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Quantum corrections to the Wigner crystal: A Hartree-Fock expansion

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The quantum corrections to the two-dimensional Wigner crystal, for filling $\nu \leq \frac{1}{3}$, are discussed by using a Hartree-Fock expansion based on wave functions which are (i) related to one another by magnetic translations, (ii) orthonormal, and (iii) strongly localized. Such wave functions are constructed in terms of Gaussians that are localized at the sites of a triangular (Wigner) lattice and have a small overlap c. The ground-state energy per particle is calculated by an expansion in $\sqrt{\nu}$ and in $\delta \equiv c^{1/4}$, which is rapidly convergent and stable under the thermodynamical limit. In particular, in this limit the cancellation of the infrared divergences occur order by order in the above expansion. The accurate control on the approximations allows a clear-cut comparison with the energy obtained by the Laughlin ansatz on the ground state and the numerical results confirm that the Wigner-crystal picture is energetically favored with respect to the Laughlin state for $\nu < \frac{1}{9}$.

I. INTRODUCTION

Classical arguments for a two-dimensional system of electrons and ions lead to a ground state in which the electrons are localized on the sites of a triangular lattice to minimize the Coulomb energy (Wigner crystal).[1] Physically this is a very intuitive and convincing picture which is actually extendable to higher dimensions.

In the presence of a magnetic field, a generalization of the Wigner crystal seems to be possible, and it has been discussed in the literature (we will return to this point below): in two dimensions, essentially one uses a Hartree-Fock ground state with Gaussian-like wave functions centered on the sites of a triangular lattice.[2] This picture seems widely used, at least for low electron filling and for low hole filling.[2,3]

On the other hand, the present wisdom on the quantum Hall effect seems to favor Laughlin's ansatz on the ground-state wave function, which appears significantly different from that of a Wigner crystal. However, recent experimental data seem to indicate that for low filling ν ($\nu < \frac{1}{5}$) and for low hole filling $\nu_h \equiv 1-\nu$ ($\nu_h < \frac{1}{5}$), two-dimensional electrons in a magnetic field do behave like a Wigner crystal.[4]

The interesting question is for which "critical" value of the filling Laughlin's picture is favored with respect to Wigner's[5] and possibly which is the mechanism which underlies the transition from one regime to the other. To approach the above questions one needs a refined treatment of the Wigner crystal, since the energy difference between the Laughlin and Wigner ground states are of the order of a few percent.

It is clear that (1) first, the classical Wigner argument is not enough, and it is crucial to consider the quantum corrections to it; and (2) second, since the problem is not exactly solvable, one needs full control of the approximations and accurate estimates of the rest (at least up to 1%).

To our knowledge, the existing literature has dealt with the quantum corrections to the Wigner crystal, but without systematic control of the approximation adopted. For example, the analysis of Ref. 2 is based on a Mayer-like expansion, i.e., on an expansion of the interaction with respect to the n-particle contributions; unfortunately, higher-order terms seem to be of the same order as the first contribution.

Similarly, the approach based on the so-called "charge-density wave" (CDW) (Refs. 3 and 5) involves assumptions and approximations which are not completely under control and which may affect the numerical results with corrections of order 10^{-3} or more.

A second type of problem arises because one would like to adopt an approach, i.e., an expansion, which allows a "smooth" thermodynamical limit. The need for considering a large number of particles N (and therefore a large volume V) is typical of many-body theory, but it is very delicate when there are long-range interactions such as the Coulomb one. Perturbative expansions or approximations which do not cope with this problem easily lead to infrared (or volume) divergences in the thermodynamical limit (see the discussion below); the point is that the subtraction of the background contribution (which diverges in the thermodynamical limit) and the long-distance cancellations of the electrostatic energies of the electrons due to the charge neutrality of the system are indeed very delicate; for example, a careful bookkeeping

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of the approximations at the level of the wave-function orthonormalization and of the Coulomb interaction is crucial to get a finite result in the thermodynamical limit. To our knowledge this aspect has not been discussed in the literature (see, e.g., Ref. 2).

Finally, as emphasized for instance in Ref. 6, it is convenient to get a description of the N-electron ground state for finite volume V, which is invariant under a subgroup of the space translations related to the Wigner crystal, in order to naturally obtain the corresponding translation invariance of the ground state in the thermodynamical limit. To this purpose the phases of the wave functions localized at the sites of the Wigner lattice have to be carefully chosen by exploiting, e.g., the magnetic translations (this point has not been exploited in, e.g., Ref. 2). We present a possible approach to the above problems whose main features are the following.

