\omega - E_p + eFd\nu + i0} \ , \tag{21}$$

$$G^{-+}_{\nu\nu'}(\omega,\mathbf{p}) = 2\pi i \rho_{\nu\nu'}(\mathbf{p},t) \delta\left(\hbar\omega - E_p + eFd\frac{\nu+\nu'}{2}\right) \ . \tag{22}$$

The substitution of these expressions in Eq. (19) gives

$$\frac{\partial \rho_{\nu\nu'}(\mathbf{p})}{\partial t} - i\frac{eFd}{\hbar}(\nu-\nu')\rho_{\nu\nu'}(\mathbf{p})$$

$$= \frac{iN_I}{\hbar} \int \frac{d\mathbf{p}'}{(2\pi)^2} |V_{\mathbf{pp}'}|^2 \sum_{\mu\mu'} \left\{ C_{\nu\nu'\mu\mu'} \left[\frac{\rho_{\mu\mu'}(\mathbf{p}')}{E_{p'} - E_p + eFd(\nu - \frac{\mu+\mu'}{2}) + i0} - \frac{\rho_{\mu\mu'}(\mathbf{p}')}{E_{p'} - E_p + eFd(\nu' - \frac{\mu+\mu'}{2}) - i0} \right] + \frac{C_{\mu\mu\mu'\nu'}\rho_{\nu\mu'}(\mathbf{p})}{E_p - E_{p'} + eFd(\mu - \frac{\nu+\mu'}{2}) - i0} - \frac{C_{\mu\mu\mu'\nu}\rho_{\mu'\nu'}(\mathbf{p})}{E_p - E_{p'} + eFd(\mu - \frac{\nu'+\mu'}{2}) + i0} \right\} . \tag{23}$$

Making use of the identity $C_{\mu+\alpha,\mu'+\alpha,\nu+\alpha,\nu'+\alpha} = C_{\mu\mu'\nu\nu'}$ it is easy to show that Eq. (23) is translationary invariant. A stationary solution to Eq. (23) has the same invariance, i.e.,

$$\rho_{\nu\nu'}(\mathbf{p}) = \wp_{\nu-\nu'}(\mathbf{p}) \ . \tag{24}$$

We will solve Eq. (23) only in the most interesting instability case when

$$\Lambda, \ eFd \ll T_e, \tag{25}$$

where T_e is the effective electron temperature which characterizes the width of the electron energy distribution. (We do not assume that a real distribution is the equilibrium one with the temperature T_e. This quantity is used only for estimates.) In the case of an effective energy relaxation $T_e \sim T$. In the case of low temperature and an appreciable heating of the electron gas $T_e > T$. The esti-

mate of T_e under different conditions will be given later in this section [Eqs. (37) and (38)].

The energy differences $E_p - E_{p'}$ in the collision operator are of the order of T_e and the collision operator can be expanded in terms of eFd/T_e. After the expansion sums with respect to numbers of the Stark levels can be calculated explicitly (see Appendix A). Keeping only terms of the first and the second order in Eq. (23) we get

$$-i\frac{eFd}{\hbar}\nu\wp_\nu(\mathbf{p}) = \frac{2\pi N_I}{\hbar}\int\frac{d\mathbf{p}'}{(2\pi)^2}|V_{\mathbf{pp}'}|^2\Big\{[\delta_{\nu,0}\wp_0(\mathbf{p}') - \wp_\nu(\mathbf{p})]\delta(E_p - E_{p'}) + \Lambda\big\{[\wp_1(\mathbf{p}') + \wp_{-1}(\mathbf{p}')]\delta_{\nu,0} - \wp_0(\mathbf{p}')(\delta_{\nu,1} + \delta_{\nu,-1}) + \wp_{\nu+1}(\mathbf{p}) + \wp_{\nu-1}(\mathbf{p})\big\}\delta'(E_p - E_{p'}) + 2\Lambda^2\delta_{\nu,0}[\wp_0(\mathbf{p}') - \wp_0(\mathbf{p})]\delta''(E_p - E_{p'}) - \frac{1}{2\pi i}\frac{eFd\nu}{(E_p - E_{p'})^2}\wp_\nu(\mathbf{p})\Big\}. \quad (26)$$

The last term in the rhs of Eq. (26) is different from the first one in the lhs only by a factor. It can be considered as a renormalizaton of the electron charge and can be neglected. Then one can see that if the terms of the order of Λ/T_e are neglected at all the equation for $\wp_0$ is separated from equations for $\wp_\nu$ with $\nu \neq 0$ and does not contain an electric field. This is natural because $\wp_0$ is the distribution function in a layer and without tunneling it does not "know" about the electric field. Along with the tunneling in this equation it is necessary to take into account inelastic scattering which was not considered so far. The operator of the inelastic collisions, $I_{\text{in}}\wp_0$, can be added into Eq. (26) without making use of the Keldysh technique.

Equations (26) with $\nu \neq 0$ show that $\wp_\nu \sim (\Lambda/T_e)^{|\nu|}\wp_0$. That is $\wp_{|\nu|}$ with $|\nu| \geq 2$ can be neglected and we come up with the following equations:

$$\frac{2\pi N_I}{\hbar}\int\frac{d\mathbf{p}'}{(2\pi)^2}|V_{\mathbf{pp}'}|^2[\wp_0(\mathbf{p}') - \wp_0(\mathbf{p})][\delta(E_p - E_{p'}) + 2\Lambda^2\delta''(E_p - E_{p'})] + \hat{I}_{\text{in}}\wp_0(\mathbf{p}) + \frac{2\pi N_I\Lambda}{\hbar}\int\frac{d\mathbf{p}'}{(2\pi)^2}|V_{\mathbf{pp}'}|^2[\wp_1(\mathbf{p}') + \wp_{-1}(\mathbf{p}') + \wp_1(\mathbf{p}) + \wp_{-1}(\mathbf{p})]\delta'(E_p - E_{p'}) = 0, \quad (27)$$

$$\mp i\frac{eFd}{\hbar}\wp_{\pm1}(\mathbf{p}) = -\frac{1}{\hbar}\Gamma(E_p)\wp_{\pm1}(\mathbf{p}) + \frac{2\pi N_I\Lambda}{\hbar}\int\frac{d\mathbf{p}'}{(2\pi)^2}|V_{\mathbf{pp}'}|^2[\wp_0(\mathbf{p}) - \wp_0(\mathbf{p}')]\delta'(E_p - E_{p'}), \quad (28)$$

where for an isotropic energy spectrum and scattering (i.e., for $|V_{\mathbf{pp}'}|^2$ depending on $|\mathbf{p} - \mathbf{p}'|$),

$$\Gamma(E_p) = 2\pi N_I\int\frac{d\mathbf{p}'}{(2\pi)^2}|V_{\mathbf{pp}'}|^2\delta(E_p - E_{p'}) \quad (29)$$

depends only on the energy.

The first term in Eq. (27) describes an elastic relaxation in separate wells and it is the largest one. If all other terms are neglected it leads to a distribution function depending only on the energy, i.e., $\wp_0(\mathbf{p}) = \wp_0(E_p)$. An equation for this function can be obtained by the averaging of Eq. (27) with respect to the energy.[32,33] The elastic relaxation is averaged out. For $\wp_0$ depending on the energy $\wp_{\pm1}$ also depends on the energy. The resulting equations are

$$\hbar\hat{I}_E\wp_0(E) + \Lambda\frac{d}{dE}\Gamma(E)\left[\wp_1(E) + \wp_{-1}(E) + 2\Lambda\frac{d\wp_0}{dE}\right] = 0, \quad (30)$$

$$\mp ieFd\wp_{\pm1}(E) = -\Gamma(E)\wp_{\pm1}(E) - \Lambda\Gamma(E)\frac{d\wp_0}{dE}, \quad (31)$$

where $\hat{I}_E$ is the operator of the energy relaxation. Equations (30) and (31) remind us very much of the Boltzmann equations for the even and odd parts of the electron distribution function with respect to the electron momentum in the theory of hot electrons in wide band semiconductors.[32,33]

Near the instability threshold and for T_e smaller than the optical-phonon energy the main relaxation mechanism is the acoustic-phonon scattering. The emission of an optical phonon without a transition between different Stark levels in this case is impossible. The optical-phonon emission in transitions between Stark levels is limited by a very wide minibands. The last possibility is discussed in detail in Sec. VII. For the energy relaxation due to acoustic-phonon scattering (compare Refs. 32 and 33)

$$\hbar\hat{I}_E\wp_0(E) = \frac{d}{dE}Q(E)\left[\wp_0(1-\wp_0) + T\frac{d\wp_0}{dE}\right] , \quad (32)$$

where

$$Q(E) = \frac{\pi^2 m\Xi^2}{2\rho_0 d_w^3}\left(1 + \frac{3Emd_w^2}{\pi^2\hbar^2}\right) , \quad (33)$$

ρ_0 is the crystal density, m is the in-plane effective mass, d_w is the width of a well, and Ξ is the deformation potential.