(a) Use of a Hartree-Fock-like approach to the ground state, based on electron wave functions localized (as much as possible, actually Gaussian) at the sites of the Wigner lattice. This seems physically very plausible in order to minimize the Coulomb interaction energy among the electrons, and it is actually strongly suggested by recent experimental data for $0<\nu<\frac{1}{5}$, $\frac{4}{5}<\nu<1$ (as mentioned above).

The translation invariance is assured by choosing as a wave function at the site (m,n) the (m,n)-magnetic translated of the wave function at the origin (0,0).

(b) To cope with the problems connected with the thermodynamical limit, and the cancellation of the infrared divergences associated with the Coulomb interaction energy, we use an orthonormal set of wave functions. Otherwise, the use of nonorthogonal single-electron wave functions ϕ_i, satisfying $\langle \phi_i,\phi_j\rangle=\delta_{ij}+c_{ij}$, leads to a ground-state wave function Ψ (given by the usual Slater determinant) with $\langle\Psi,\Psi\rangle=1+O(N)$ with a diverging behavior in the thermodynamical limit which, among other things, would complicate the problem of the infrared cancellations.

(c) To compute the energy of the ground state with the choices (a) and (b), some sort of expansion (and/or approximation) is needed, and we choose an expansion which has the classical Wigner crystal as the zero order. More specifically, we consider the parameter c defined as the (modulus of the) overlap between the Gaussian centered at the origin and the Gaussian centered in one of the neighboring points: $c\approx\exp[-a_o^2/(4\nu l^2)]$, where ν is the filling, a_o is the lattice spacing for $\nu=1$, and l is the magnetic length $[l=(\hbar c/eH)^{1/2}]$. Then we set up an expansion in $\delta\equiv c^{1/4}$ (see below). Clearly such an expansion parameter goes to zero when $4\nu l^2/a_o^2$ goes to zero and therefore such an expansion appears justified for small ν (alternatively, if the hole picture is adopted, for $\nu_h\rightarrow 1$) or as an expansion for high magnetic field.

Incidentally, since $l^2=\hbar c/eH$, the expansion can also be interpreted as an expansion in $\hbar$ with the zero order given by the classical Wigner crystal; in this respect, it qualifies itself as the natural expansion for evaluating the quantum corrections to the Wigner crystal. It should be remarked, however, that the matrix elements of the Coulomb interaction are, for small ν, entire functions of the square root of the filling, $\sqrt{\nu}$, and of the "overlap parameter" $\delta\equiv c^{1/4}\approx\exp(-A/\nu)$ [see Sec. III Eqs. (3.4) and (3.8)] and therefore they are not analytic functions of $\sqrt{\nu}$. Hence, an expansion in $\sqrt{\nu}$ is not allowed. In our approach the classical Wigner crystal is obtained by taking the zeroth order in δ and then keeping only the leading order in $\sqrt{\nu}$. The quantum corrections to the Wigner crystal are therefore given by the further terms in the expansion in $\sqrt{\nu}$ and in δ. As we will see, in fact, the δ expansion starts with $c^2=\delta^8$ and the numerical calculations show that such expansion is extremely good, namely, that the corrections to the Coulomb interaction energy beyond the first order are extremely small (typically of the order of 10^{-4} or smaller, in our units).

Our numerical results do not differ significantly from the results of Refs. 2, 3, and 7; however, with our approach we are able to estimate the correction to their calculations and approximations (which essentially neglect the overlap) and we find precise estimates of the smallness of the exchange interaction term and of the overlap contribution. Such accurate estimates allow a clear-cut comparison with the energy obtained by Laughlin ansatz (as remarked above, the differences with respect to the Wigner crystal are of a few percent).

Our calculations confirm that the Laughlin state is favored with respect to that of the Wigner crystal for fillings $\frac{1}{9}<\nu<\frac{8}{9}$. Our approximation is still not able to favor the Wigner crystal in the regions $\frac{1}{9}<\nu<\frac{1}{5}$ and $\frac{4}{5}<\nu<\frac{8}{9}$, as recent experiments indicate. We hope that possible refinements of the quantum corrections to the Wigner crystal (e.g., by exploiting charge screening effects) may possible lead to the identification of $\nu=\frac{1}{5}$ as the transition point between the Wigner and the Laughlin regimes.