Equation (31) gives

$$\wp_{\pm1}(E) = -\frac{\Lambda\Gamma(E)}{\Gamma(E) \mp ieFd}\frac{d\wp_0}{dE} , \quad (34)$$

and then Eq. (30) with the help of Eq. (32) is reduced to

$$Q(E)\left[\wp_0(1-\wp_0) + T\frac{d\wp_0}{dE}\right] + 2\Lambda^2(eFd)^2\frac{\Gamma(E)}{\Gamma^2(E)+(eFd)^2}\frac{d\wp_0}{dE} = 0 . \quad (35)$$

The solution to this equation is

$$\wp_0 = \left\{\exp\left[\int_0^E\left(T + \frac{2\Lambda^2(eFd)^2}{Q(E)}\frac{\Gamma(E)}{\Gamma^2(E)+(eFd)^2}\right)^{-1}dE - \zeta\right] + 1\right\}^{-1} , \quad (36)$$

where ζ is a normalization constant. Even without an exact calculation Eq. (36) allows us to estimate the importance of the heating of the electron gas near the instability threshold. We can estimate the momentum and energy relaxation times as $\tau_p \sim \hbar/\Gamma$ and $\tau_E \sim \hbar T_e/Q$, respectively. Near the instability threshold, when $eFd \sim \Gamma$, the width of the electron energy distribution T_e in the case of a weak heating and the condition for weak heating are

$$T_e \sim T , \qquad \frac{\Lambda^2}{T^2}\frac{\tau_E}{\tau_p} \ll 1 . \quad (37)$$

Because of $\tau_E \gg \tau_p$ the condition Eq. (37) is satisfied only for $\Lambda \ll T$. For the band width $4\Lambda \sim 0.5$ meV the last condition is satisfied for temperatures about 30 K and higher. This estimate shows that for lower temperature or for a wider band one can expect an appreciable heating of the electron gas. In such a case

$$T_e \sim \Lambda\left(\frac{\tau_E}{\tau_p}\right)^{1/2} , \qquad \frac{\Lambda^2}{T^2}\frac{\tau_E}{\tau_p} \gg 1 . \quad (38)$$

Now it is possible to justify Eq. (25). In the estimate we assume that $eFd \sim \Gamma$ because this region of the electric field is important for the instability, and $\Gamma^2/\Lambda^2 \ll \tau_E/\tau_p$ because in the opposite case the resonance tunneling is smeared so much that the width of the resonance can be of the order of the separation between the levels in a well. Then for a weak heating, Eq. (37), $T_e \sim T \geq \Lambda(\tau_E/\tau_p)^{1/2} \gg \Lambda, \Gamma$. In the case of a strong heating, Eq. (38), $\Gamma/T_e \ll (\Gamma/\Lambda)(\tau_p/\tau_E)^{1/2} \ll 1$ and $\Lambda/T_e \sim (\tau_p/\tau_E)^{1/2} \ll 1$. That is, in both cases Eq. (25) is satisfied.

For a uniform electron distribution the expression for current density Eq. (20) can be simplified with the help of Eq. (A2) of Appendix A and takes the form

$$j(F) = \frac{2ie\Lambda}{\hbar}\int\frac{d\mathbf{p}}{(2\pi)^2}(\wp_{-1} - \wp_1) = -\frac{2me^2Fd\Lambda^2}{\pi\hbar^3}\int_0^\infty\frac{\Gamma(E)}{\Gamma^2(E)+(eFd)^2}\frac{d\wp_0}{dE}dE , \quad (39)$$

where m is an in-plane effective mass. This expression shows that the Ohm's law is satisfied for $eFd \ll \Gamma$ and $j \propto 1/F$ in the opposite case.

V. STABILITY OF THE STATIONARY REGIME

A. Kinetic equation

In the stability problem we have to study a nonuniform distribution of electrons along the superlattice and cannot assume that $\rho_{\nu\nu'}$ depends only on $\nu-\nu'$. We have also to take into account the self-consistent potential Φ. It is important to note that the Φ modifies the electron spectrum so that, strictly speaking, Eqs. (21) and (22) are not justified in the presence of a nonuniform electric field. However, the corrections to the spectrum are of the order of $e\Phi/T$ (see Appendix B) and we neglect them.

The difference of the subscripts $\nu-\nu'$ in $\rho_{\nu\nu'}$ characterizes the widths of the electron wave packet and its position is $(\nu+\nu')/2$. It is convenient to study nonuniform perturbation in a uniform system in the Fourier representation. For that reason we introduce the Fourier transform of $\rho_{\nu\nu'}$ with respect to $(\nu+\nu')/2$,

$$\delta\rho_{\nu,k}(E_p,t) = \sum_{\nu'-\nu''=\nu} e^{-ikd\frac{\nu'+\nu''}{2}}\rho_{\nu'\nu''}(t,E_p) . \quad (40)$$

Sums containing Bessel functions are calculated with the help of Eq. (A1) and after the transformation Eq. (23) takes the form

$$\left(\hbar\frac{\partial}{\partial t}-ieFd\nu\right)\rho_{\nu,k}(\omega,\mathbf{p})=-ie[\hat{\Phi},\rho]_{\nu,k}+iN_I\int\frac{d\mathbf{p}'}{(2\pi)^2}|V_{\mathbf{pp}'}|^2$$
$$\times\sum_\mu\left\{\rho_{\mu,k}(\omega,\mathbf{p}')\;A_\nu\left(k-ieF\frac{\partial}{\partial E_p}\right)\;A_\mu\left(-k+ieF\frac{\partial}{\partial E_p}\right)\right.$$
$$\times\left[\frac{1}{E_{p'}-E_p+eFd\nu/2+i0}-\frac{1}{E_{p'}-E_p-eFd\nu/2-i0}\right]$$
$$-\rho_{\mu,k}(\omega,\mathbf{p})\;A_{\nu-\mu}\left(-ieF\frac{\partial}{\partial E_p}\right)\;A_0\left(ieF\frac{\partial}{\partial E_p}\right)$$
$$\left.\times\left[\frac{e^{ikd(\nu-\mu)/2}}{E_{p'}-E_p+eFd\nu/2+i0}-\frac{e^{-ikd(\nu-\mu)/2}}{E_{p'}-E_p-eFd\nu/2-i0}\right]\right\}. \tag{41}$$

The commutator in the rhs of Eq. (41) is

$$[\hat{\Phi},\rho]_{\nu,k}=\sum_\mu\sum_{k'}2i\sin\left(\frac{k\mu-k'\nu}{2}d\right)\;A_{\nu-\mu}(k-k')\;\Phi_{k-k'}\rho_{\mu,k'}(\mathbf{p})\;. \tag{42}$$

Here we introduced the notations

$$A_\nu(k)=i^{-\nu}J_\nu\left(\frac{4\Lambda}{eFd}\;\sin\frac{kd}{2}\right) \tag{43}$$

and

$$\Phi_k=\sum_\alpha\Phi_{\alpha\alpha}e^{-ikd\alpha}\;. \tag{44}$$

The stability of the uniform distribution of the electric field and electrons along the superlattice is determined by the evolution of a small nonuniform perturbation $\delta\rho_{\nu,k}$ of the density matrix,

$$\rho_{\nu,k}(\mathbf{p},t)=\delta_{k,0}\;\wp_\nu(\mathbf{p})+\delta\rho_{\nu,k}(\mathbf{p},t)\;. \tag{45}$$

We linearize Eq. (41) with respect to $\delta\rho_{\nu,k}$ and put $\delta\rho_{\nu,k}(\mathbf{p},t)=\delta\rho_{\nu,k}(\mathbf{p})e^{-i\omega t}$. We also expand this equation in terms of eFd/T_e and average over the surface with a constant energy in the same way as we did this in the uniform case. The result has the form

$$\left[-i\hbar\omega-ieFd\nu+\Gamma(E)\right]\delta\rho_{\nu,k}(\omega,E)$$
$$=-ie[\hat{\Phi},\wp]_{\nu,k}+\Gamma(E)\;A_\nu(k)\;\sum_\mu A^*_\mu(k)\delta\rho_{\mu,k}(E)$$
$$+\Lambda\Gamma(E)\frac{\partial}{\partial E}\left\{\left[\delta\rho_{1,k}(E)+\delta\rho_{-1,k}(E)\right]\delta_{\nu,0}-\delta\rho_{0,k}(E)\left(-4ik\frac{\Lambda}{eF}\delta_{\nu,0}+\delta_{\nu,1}+\delta_{\nu,-1}\right)\right\}$$
$$+\frac{\Lambda}{2}\;\frac{d\Gamma(E)}{dE}\left\{\left[\delta\rho_{1,k}(E)+\delta\rho_{-1,k}(E)\right]\delta_{\nu,0}-\delta\rho_{0,k}(E)\left(-4ik\frac{\Lambda}{eF}\delta_{\nu,0}+\delta_{\nu,1}+\delta_{\nu,-1}\right)\right.$$
$$\left.+\delta\rho_{\nu+1,k}(E)+\delta\rho_{\nu-1,k}(E)\right\}+2\delta_{\nu,0}\Lambda^2\frac{\partial}{\partial E}\Gamma(E)\frac{\partial\delta\rho_{0,k}(E)}{\partial E}+\delta_{\nu,0}\hat{I}_E\delta\rho_{0,k}\;. \tag{46}$$

Derivatives with respect to E in these equations are taken into account in order to consider the perturbation of the heating of the electron gas (compare with the case of the stationary conductivity, Sec. IV). This perturbation is important only for a small enough k (see Sec. V C). Because of that reason all terms containing these derivatives are calculated in the lowest order on magnitude in k. Also, a linearized operator of energy relaxation is added in the equation with $\nu=0$.

The commutator in the rhs is simplified to

$$[\hat{\Phi},\wp]_{\nu,k}=-2i\sin(kd/2)\left(A_{\nu+1}(k)\wp_{-1}-A_{\nu-1}(k)\wp_1\right)\Phi_k$$
$$=-\frac{eFd}{\Lambda}\left[\nu A_\nu(k)\;\mathrm{Re}\wp_1\right.$$
$$\left.+i\left(\Lambda\frac{\partial}{\partial\Lambda}A_\nu(k)\right)\;\mathrm{Im}\wp_1\right]\Phi_k\;. \tag{47}$$

Equation (46) has to be supplemented with the equation

for the self-consistent electric potential Φ which is determined from the Poisson equation.

B. Solution of the Poisson equation

The electric potential Φ results from the redistribution of electrons between layers of the superlattice. It is also modified as a result of a polarization of electrons in each layer. That is, Φ satisfies the equation

$$\epsilon \frac{d^2\Phi}{dx^2} = 0 \tag{48}$$

within a barrier and

$$\epsilon' \frac{d^2\Phi}{dx^2} = -4\pi e\left[n_\alpha |\psi_\alpha(x-\alpha d)|^2 - n w_1^2(x-\alpha d)\right] \tag{49}$$

within the αth well. Here ϵ and ϵ' are dielectric constants in barriers and wells, respectively, $\psi_\alpha(x)$ is an electron wave function perturbed by Φ, and n is an average electron concentration in a well. The actual electron concentration

$$n_\alpha \equiv n + \delta n_\alpha = \sum_{\nu\nu'} \int \frac{2d\mathbf{p}}{(2\pi)^2} J_{\nu-\alpha} J_{\nu'-\alpha} \rho_{\nu\nu'} \ . \tag{50}$$

The integration of Eq. (48) leads to a potential drop across the αth barrier ($\alpha d - d + d_w/2 < x < \alpha d - d_w/2$)

$$\phi_\alpha^- - \phi_{\alpha-1}^+ = -D_\alpha \frac{d-d_w}{\epsilon} \ , \tag{51}$$

where $\phi_\alpha^\pm = \Phi(\alpha d \pm d_w/2)$ and D_α is an electric displacement which is constant within any barrier.

The integration of Eq. (49) gives the following expressions for the displacement and potential in the αth well, $\alpha d - d_w/2 < x < \alpha d - d_w/2$,

$$\begin{aligned} D(x) &= D_\alpha + 4\pi e \int_{\alpha d - d_w/2}^{x} dx' \left\{ n_\alpha |\psi_\alpha(x'-\alpha d)|^2 - n w_1^2(x'-\alpha d) \right\} \\ &= D_{\alpha+1} - 4\pi e \int_{x}^{\alpha d + d_w/2} dx' \left\{ n_\alpha |\psi_\alpha(x'-\alpha d)|^2 - n w_1^2(x'-\alpha d) \right\} \ , \end{aligned} \tag{52}$$

$$\Phi(x) = \phi_\alpha^- - \frac{1}{\epsilon'} \int_{\alpha d - d_w/2}^{x} D(x')dx' \ = \phi_\alpha^+ + \frac{1}{\epsilon'} \int_{x}^{\alpha d + d_w/2} D(x')dx' \ , \tag{53}$$

and

$$D_{\alpha+1} - D_\alpha = 4\pi e \delta n_\alpha \ . \tag{54}$$

Under the condition $eFd \ll \mathcal{E}_2 - \mathcal{E}_1$ the perturbation of the wave function in the first level is a linear functional of the potential inside the well, and

$$|\psi_\alpha(x)|^2 = w_1^2(x) + \hat{K}\Phi \ . \tag{55}$$

The operator $\hat{K}$ can be found explicitly with the help of quantum-mechanical perturbation theory in terms of the eigenfunctions of the separate well.[34]

Even in the linear approximation the solution to Eqs. (52), (53), and (55) is very cumbersome and requires wave functions of all levels in the well. Actually this solution is not necessary. We are going to show that each well can be characterized by only two numbers, effective dielectric constant ϵ_{eff} and effective capacitance C_{eff} . For these numbers we will obtain formal analytic expressions in an operator form.