In conclusion, the aim of the present paper is to provide an accurate discussion of the quantum corrections to the Wigner crystal by using a Hartree-Fock expansion with a full control on the approximations [see Eq. (3.9) below].

II. THE PICTURE OF THE WIGNER CRYSTAL

Semiclassical considerations indicate that (in two dimensions) a system of electrons in a positive uniform background should "crystallize" in a triangular lattice *(Wigner crystal)* to minimize the electron Coulomb interaction.[1,8] In the extreme classical limit, therefore, the squared modulus of electron wave functions should be δ functions centered at the lattice sites. Clearly quantum corrections lead to a spread of the wave functions, and one is led to consider Gaussians centered at the lattice points. One then expects quantum corrections arising both from the finite width of the electron wave functions as well as from the nonvanishing overlap between wave functions centered around different points. Our aim is to treat systematically such quantum corrections to the Wigner crystal in the case of electrons in the presence of a (large) magnetic field.

Our strategy will be to consider the electrons in a finite lattice V (with periodic boundary conditions) and to consider the thermodynamical limit at the end. The first problem is that we would like to have a ground-state invariant under a discrete subgroup of translations (corresponding to the lattice structure). This is one of the points where the magnetic field enters in a significant way. In fact, the above translation invariance requires that the electron wave function at the lattice site (i,j) is obtained as the (i,j) translation of the wave function localized around the origin. This requirement, however, meets with the problem that the magnetic translations along different axes commute with the Hamiltonian but not among themselves.

In fact, the single-electron Hamiltonian has the form

$$H_0=\frac{\Pi^2}{2M},\quad \mathbf{\Pi}=\mathbf{p}-\frac{e}{c}\mathbf{A}\ , \tag{2.1}$$

where, in the symmetric gauge,

$$\mathbf{A}=-\tfrac{1}{2}\mathbf{H}\times\mathbf{r}\ . \tag{2.2}$$

We use units such that $\hbar=M=e|\mathbf{H}|/c=1$, which also implies that $\omega_c=e|\mathbf{H}|/Mc$ and the magnetic length $l=(\hbar c/e|\mathbf{H}|)^{1/2}$ are both equal to 1, so that the Hamiltonian takes the form

$$H_o=\tfrac{1}{2}(p_x-y/2)^2+\tfrac{1}{2}(p_y+x/2)^2\ .$$

It is convenient to introduce the following pairs of canonical variables,[9] which, in our gauge, have the form

$$\Pi_x=p_x-y/2,\quad \Pi_y=p_y+x/2\ , \tag{2.3}$$

$$\Pi_{cx}=p_x+y/2,\quad \Pi_{cy}=p_y-x/2\ . \tag{2.4}$$

The nonvanishing commutation relations are

$$[\Pi_y,\Pi_x]=[\Pi_{cx},\Pi_{cy}]=i\ . \tag{2.5}$$

The magnetic translations are

$$T(\mathbf{a}_j)=\exp\{i\mathbf{\Pi}_c\cdot\mathbf{a}_j\}\ , \tag{2.6}$$

where the vectors $\mathbf{a}_j$, $j=1,2$ are the lattice basis. These operators commute with H_o and satisfy the following commutation relation:

$$T(\mathbf{a}_1)T(\mathbf{a}_2)=T(\mathbf{a}_2)T(\mathbf{a}_1)\exp[i(a_{1x}a_{2y}-a_{1y}a_{2x})]\ ,$$

so that the noncommutativity disappears if the lattice cell is chosen to satisfy the "rationality condition"

$$(a_{1x}a_{2y}-a_{1y}a_{2x})=2\pi M,\quad M\in\mathbb{Z}\ . \tag{2.7}$$

For a triangular lattice $\mathbf{a}_1=a(1,0)$, $\mathbf{a}_2=a(\frac{1}{2},\sqrt{3}/2)$, with a the lattice spacing, so that for $M=1$ (the minimal triangular lattice), Eq. (2.7) gives $a^2=(4\pi/\sqrt{3})l^2\equiv a_o^2$. Since, in general, the filling ν is equal to a_o^2/a^2, Eq. (2.7) is fulfilled for the $\nu=1/M$, M integer.