Equation (52) can be written in a symmetric form

$$D(x) = \frac{D_{\alpha+1} + D_\alpha}{2} + 4\pi e \delta n_\alpha \hat{P} w_1^2 + 4\pi e n \hat{P}\ \hat{K}\Phi \ , \tag{56}$$

where the operator $\hat{P}$ is defined by the relation

$$\hat{P}f = \frac{1}{2}\left(\int_{\alpha d - d_w/2}^{x} dx' - \int_{x}^{\alpha d + d_w/2} dx' \right) f(x') \ . \tag{57}$$

With the help of this operator the symmetrized Eq. (53) has the form

$$\Phi(x) = \frac{\phi_\alpha^+ + \phi_\alpha^-}{2} - \frac{1}{\epsilon'} \hat{P} D \ . \tag{58}$$

The substitution of Eq. (58) in Eq. (56) leads to an integral equation for $D(x)$. In this substitution one has to take into account that a constant potential does not change wave functions in a well, so that $\hat{K}\text{const} = 0$ while $\hat{P}\ 1 = (x - \alpha d) \neq 0$. A formal solution to the obtained equation is

$$\begin{aligned} D(x) = \frac{D_{\alpha+1} + D_\alpha}{2} \left(1 + \frac{4\pi e n}{\epsilon'} \hat{P}\ \hat{K}\ \hat{P} \right)^{-1} 1 \\ + 4\pi e \delta n_\alpha \left(1 + \frac{4\pi e n}{\epsilon'} \hat{P}\ \hat{K}\ \hat{P} \right)^{-1} \hat{P} w_1^2 \ , \end{aligned} \tag{59}$$

which gives the following expression for the potential:

$$\Phi(x) = \frac{\phi_\alpha + \phi_{\alpha+1}}{2} - \frac{D_{\alpha+1} + D_\alpha}{2\epsilon'} \hat{P} \left(1 + \frac{4\pi e n}{\epsilon'} \hat{P}\,\hat{K}\,\hat{P}\right)^{-1} 1 - \frac{4\pi e \delta n_\alpha}{\epsilon'} \hat{P} \left(1 + \frac{4\pi e n}{\epsilon'} \hat{P}\,\hat{K}\,\hat{P}\right)^{-1} \hat{P} w_1^2 \ . \quad (60)$$

We assume that each well is symmetric with respect to its center. Then wave functions in a well are odd or even. The operator $\hat{K}$ being applied to an odd or even function keeps its symmetry while the operator $\hat{P}$ changes it. Then for $x = \alpha d \pm d_w/2$ Eq. (60) gives

$$\phi_\alpha^- - \phi_\alpha^+ = \frac{d_w}{\epsilon_{\text{eff}}} \frac{D_\alpha + D_{\alpha+1}}{2} \ , \quad (61)$$

where

$$\frac{1}{\epsilon_{\text{eff}}} = \frac{1}{\epsilon'} \left\langle \left(1 + \frac{4\pi e n}{\epsilon'} \hat{P}\,\hat{K}\,\hat{P}\right)^{-1} 1 \right\rangle , \quad (62)$$

and we introduced another notation

$$\langle f \rangle = \frac{1}{d_w} \int_{-d_w/2}^{d_w/2} f(x')dx' \ . \quad (63)$$

We can calculate the diagonal matrix element of the potential, $\Phi_\alpha \equiv \Phi_{\alpha\alpha}$, using Eq. (60) and Eq. (59). The result is

$$\Phi_\alpha + \frac{e}{C_{\text{eff}}} \delta n_\alpha = \phi_\alpha^- - \frac{d_w}{2\epsilon_{\text{eff}}} D_\alpha = \phi_\alpha^+ + \frac{d_w}{2\epsilon_{\text{eff}}} D_{\alpha+1} \ , \quad (64)$$

where

$$\frac{1}{C_{\text{eff}}} = \frac{\pi d_w}{\epsilon_{\text{eff}}} - \frac{4\pi}{\epsilon'} \left\langle (\hat{P} w_1^2) \left(1 + \frac{4\pi e n}{\epsilon'} \hat{P}\,\hat{K}\,\hat{P}\right)^{-1} (\hat{P} w_1^2) \right\rangle . \quad (65)$$

In the derivation of Eq. (65) we used the relation $\langle f_1 \hat{P} f_2 \rangle = -\langle f_2 \hat{P} f_1 \rangle$.

The explicit form of the operator in the right-hand side of Eq. (65) can be easily found in terms of the eigenfunctions of the operator $\hat{P}\hat{K}\hat{P}$. The calculation of the eigenfunctions can be made numerically for any specific structure.

Eliminating the displacement and $\phi_\alpha^\pm$ from Eqs. (51), (54), and (64) we obtain a relation between the potential matrix elements and redistribution of electrons in wells,

$$\Delta_\alpha[\Phi] + \frac{e}{C_{\text{eff}}} \Delta_\alpha[\delta n] = -\frac{4\pi e d}{\bar{\epsilon}} \delta n_\alpha \ , \quad (66)$$

reminding us of the usual Poisson equation. Here

$$\frac{d}{\bar{\epsilon}} = \frac{d - d_w}{\epsilon} + \frac{d_w}{\epsilon_{\text{eff}}} \quad (67)$$

is an average dielectric constant, and we introduced the notation $\Delta_\alpha[f] = f(\alpha+1) + f(\alpha-1) - 2f(\alpha)$.

Equation (66) shows that an equivalent electrostatic circuit for a superlattice is not just a chain of capacitors connected in series as it was assumed in Ref. 12. It is reduced to the chain of capacitors if the concentration n increases and both ϵ_{eff} and C_{eff} go to infinity. Equation (66) in this case is reduced to

$$\Delta_\alpha[\Phi] = -\frac{4\pi e(d - d_w)}{\epsilon} \delta n_\alpha \ . \quad (68)$$

The appearance of the second term in the lhs in Eq. (66) can be explained in the following way. Equation (68) describes the potential differences between the layers in an external electric field. The sources of that field are the charges of the layers. That is, each charged layer brings about an electric field which induces the potential differences between all other layers. However, a charge not only brings about a field. It also changes the potential of the charged layer according to its capacitance. This second effect is taken into account in the second term in the lhs in Eq. (66). A valid equivalent circuit (Fig. 2) contains an element which accounts for this effect. A shift of the potential in a layer proportional to the perturbation of the electron concentration can be represented by a capacitor which separates the point where electrons are confined (A) from that where the potential is measured (B). The using of this circuit is limited by the condition $eFd \ll \mathcal{E}_2 - \mathcal{E}_1$.

Equation (66) gives the following expression for the Fourier component of the potential:

$$\Phi_k = \left(\frac{\pi d}{\bar{\epsilon} \sin^2(kd/2)} - \frac{1}{C_{\text{eff}}} \right) e d \delta n_k \ , \quad (69)$$

where

$$\delta n_k = \frac{1}{d} \sum_\alpha \mathrm{e}^{-ikd\alpha} \delta n_\alpha = \frac{1}{d} \sum_\nu A_\nu^*(k) \int \frac{2d\mathbf{p}}{(2\pi)^2} \rho_{\nu,k}(E_p) \ . \quad (70)$$

The factor $1/d$ in Eq. (70) allows us to consider δn_k as the Fourier component of a three-dimensional electron concentration in the superlattice. The expression in the parentheses in Eq. (69) is positive because $1/C_{\text{eff}} < \pi d/\bar{\epsilon}$ according to Eqs. (65) and (67).

C. Stability with respect to long-wavelength perturbations

The exact solution of Eqs. (46) and (69) is very complicated and we start with the case of perturbations with a long wavelength $2\pi/k$. We have a few scales for the wavelength and corresponding scales for k. The minimal length scale is the period of the superlattice d, and the corresponding $k \sim 1/d$. The next scale is the size of the Stark states Λ/eF and the corresponding $k \sim eF/\Lambda$. If the perturbation wavelength is larger than both of those scales,

$$kd \ll 1, \frac{eFd}{\Lambda}, \tag{71}$$

then the superlattice can be considered as a continuous medium and the equations for the density-matrix elements can contain only the first and the second powers of k which correspond to the first and the second coordinate derivatives. Equation (71) allows us to simplify the coefficients $A_\nu(k)$,

$$A_0(k) = 1 - \left(\frac{\Lambda}{eF}\, k\right)^2 , \; A_{\pm 1}(k) = -ik\,\frac{\Lambda}{eF} . \tag{72}$$

We will see that for small k a characteristic frequency of instable fluctuations is also small and we can neglect $\hbar\omega$ compared to Γ. Because of the same Eq. (72) we can keep of all Eqs. (46) only those with $\nu = 0, \pm 1$, which takes the form

$$-i\hbar\omega\delta\rho_{0,k} = -kd\Lambda(\delta\rho_{1,k} - \delta\rho_{-1,k}) + \frac{\partial}{\partial E}\left\{\Lambda\Gamma(E)\left(\delta\rho_{1,k} + \delta\rho_{-1,k} + 2\Lambda\frac{\partial\delta\rho_{0,k}}{\partial E}\right) + Q(E)\left((1-2\wp_0)\delta\rho_{0,k} + T\frac{\partial\delta\rho_{0,k}}{\partial E}\right)\right\}, \tag{73}$$

$$[\mp ieFd + \Gamma(E)]\,\delta\rho_{\pm 1,k} = \pm ekd\wp_{\pm 1}\Phi_k - ik\Gamma(E)\,\frac{\Lambda}{eF}\delta\rho_{0,k} - \Lambda\Gamma(E)\,\frac{\partial\delta\rho_{0,k}}{\partial E} . \tag{74}$$

The simple form of the first term in the rhs in Eq. (73) is obtained with the help of Eq. (74).

Equations (69) and (70) in this case also take a more simple form,

$$\Phi_k = \frac{4\pi e}{\bar{\epsilon}k^2}\,\delta n_k , \tag{75}$$

$$\delta n_k = \frac{1}{d}\int \frac{2d\mathbf{p}}{(2\pi)^2}\delta\rho_{0,k}(\omega, E_p) , \tag{76}$$

which is equivalent to the Poisson equation.

It is worth noting that Eqs. (73) and (74) are very similar to the kinetic equation describing fluctuations in a uniform semiconductor under a strong uniform electric field (see, e.g., Ref. 35).

Even in the continuous-medium approximation the competition of the electron transfer in the real space and in the energy makes the study of fluctuations very difficult. Because of this reason we consider only extreme cases when one of those phenomena dominates. We start with the case of the very-long-wavelength fluctuations when the transfer in the real space, i.e., the first term in the rhs in Eq. (73), can be neglected compared to the energy relaxation, i.e., the last term in the braces in Eq. (73). In this case a strong energy relaxation prevents fluctuation of the electron energy distribution. Only the space electron distribution can fluctuate,

$$\delta\rho_{0,k} = \frac{\partial\wp_0}{\partial n}\, d\delta n_k . \tag{77}$$

The space fluctuations determine all time scales so that the condition for Eq. (77) can be written as $\omega\tau_E \ll 1$. The last inequality also justifies the omission of $\hbar\omega$ in the parentheses in the lhs in Eq. (74) because $\hbar\omega \ll \hbar/\tau_E \ll \hbar/\tau_p = \Gamma$.

Substituting Eq. (77) in Eq. (74) we can calculate the electric current density perturbation

$$\delta j_k = \sum_\alpha j_{\alpha,\alpha+1}\, e^{-ikd\alpha} \cong \frac{2ie\Lambda}{\hbar}\int \frac{d\mathbf{p}}{(2\pi)^2}(\delta\rho_{-1,k} - \delta\rho_{1,k}) \tag{78}$$

[compare Eq. (20)]. In the second equation we neglected small terms containing $\delta\rho_{\nu,k}$ with $|\nu| > 1$. The result is identical with the result of the phenomenological theory in a uniform semiconductor [see Eq. (7.2) in Ref. 36 and Eq. (2.41) in Ref. 35],

$$\delta j_k = -\sigma_{\rm dif}\, ik\Phi_k + ev_d\delta n_k - e\mathcal{D}ik\delta n_k , \tag{79}$$

and the only specifics is in the expressions for the coefficients, the differential conductivity,

$$\sigma_{\rm dif} = \left(\frac{\partial j}{\partial F}\right)_{\wp_0} = -\frac{2me^2d\Lambda^2}{\pi\hbar^3}\int \Gamma(E)\frac{\Gamma^2(E) - (eFd)^2}{[\Gamma^2(E) + (eFd)^2]^2}\,\frac{\partial\wp_0}{\partial E}\, dE , \tag{80}$$

the drift velocity

$$v_d = \frac{d}{e}\frac{\partial j}{\partial n} = -\frac{2meFd^2\Lambda^2}{\pi\hbar^3}\int \frac{\Gamma(E)}{\Gamma^2(E) + (eFd)^2}\,\frac{\partial^2\wp_0}{\partial n\partial E}\, dE , \tag{81}$$

and the diffusion coefficient,

$$\mathcal{D} = \frac{2md^2\Lambda^2}{\pi\hbar^3}\int \frac{\Gamma(E)}{\Gamma^2(E) + (eFd)^2}\,\frac{\partial\wp_0}{\partial n}\, dE . \tag{82}$$

The derivative with respect to F in Eq. (80) is calculated for a constant $\wp_0$, i.e., only with respect to explicit F in Eq. (39). It is identical with the derivative without such a limitation in the case when the heating of the electron gas can be neglected.

By the integrating of Eq. (73) we obtain the conservation law for the electric charge,

$$-ie\omega\delta n_k + ik\delta j_k = 0 . \tag{83}$$

Equations (79), (83), and (75) lead to the well-known dispersion relation (see Refs. 36, 9, and 18)

$$\omega(k) = kv_d - i\left(\frac{4\pi}{\bar{\epsilon}}\sigma_{\rm dif} + \mathcal{D}k^2\right) . \quad (84)$$

Equation (84) shows that the instability is related to the negative differential conductivity which eventually takes place for high enough electric field. Near the instability threshold, when $|\sigma_{\rm dif}|$ is small, only fluctuations in the region $0 < k < k_m$ are unstable, where $k_m = (4\pi|\sigma_{\rm dif}|/\bar{\epsilon}\,\mathcal{D})^{1/2}$ is limited by diffusion. The growing fluctuations drift with the velocity v_d.

Now we consider the second extreme case, $\omega\tau_E \gg 1$. This condition allows us to neglect all the terms with the derivatives with respect to E in Eqs. (73) and (74). Then eliminating $\delta\rho_{\pm1,k}$ from Eqs. (73) and (74) we get

$$\left(-i\hbar\omega + 2(kd)^2\frac{\Lambda^2\Gamma(E)}{\Gamma^2(E)+(eFd)^2}\right)\delta\rho_{0,k} = 2(kd)^2\Lambda^2\Gamma(E)\,\frac{\Gamma^2(E)-(eFd)^2}{[\Gamma^2(E)+(eFd)^2]^2}\,\frac{d\wp_0}{dE}\,e\Phi_k . \quad (85)$$

One can express δn_k in terms of Φ_k using Eqs. (85) and (76). The comparison of this expression with Eq. (75) results in the dispersion relation

$$\frac{8me^2d\Lambda^2}{\bar{\epsilon}\hbar^2}\int_0^\infty \Gamma(E)\,\frac{\Gamma^2(E)-(eFd)^2}{[\Gamma^2(E)+(eFd)^2]^2}\,\frac{d\wp_0}{dE} \times\frac{dE}{-i\hbar\omega + 2(kd)^2\frac{\Lambda^2\Gamma(E)}{[\Gamma^2(E)+(eFd)^2]}} = 1 . \quad (86)$$

In Eq. (86) the functional dependence $\omega(k)$ is determined not only by the distribution function $\wp_0(E)$ but also by $\Gamma(E)$ and it cannot be presented in a more simple form. In the case when the energy scale of the variation of $\Gamma(E)$ is larger than T_e we can put $\Gamma(E) = \Gamma(0)$ in Eq. (86) and

$$\omega(k) = -i\left(\frac{4\pi}{\bar{\epsilon}}\sigma_{\rm dif} + \mathcal{D}k^2\right) , \quad (87)$$

which reminds us of Eq. (84). A weak dependence of Γ on E leads to a renormalization of the diffusion coefficient. The energy dependence of Γ is determined by the scattering potential [see Eq. (29)]. Typically $|V_{\mathbf{pp}'}|^2$ monotonously decreases with $|\mathbf{p}-\mathbf{p}'|$ and in such a case $\Gamma(E)$ is also a monotonously decreasing function. Our numerical calculation show that then there can be only one unstable solution to Eq. (86). For this solution $\omega = i{\rm Im}\omega$ and ${\rm Im}\omega(k)$ is a monotonous function. $\omega(0) = -4\pi i\sigma_{\rm dif}/\bar{\epsilon}$ is an exact solution, so the instability threshold is $\sigma_{\rm dif} = 0$ in any case. If $\sigma_{\rm dif} < 0$ then fluctuations are unstable in the interval $0 < k < k_m$, where

$$k_m^2 = \frac{4me^2}{\bar{\epsilon}\hbar^2 d}\int_0^\infty \frac{\Gamma^2(E)-(eFd)^2}{\Gamma^2(E)+(eFd)^2}\,\frac{d\wp_0}{dE}\,dE \quad (88)$$

is an exact solution to Eq. (86). That is, the qualitative behavior of $\omega(k)$ in the case of $\omega\tau_E \gg 1$ is different from the phenomenological case only by the lack of the drift.