Under condition (2.7), we can easily implement the requirement of translation invariance, which completely fixes the electron wave function at different lattice sites, provided we specify the wave function at the origin $\Psi_{o,o}(x,y)$:

$$\Psi_{n,m}(x,y)\equiv T(\mathbf{a}_1)^nT(\mathbf{a}_2)^m\Psi_{o,o}(\mathbf{r})\ ,$$

and, therefore,

$$\Psi_{n,m}(x,y)=(-1)^{nm/\nu}e^{i(Y_{n,m}x-X_{n,m}y)/2}\Psi_{o,o}(\mathbf{r}-\mathbf{R}_{n,m})\ , \tag{2.8}$$

where the integers m and n label the lattice points, and

$$\mathbf{R}_{n,m}=(X_{n,m},Y_{n,m})=-a(n+m/2,\sqrt{3}/2m)\ . \tag{2.9}$$

The wave function $\Psi_{o,o}(\mathbf{r})$ is chosen as an eigenfunction of H_o (belonging to the lowest Landau level), with the maximum localization allowed (actually an exponential one) to minimize the Coulomb interaction energy and to be as close as possible to the classical Wigner crystal:

$$\Psi_{o,o}(x,y)=(2\pi)^{-1/2}\exp[-(x^2+y^2)/4]\ . \tag{2.10}$$

As discussed in Appendix A, condition (2.7) with $M=1$ guarantees that the set of states $\{\Psi_{n,m}\}$ form a complete set for the first Landau level. Actually the set $\{\Psi_{n,m}\}$ is unitarily equivalent to the set $\{\Psi_{jk}^{\mathrm{VN}}(z)\Psi_0(\Pi_x)\}$ where $\Psi_0(\Pi_x)=\exp(-\Pi_x^2/2)/\pi^{1/4}$ is the ground state of the Hamiltonian (2.1) and $\Psi_{jk}^{\mathrm{VN}}(z)$ are the coherent states associated to the points of the Von Neumann lattice in the phase space defined by the canonical variables (Π_{cx},Π_{cy}) (see Ref. 10). It is worthwhile to mention that the rationality condition (2.7) for the configuration space cell, which enters in the definition of the $\Psi_{n,m}$'s, is equivalent to the Von Neumann condition on the cell in the phase space which characterizes the completeness of the states $\Psi_{jk}^{\mathrm{VN}}(z)$.

More generally, in the following we will consider the case $M>1$, in which the electron wave functions are centered at the sites of a triangular lattice with lattice spacing a larger than the one of the minimal lattice: $a^2/a_o^2=\nu^{-1}=M$. This will be relevant for discussing the cases of fillings $\nu<1$. In this case, the set $\{\Psi_{n,m}\}$ is no longer complete for the whole first Landau level, but it is still enough for expanding wave functions "localized" at the sizes of the triangular lattice [see Eq. (2.15) and Appendix B]. Equation (2.8) defines our Hartree-Fock states for any filling $\nu=1/m$, m integer. For these values of ν, the Hartree-Fock ground state is strictly invariant under lattice space translations (no phase arises).

Clearly, the same formulation applies if the Hartree-Fock ground state is described in terms of hole (rather than electron) wave functions.

The electron wave functions in (2.8) are, apart the phase factor $(-1)^{nm/\nu}$, the same as those used in Ref. 2. The presence of such a factor is crucial for implementing the translation invariance we have stressed above; in fact, we have

$$(\Psi_{k,l},\Psi_{n,m})=(\Psi_{o,o},\Psi_{n-k,m-l})\equiv d_{n-k,m-l}\ , \tag{2.11}$$

whereas this property is not shared by the functions used in Ref. 2.

In the following we will often use a multi-index notation $n=(n_1,n_2)$ to label the lattice points, so that, for example, the electron wave function localized at (n_1,n_2) will be simply denoted by Ψ_n. A plain use of the wave functions Ψ_n leads to difficulties in the thermodynamical limit. This is due to the lack of orthogonality

$$(\Psi_n,\Psi_k)=\delta_{n-k,0}+c_{n-k}\ , \tag{2.12a}$$

$$c_o=0\ , \tag{2.12b}$$

(nonvanishing overlap) so that if Λ_o^N denotes the ground state for the noninteracting N electrons, Λ_o^N is given by the Slater determinant