Equation (80) shows that the electric field at the instability threshold $F_{\rm th}$ satisfies the condition $eF_{\rm th}d \sim \Gamma$. If in the integral in Eq. (88) $\Gamma - eFd \sim \Gamma$ and $\wp_0(0) \approx 1$ (a high enough doping) then $k_m^2 \sim q_s/d$, where $q_s = 2me^2/\bar{\epsilon}\hbar^2$ is the screening parameter. Usually $q_s d \geq 1$ and the condition Eq. (71) is violated. It means that the condition Eq. (71) is satisfied only in a small vicinity near the threshold, $F - F_{\rm th} \ll F_{\rm th}$. The same condition also justifies the omission of $\hbar\omega$ in the parentheses in the lhs in Eq. (74). Indeed, near the instability threshold Eq. (80) gives the estimate

$$\sigma_{\rm dif} \sim \frac{me^2d\Lambda^2}{\hbar^3\Gamma}\,\frac{F-F_{\rm th}}{F_{\rm th}}, \quad (89)$$

which means that $\hbar\omega/\Gamma \sim q_s d(\Lambda/\Gamma)^2(F-F_{\rm th})/F_{\rm th}$.

D. Dispersion relation for short-wavelength fluctuation

The results of Sec. V C can be generalized for large values of k. For this purpose we neglect in Eq. (46) the energy relaxation operator and all terms containing derivatives with respect to the energy. Then we obtain

$$\left[-i\hbar\omega - ieFd\nu + \Gamma(E)\right]\delta\rho_{\nu,k}(\omega,E) = -ie[\hat{\Phi},\wp]_{\nu,k} + \Gamma(E)\,A_\nu(k)\sum_\mu A_\mu^*(k)\delta\rho_{\mu,k}(E) , \quad (90)$$

where the commutator in the rhs is defined in Eq. (47). Because of the lack of a scattering potential correlation in different wells the integral operator in the rhs of Eq. (90) appears to be degenerate. It means that Eq. (90) easily can be solved with respect to the sum in the rhs. The result is

$$\sum_\mu A_\mu^*(k)\delta\rho_{\mu,k}(\omega,E_p) = -ie\,\frac{-i\hbar\omega+\Gamma(E_p)}{-i\hbar\omega+\Gamma(E_p)g(\omega,k,E_p)}\sum_\nu \frac{A_\nu^*(k)[\hat{\Phi},\wp]_{\nu,k}}{-i\hbar\omega - ieFd\nu + \Gamma(E_p)} , \quad (91)$$

where we introduced a notation

$$g(\omega,k,E) = 1 - \sum_\nu A_\nu^*(k)\frac{\Gamma(E)-i\hbar\omega}{\Gamma(E)-i\hbar\omega-ieFd\nu}A_\nu(k) . \quad (92)$$

The rhs of Eq. (91) is also expressed in terms of $g(\omega,k,E)$ and we get

$$\sum_\mu A^*_\mu(k)\delta\rho_{\mu,k}(\omega,E_p)=\frac{1}{\Lambda}\ \frac{e\Phi_k}{-i\hbar\omega+\Gamma(E_p)g(\omega,k,E_p)} \times\left\{-[\Gamma(E_p)-i\hbar\omega]g(\omega,k,E_p)\mathrm{Re}\wp_1+\frac{eFd}{2}\left(\Lambda\frac{\partial}{\partial\Lambda}g(\omega,k,E_p)\right)\mathrm{Im}\wp_1\right\}. \tag{93}$$

The integral of the left-hand side of Eq. (93) with respect to $\mathbf{p}$ is the Fourier transform of the perturbation of the concentration, Eqs. (70). It means that integrating Eq. (93) we obtain a relation between δn_k and Φ_k,

$$\delta n_k=\mathcal{G}_k(\omega)e\Phi_k\ , \tag{94}$$

where

$$\mathcal{G}_k(\omega)=\frac{m}{\pi\hbar^2\Lambda d}\int_0^\infty\frac{dE}{-i\hbar\omega+\Gamma(E)g(\omega,k,E)}\left\{-[\Gamma(E)-i\hbar\omega]g(\omega,k,E)\mathrm{Re}\wp_1+\frac{eFd}{2}\left(\Lambda\frac{\partial}{\partial\Lambda}g(\omega,k,E)\right)\mathrm{Im}\wp_1\right\}. \tag{95}$$

The comparison of this relation with Eq. (69) leads to the dispersion relation between ω and k,

$$e^2d\left(\frac{\pi d}{\bar\epsilon\sin^2(kd/2)}-\frac{1}{C_{\mathrm{eff}}}\right)\mathcal{G}_k(\omega)=1\ . \tag{96}$$

Under the condition of Eq. (71) and $\hbar\omega\ll\Gamma$ Eq. (96) is reduced to Eq. (86).

Equation (96) is very complicated and cannot be solved analytically even in the case of a constant Γ. It is possible to obtain an analytical solution for eFd much larger and much smaller than Γ. Both cases, however, do not have much practical interest and instead we show some results of the numerical solution.

In all calculations we considered an infinitely long superlattice. The main result is that as the field increases the instability comes about first for small values of k. The instability region of k extends with further increase of the field. Typical results are shown in Fig. 3. Positive values of $\mathrm{Im}\omega$ correspond to unstable fluctuations. We failed to find reasonable values of parameters for which fluctuations with finite value of k become unstable at a smaller field than fluctuations with infinitesimally small k.

The instability at small k seems quite unexpected compared to a high-field domain which results from the development of unstable fluctuations and has a characteristic length of one period. It is very interesting in this relation to follow the nonlinear stage of this development. Some simple study of the nonlinear problem is made in the next section.

VI. DEVELOPMENT OF INSTABILITY FOR $F > F_{\mathrm{th}}$

The development of the instability for $F > F_{\mathrm{th}}$ is determined by nonlinear effects. There are two main sources of nonlinear effects in superlattices. The first is the nonlinear susceptibility of each layer, in other words, the effect of upper levels. The second is the self-consistent redistribution of the electric field along the superlattice. Here we neglect the effect of the nonlinear susceptibility because of $eFd\ll\mathcal{E}_2-\mathcal{E}_1$.

In the analysis of growing fluctuations it is important to keep in mind that the superlattice has a finite length and values of k are quantized. Nonuniform fluctuations start to grow not at $F > F_{\mathrm{th}}$, when $\mathrm{Im}\omega(0) > 0$, but at a larger field, $F > F'_{\mathrm{th}}$ when $\mathrm{Im}\omega(k) > 0$, where $k \geq \pi/L$ and L is the length of the superlattice. For a long enough superlattice the minimal possible value of $k = \pi/L$ appears in the phenomenological region. This case is well studied (see Ref. 9 and the literature therein). It was shown that nonlinear effects limit the fluctuation growth and eventually lead to the formation of high-field domains drifting in the direction of the electron drift. We will consider here the opposite case, $\omega\tau_E\gg 1$.

The evolution of growing fluctuations is determined by nonlinear effects which couple harmonics with different values of k. The study of the evolution of many growing and interacting harmonics is a very difficult nonlinear problem. We will consider here a situation very close to the instability threshold when only the harmonic with the minimal possible value of k appears in the region $0 < k < k_m$ and is growing up. Such a limitation allows us to consider nonlinear effects in the lowest approximation. The main question which we are interested in is whether nonlinear effects intensify or hinder the growth of this harmonic.

For the solution of the nonlinear problem we use Eq. (83) in t representation, where current in the rhs is expressed in terms of $\delta\rho_{\pm1,k}$, Eq. (78). It is necessary now to find an expression for $\delta\rho_{\pm1,k}$ taking into account nonlinear effects. The only source of nonlinear effects is the commutator in the rhs of Eq. (41). The nonlinear part that we need is

$$-ie[\Phi,\delta\rho]_{\pm1,k}=\pm ed\sum_{k'}(k-k')\Phi_{k-k'}\delta\rho_{\pm1,k'}\ . \tag{97}$$

In Eq. (97) we keep only terms of the lowest order in k. To find a nonlinear expression for $\delta\rho_{\pm1,k}$ we have to add Eq. (97) in the rhs of Eq. (74). The iteration of the

resulting equation leads to the necessary expression for $\delta\rho_{\pm 1,k}$. Equation (75) shows that in the lowest order in k main terms are those containing Φ_k . The result is

$$\delta j_k = e\frac{\omega(k)}{k}\delta n_k - \frac{1}{2}\frac{\partial^2 j}{\partial F^2}\sum_{k'}(k-k')\Phi_{k-k'}\ k'\Phi_{k'} + \frac{i}{6}\frac{\partial^3 j}{\partial F^3}\sum_{k'k''}(k-k')\Phi_{k-k'}\ (k'-k'') \times \Phi_{k'-k''}\ k''\Phi_{k''}\ , \quad (98)$$

which essentially means the expansion of the current in terms of the electric-field perturbation. Here, as in Eq. (80), current derivatives are calculated for a constant $\wp_0$, i.e.,

$$\frac{\partial^n j(F)}{\partial F^n} = -\frac{2me^2 d\Lambda^2}{\pi\hbar^3}\int_0^\infty dE\ \frac{d\wp_0}{dE}\ \frac{\partial^n}{\partial F^n} \times \frac{\Gamma(E)F}{\Gamma^2(E)+(eFd)^2}\ . \quad (99)$$

$\omega(k)$ in Eq. (98) is the solution to Eq. (86).

Fluctuations of the electron concentration and electric potential satisfy some boundary conditions. For simplicity we assume that an external circuit maintains a constant potential drop across the superlattice. Then phases of δn_k can be chosen in such a way that $\delta n_k^* = \delta n_{-k}$.

Because only the Fourier harmonic δn_k with the minimum possible value of k is growing in the linear approximation, it is enough to consider its coupling only with the second harmonic δn_{2k} and neglect higher ones. Then we come up with the well-known equations,

$$\dot{n}_k = -i\omega_1 n_k - i\beta n_{2k} n_k^* - \alpha|n_k|^2 n_k, \quad (100)$$

$$\dot{n}_{2k} = -i\omega_2 n_{2k} + 4i\beta n_k^2\ , \quad (101)$$

where $\omega_1 = \omega(k)$, $\omega_2 = \omega(2k)$,

$$\beta = \left(\frac{4\pi}{\bar{\epsilon}}\right)^2 \frac{e}{4k}\ \frac{\partial^2 j}{\partial F^2}\ , \quad \alpha = \left(\frac{4\pi}{\bar{\epsilon}}\right)^3 \frac{e^2}{6k^2}\ \frac{\partial^3 j}{\partial F^3}\ . \quad (102)$$

Near the instability threshold $\omega_j = -i\gamma_j$, $\gamma_1 < 0$, and $\gamma_2 > 0$. In this case it easy to show that nonlinear effects can limit the growth of the fluctuations only if

$$-\frac{4\beta^2}{\gamma_2} + \alpha\ >\ 0. \quad (103)$$

We can estimate the lhs of Eq. (103) in the case of a constant Γ and very close to the instability threshold, when $k_m - k \ll k_m$. Then $\gamma_2 \approx 3\mathcal{D}k_m^2$, and Eq. (103) is satisfied for $eF'_{\rm th}d \geq 1.1\Gamma$. For so high field, however, $k_m^2 \sim q_s/d$, Eq. (71) is violated, and we cannot use the expansion in k. That is, in general, near the instability threshold nonlinear effects intensify the fluctuation growth.

In general, the situation can be different, i.e., nonlinear effects can limit the growth of fluctuations, in superlattices with a small doping, when $\wp_0(0) = \pi\hbar^2 n/mT \ll 1$. We do not elaborate this case because in an experiment there may be problems related to a nonuniform field distribution due to electron injection from highly doped contacts.

We want to note, however, that in such a case near the threshold nonlinear effects determine only the amplitude of the resulting steady state while the field distribution corresponds to the most unstable harmonic. As we have seen, in the most realistic situation nonlinear effect near the threshold accelerates the growth of fluctuation and further increases the nonlinearity. Such a development eventually leads to the growth of high harmonics which effectively reduces the characteristic length scale of the fluctuations. This mechanism can explain the contradiction between the long-wavelength instability and short high-field domains detected in experiments.

VII. DISCUSSION

The most intriguing result of our paper is that near the stability threshold only long-wavelength fluctuations appear to be unstable. This conclusion does not contradict the formation of high-field domains with the length of one period. Near the threshold nonlinear effects accelerate the growth of unstable fluctuations which eventually leads to the generation of high harmonics and the reduction of the characteristic length. We are not aware of any experiment where the spectrum of unstable fluctuations was measured. A possible check of our results could be made in microwave experiments just below the stability threshold. For large enough fluctuations the nonlinear amplification overcomes the linear attenuation. So a large enough microwave signal can excite linearly stable fluctuations. The frequency dependence of the amplitude necessary for the excitation can give some information about the stability in different regions of the spectrum.

The main limitation of our theory is related to the possibility of deriving an equation for the electron density matrix. As we already discussed in Sec. II an equation for the density matrix can be easily derived in cases of $\Gamma \ll eFd$ (in the Stark representation) and $eFd \ll \Gamma \ll 4\Lambda$ (in k representation). In the intermediate case, when $\Gamma \sim eFd$, an equation for the density matrix cannot be derived in a simple way in spite of k being a good quantum number in a periodic system. The problem is that in this case the collision operator cannot be written in the Born approximation and high-order corrections have to be taken into account. We got around this difficulty by considering a wide enough electron energy distribution where the Born approximation is justified.

In the calculation of the conductivity in Sec. IV we considered only one energy relaxation mechanism, acoustic-phonon scattering. The reason for this is that the typical width of the electron energy distribution of 2D electron gas is smaller than the optical-phonon energy $\hbar\omega_0$. In general, the energy relaxation by the optical-phonon emission is possible in a superlattice under electric field if after phonon emission an electron goes to another Stark level. If the miniband width is smaller than the optical-phonon energy then this mechanism is reduced near the instability threshold for the following reason. Usually $eFd \sim \Gamma \ll \hbar\omega_0$. This means that the emission of an optical phonon is possible only for a transition from a Stark

level with the quantum number ν_1 to a level with the quantum number ν_2 such that $\nu_1 - \nu_2 \equiv \nu \geq \hbar\omega_0/eFd$. The distance between the centers of these levels is νd. The transition is possible only in the case of an appreciable space overlap between the Stark states, i.e., when their size Λ/eF is larger than the distance between their centers. We get the system of conditions

$$\frac{\Lambda}{eF} > \nu d \geq \frac{\hbar\omega_0}{eF}, \tag{104}$$

which requires $\Lambda > \hbar\omega_0$. That is, the energy relaxation by the optical-phonon emission can be neglected in superlattices with minibands narrower than the optical-phonon energy.

Our result concerning the NDC threshold at $eFd \sim \Gamma$ seems to be different from that of Tsu and Döhler[23] (see also Ref. 29). In Ref. 23 the current falls off for eFd larger than some energy which is sometimes close to the width of the band, and there are oscillations of the I-V characteristic even below this point.