$$\Lambda_o^N(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$$

$$=\frac{1}{\sqrt{N!}}\begin{vmatrix}\Psi_1(1) & \Psi_2(1) & \cdots & \Psi_N(1)\\ \Psi_1(2) & \Psi_2(2) & \cdots & \Psi_N(2)\\ \cdots & \cdots & \cdots & \cdots\\ \Psi_1(N) & \Psi_2(N) & \cdots & \Psi_N(N)\end{vmatrix} \tag{2.13}$$

and

$$(\Lambda_o^N,\Lambda_o^N)=1+O(N)\ . \tag{2.14}$$

The norm of Λ_o^N is divergent in the thermodynamical limit ($V\rightarrow\infty, N\rightarrow\infty$). This makes things difficult when approximations are made in computing the Coulomb energy, since the cancellation of infrared divergences becomes less obvious (see below).

To overcome these problems, starting from Ψ_n we construct an orthonormal set of wave functions Ω_n, still keeping the translational covariance

$$\Omega_n(\mathbf{r})=\sum_i f_i\Psi_{n+i}(\mathbf{r})\ . \tag{2.15}$$

The existence of such an expansion, which manifestly preserves the translational covariance, is not obvious if the $\{\Psi_n\}$ merely form a complete set, since the dependence of n on the right-hand side of Eq. (2.15) is very special. The existence of expansion (2.15), i.e., the proof of the convergence of the right-hand side of Eq. (2.15) for fillings $\nu<1$, is given in Appendix B. The case $\nu=1$ is much more delicate and, since it falls outside the regime we are interested in, it will not be discussed here. For our present purposes the case $\nu\rightarrow 1$ can be obtained from the case $\nu_h\rightarrow 0$ by exploiting the electron-hole symmetry.

The coefficients f_i are chosen to yield orthogonal wave functions

$$(\Omega_n,\Omega_m)=\delta_{n,m}\ , \tag{2.16}$$

and one gets the following expansion:

$$\Omega_i=N_i\left\{\Psi_i-\tfrac{1}{2}\sum_n c_n\Psi_{i+n}+\tfrac{3}{8}\sum_n\sum_s c_{n-s}c_s\Psi_{i+n}+\cdots\right\}, \tag{2.17}$$

where N_i is a normalization constant, $N_i=1+O(c^3)$, and

$$\begin{aligned}c_i&=(\Psi_o,\Psi_i)=(-1)^{i_1i_2/\nu}e^{-\mathbf{R}_i^2/4},\quad i\neq 0\ ,\\ c_o&=0\ .\end{aligned} \tag{2.18}$$

As discussed in Sec. I, our approach is to compute the ground-state energy for a two-dimensional electron gas, in the presence of a magnetic field, by using as the ground-state wave function the Hartree-Fock wave function

$$\Lambda_o^N(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$$

$$=\frac{1}{\sqrt{N!}}\begin{vmatrix}\Omega_1(1) & \Omega_2(1) & \cdots & \Omega_N(1)\\ \Omega_1(2) & \Omega_2(2) & \cdots & \Omega_N(2)\\ \cdots & \cdots & \cdots & \cdots\\ \Omega_1(N) & \Omega_2(N) & \cdots & \Omega_N(N)\end{vmatrix}\ , \tag{2.19}$$

which clearly satisfies

$$\langle\Lambda_o^N,\Lambda_o^N\rangle=1,\quad\forall N\ .$$

The advantage of such a Hartree-Fock approach is that of being as close as possible to the Wigner crystal, and to allow a systematic control of the quantum corrections arising from an expansion in $\nu^{1/2}$ and in the electron overlap parameter δ. In our opinion, this may be considered as an improvement with respect to other Hartree-Fock approaches to the Wigner crystal discussed in the literature; for example, with respect to Ref. 11, we have better localization and electron wave functions symmetric under rotations of $\pi/3$.

III. ENERGY MATRIX ELEMENTS IN THE THERMODYNAMICAL LIMIT

In this section, we use the ground-state wave function (2.19) to compute the matrix elements of the total Hamiltonian

$$H^{(N)}=\sum_i^N H_o^{(i)}+H_c^{(N)}+H_B^{(N)}\ ,$$

where $H_o^{(i)}$ is the single-electron Hamiltonian defined by (2.1), $H_c^{(N)}$ is the Coulomb Hamiltonian

$$H_c^{(N)}=\frac{1}{2}\sum_{1\neq j}^N\frac{1}{|\mathbf{r}_i-\mathbf{r}_j|}\ ,$$

and $H_B^{(N)}$ is the interaction with the background of uniform density $\rho_B=-Ne$:

$$\begin{aligned}H_B^{(N)}&=e\sum_i^N\rho_B\int_V\frac{1}{|\mathbf{r}_i-\mathbf{y}|}d^2y+\rho_B^2\int_V\frac{1}{|\mathbf{x}-\mathbf{y}|}d^2x\,d^2y\\ &=\frac{e}{2}\sum_i^N\rho_B\int_V\frac{1}{|\mathbf{r}_i-\mathbf{y}|}d^2y\end{aligned}$$

(the same also holds in the case in which one considers a periodic background).

One can prove that in the thermodynamical limit the difference

$$\frac{1}{N}(\langle H_B^{(N)}\rangle_{\Lambda_o}-H_B^{(N)\mathrm{cl}})$$

$$\equiv\frac{1}{N}\left[\langle H_B^{(N)}\rangle_{\Lambda_o}-\frac{e}{2}\sum_i^N\rho_B\int_V\frac{1}{|\mathbf{R}_i-\mathbf{y}|}d^2y\right]$$

tends to zero (see Appendix C). Hence, since Λ_o^N is an eigenfunction of $\sum_i^N H_o^{(i)}$, we are left with the problem of computing the matrix elements of H_c.

As already mentioned, one has to carefully control the cancellation of infrared (or infinite volume) divergences arising both by letting $N \to \infty$ in the sum over i's and by letting $V \to \infty$ in the Coulomb integrals. To this purpose, we first note that one is actually interested in the energy per particle, i.e.,

$$\lim_{N\to\infty} \frac{\langle H^{(N)}\rangle_{\Lambda_o}}{N} = E_{\text{kin}} + E_o \ .$$

Second, it is convenient to split the energy matrix elements as

$$\begin{aligned} E_o &= \lim_{N\to\infty} \left\{ \frac{E(N)+E_B(N)}{N} \right\} \\ &= \lim_{N\to\infty} \left\{ \frac{E(N)-E_W(N)}{N} + \frac{E_W(N)+E_B(N)}{N} \right\} \\ &\equiv \delta E + E_W \ , \end{aligned} \tag{3.1}$$

where

$$E(N) \equiv \langle \Lambda_o^N, H_c^{(N)} \Lambda_o^N \rangle, \quad E_B(N) \equiv \langle \Lambda_o^N, H_B^{(N)} \Lambda_o^N \rangle \ ,$$

and $E_W(N)$ is the Coulomb energy of N (classical) electrons sitting on a triangular lattice with no background. Thus in the limit $N \to \infty$, $[E_W(N)+E_B(N)]/N$ becomes the well-known energy per particle of the classical Wigner crystal:

$$E_W = \lim_{N\to\infty} \left\{ \frac{E_W(N)+E_B(N)}{N} \right\} = -0.7821\sqrt{\nu} \ . \tag{3.2}$$

We are left with the first term in Eq. (3.1), where the infinite volume divergences due to the Coulomb tail are subtracted by $E_W(N)$. It is important to devise approximations or expansions for which such cancellation occurs order by order. This is one of the main motivations for the choice of orthogonal electron wave functions, since then the expansion in $\nu^{1/2}$ and δ preserve the neutrality of matter order by order. This in turn guarantees the cancellation of the Coulomb divergences (see below). We briefly sketch our calculation. Denoting by E_d (E_{ex}) the direct (exchange) Coulomb integral

$$\begin{aligned} \delta E &= \lim_{N\to\infty} \left\{ \frac{E(N)-E_W(N)}{N} \right\} \\ &= \lim_{N\to\infty} \frac{1}{N} \left\{ \frac{1}{2} \sum_{i\neq j}^{N} [E_d(i,j) - E_{\text{ex}}(i,j)] - E_W(N) \right\} \\ &= \lim_{N\to\infty} \left\{ \frac{1}{2} \sum_{i\neq 0}^{N} [E_d(i,0) - E_{\text{ex}}(i,0)] - \frac{E_W(N)}{N} \right\} , \end{aligned} \tag{3.3}$$

where

$$E_d(i,0) = \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{|\Omega_i(\mathbf{r}_1)|^2 |\Omega_0(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

and

$$E_{\text{ex}}(i,0) = \int d\mathbf{r}_1 d\mathbf{r}_2 \frac{\Omega_i^*(\mathbf{r}_1)\Omega_0^*(\mathbf{r}_2)\Omega_0(\mathbf{r}_1)\Omega_i(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \ .$$

In computing the above integrals, we will expand the wave function Ω_i in terms of the Ψ_i [Eq. (2.17)].