First, one has to notice that the oscillations are obtained only if the transitions between adjacent wells are taken into account. Tsu and Döhler showed that the inclusion of transitions to the next nearest wells eliminated the oscillations. Our expressions for the current, Eq. (20), and the first line of Eq. (39) are exact. In the case of hopping conductivity the last equation can be written in another form containing diagonal density-matrix elements and the transition probabilities between all of the wells.[26] That is, we cannot expect the oscillatory behavior in our results.

The comparison of the critical value of eFd is difficult. Tsu and Döhler[23] considered electron–acoustic-phonon scattering and implicitly assumed that the phonon energy transferred in one scattering event is larger than the energy uncertainty due to scattering. Indeed, they used a δ function including the phonon energy for the energy conservation. Our situation is opposite. We neglect the energy transfer in the calculation of Γ. Actually, for the electron (in-plane) wave vector 2×10^6 cm$^-1$ and the sound velocity 5×10^5 cm/s the characteristic phonon energy is less than 1 meV. A typical value of Γ is usually a few meV or larger. So we consider that our calculations are more close to typical experimental conditions.

Recently Sibille *et al.*[15,37,38] observed the NDC in GaAs/AlAs superlattices with very wide minibands. We make an estimate for one of their samples for which we found all necessary data. An estimate Γ is obtained with the help of Eq. (39). For a low electric field and Γ independent of the electron energy the integral in the rhs gives $\wp_0(0)/\Gamma$. $\wp_0(0)$ is determined by Eq. (36) where in low field $\exp(\zeta) = \exp(E_F/T) - 1$. As a result we obtain the following expression for the conductivity:

$$\sigma = \frac{2me^2\Lambda^2 d}{\pi\hbar^3\Gamma}\left(1 - e^{-E_F/T}\right). \tag{105}$$

For the nondegenerate electron gas Eq. (105) results in the mobility

$$\mu = \frac{2e\Lambda^2 d^2}{\hbar\Gamma T}. \tag{106}$$

For a superlattice with 13 monolayers of GaAs and seven monolayers of AlAs per period $d = 57$ Å, the calculated width of the first miniband is 12 meV,[39,40] and the mobility $\mu \sim 40$ cm^2/V s. The measurements were made at room temperature. This gives $\Lambda = 3$ meV Eq. (106) and $\Gamma \approx 8.6$ meV. The threshold field $F = 22$ kV/cm gives $eFd = 12.5$ meV. There are two possible reasons for the difference between Γ and eFd values. One is the difference between calculated and actual values of the band width by 20%. The other is the simplification which we made in the derivation of Eq. (105), actually Γ depends on E. This dependence is different for different scattering mechanisms and different doping (because of screening). It is possible that different instability thresholds in different samples[41] are explained by differences in the scattering mechanism.

In conclusion we would like to make a remark concerning short superlattices with a very low doping. In such superlattices the NDC can exist without any instability of nonuniform fluctuations. Really, $\sigma_{\rm dif}$ in Eq. (87) is proportional to the electron concentration while the diffusion coefficient does not depend on it. That is, for a low concentration and large minimum value of $k = \pi/L$ the diffusion term in the rhs of Eq. (87) is greater than the first term and the instability threshold is not reached. The suppression of the NDC instability by diffusion is well known in continuous medium.[9]

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APPENDIX A

A specific case of the addition theorem for the Bessel functions has the form [see, e.g., Ref. 42 Eq. (8.530.1)]

$$\sum_k J_k(z)J_{k+m}(z)e^{ik\phi} = J_m\left(2z\sin\frac{\phi}{2}\right)e^{im(\pi-\phi)/2}. \tag{A1}$$

From this equation it is easy to obtain two identities necessary for the calculation of the sums over Stark levels,

$$\sum_k J_k(z)J_{k+m}(z) = \delta_{m,0}, \tag{A2}$$

$$\sum_k kJ_k(z)J_{k+m}(z) = \frac{z}{2}(\delta_{m,1} + \delta_{m,-1}). \tag{A3}$$

Making use of Eqs. (17), (A2), and (A3) we get

$$\sum_{\mu\mu'} C_{0\nu\mu\mu'}\wp_{\mu-\mu'}(\mathbf{p}') = \delta_{\nu,0}\wp_0(\mathbf{p}'), \tag{A4}$$

$$\sum_{\mu\mu'} C_{0\mu\mu\mu'}\wp_{\nu-\mu'}(\mathbf{p}) = \sum_{\mu\mu'} C_{\nu\mu\mu\mu'}\wp_{\mu'}(\mathbf{p}) = \wp_\nu(\mathbf{p}) \ , \tag{A5}$$

$$\sum_{\mu\mu'} C_{\nu 0\mu\mu'}\wp_{\mu-\mu'}(\mathbf{p}')\left(\nu - \frac{\mu+\mu'}{2}\right) = \frac{z}{2}\{(\delta_{\nu,1}+\delta_{\nu,-1})\wp_0(\mathbf{p}') - \delta_{\nu,0}[\wp_1(\mathbf{p}')+\wp_{-1}(\mathbf{p}')]\} \ , \tag{A6}$$

$$\sum_{\mu\mu'} C_{\nu 0\mu\mu'}\wp_{\mu-\mu'}(\mathbf{p}')\frac{\mu+\mu'}{2} = -\frac{z}{2}\{(\delta_{\nu,1}+\delta_{\nu,-1})\wp_0(\mathbf{p}') - \delta_{\nu,0}[\wp_1(\mathbf{p}')+\wp_{-1}(\mathbf{p}')]\} \ , \tag{A7}$$

$$\sum_{\mu\mu'} C_{0\mu\mu\mu'}\wp_{\nu-\mu'}(\mathbf{p})\left(\mu - \frac{\nu+\mu'}{2}\right) = -\frac{z}{2}[\wp_{\nu-1}(\mathbf{p})+\wp_{\nu+1}(\mathbf{p})] - \frac{\nu}{2}\wp_\nu(\mathbf{p}) \ , \tag{A8}$$

$$\sum_{\mu\mu'} C_{\nu\mu\mu\mu'}\wp_{\mu'}(\mathbf{p})\left(\mu - \frac{\mu'}{2}\right) = -\frac{z}{2}[\wp_{\nu-1}(\mathbf{p})+\wp_{\nu+1}(\mathbf{p})] + \frac{\nu}{2}\wp_\nu(\mathbf{p}) \ . \tag{A9}$$

APPENDIX B

$\hat{S}$ is a matrix which diagonalizes the Hamiltonian $\mathcal{H}_1$, i.e.,

$$\sum_{\nu\nu'} S^{-1}_{\chi\nu}\left[(E_p - eFd\nu)\delta_{\nu\nu'} + e\Phi_{\nu\nu'}\right]S_{\nu'\chi'} = (E_p + E_\chi)\delta_{\chi\chi'} \ . \tag{B1}$$

In this new representation we can neglect the effect of scattering on the spectrum and use expressions for Green functions similar to Eqs. (21) and (22), e.g.,

$$G^{-+}_{\chi\chi'}(\omega,\mathbf{p}) = 2\pi i\rho_{\chi\chi'}(\mathbf{p},t)\delta\left(\hbar\omega - E_p - \frac{E_\chi + E_{\chi'}}{2}\right) \ . \tag{B2}$$

Then the inverse transformation to the Stark representation gives

$$G^{-+}_{\mu\mu'}(\omega,\mathbf{p}) = 2\pi i \sum_{\nu\nu'\chi\chi'} S_{\mu\chi}\rho_{\chi\chi'}(\mathbf{p},t)\delta\left(\hbar\omega - E_p - \frac{E_\chi + E_{\chi'}}{2}\right)S^{-1}_{\chi'\mu'} \ . \tag{B3}$$

If the energy corrections $E_\chi + E_{\chi'}$ in the δ function can be neglected compared to a characteristic electron energy then Eq. (B3) is reduced to Eq. (22).

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