Now, by Fourier transforming the Coulomb potential and by performing the Fourier transform of the Gaussians, we have

$$\begin{aligned} V_{ijkl} &\equiv \int \frac{d\mathbf{r}_1 d\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \Psi_i^*(\mathbf{r}_1)\Psi_j(\mathbf{r}_1)\Psi_k^*(\mathbf{r}_2)\Psi_l(\mathbf{r}_2) \\ &= (-1)^{(i_1 i_2 + j_1 j_2 + k_1 k_2 + l_1 l_2)/\nu} \left[\frac{1}{2\pi} \right]^2 e^{-(A_1+A_2)/4} \frac{2}{\pi} \Phi_1 \\ &\quad \times \int \frac{d\mathbf{k}}{|\mathbf{k}|} e^{-i\mathbf{k}\cdot(\mathbf{q}_1-\mathbf{q}_2)} \int d\mathbf{P}_1 e^{i\sqrt{2}[(1/2)\boldsymbol{\Theta}_1 - \mathbf{k}]\cdot\mathbf{P}_1} e^{-\mathbf{P}_1^2} \int d\mathbf{P}_2 e^{i\sqrt{2}[(1/2)\boldsymbol{\Theta}_2 + \mathbf{k}]\cdot\mathbf{P}_2} e^{-\mathbf{P}_2^2} \\ &= (-1)^{(i_1 i_2 + j_1 j_2 + k_1 k_2 + l_1 l_2)/\nu} \frac{\Phi_1}{2\pi} e^{-(A_1+A_2)/2} \int \frac{d\mathbf{k}}{|\mathbf{k}|} e^{\mathbf{k}\cdot\mathbf{v}_c} e^{-\mathbf{k}^2} \ , \end{aligned}$$

where

$$A_1 = \tfrac{1}{2}(\mathbf{R}_i - \mathbf{R}_j)^2; \quad A_2 = \tfrac{1}{2}(\mathbf{R}_1 - \mathbf{R}_k)^2 \ ;$$

$$\mathbf{q}_1 = \tfrac{1}{2}(\mathbf{R}_i + \mathbf{R}_j); \quad \mathbf{q}_2 = \tfrac{1}{2}(\mathbf{R}_k + \mathbf{R}_1) \ ;$$

$$\boldsymbol{\Theta}_1 = (Y_j - Y_i, X_i - X_j); \quad \boldsymbol{\Theta}_2 = (Y_l - Y_k, X_k - X_l) \ ;$$

$$\Phi_1 = \exp\left\{ \frac{i}{2}(\mathbf{q}_1\cdot\boldsymbol{\Theta}_1 + \mathbf{q}_2\cdot\boldsymbol{\Theta}_2) \right\} \ ;$$

$$\mathbf{v}_c = \tfrac{1}{2}(\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2) - i(\mathbf{q}_1 - \mathbf{q}_2) \ .$$

By using polar coordinates, the last integral reduces to the integral

$$\int_0^\infty d|k| e^{-|k|^2} I_o(-|k| v_c) = \left[\frac{\pi}{4} \right]^{1/2} \exp\left[\frac{v_c^2}{8} \right] I_o \left[\frac{v_c^2}{8} \right] , \tag{3.4}$$

where I_o is the modified Bessel function.[12]

The indices i, j, k, and l, over which we have to sum, are hidden in $\mathbf{v}_c$, A_1, A_2, and Φ_1, and one cannot perform the sum explicitly. Thus some sort of expansion and/or approximation is needed. From Eq. (3.4) and the asymptotic expansion of $I_o(z)$, we know that, for small $\nu \approx 1/z$, the matrix elements V_{ijkl} are analytic functions of $\nu^{1/2}$ and of the overlap parameter $\delta \equiv c^{1/4} \approx \exp(-A/\nu)$ and therefore it is justified to perform our expansion of the ground-state energy in the parameters $\nu^{1/2}$ and δ.

First we remark that, by computations similar to those leading to (3.4), one gets the estimate

$$|V_{ijkl}| \le k e^{-(1/4)(A_1+A_2)} , \tag{3.5}$$

k being a constant, so that for fixed pair of indices [which for simplicity we take at the origin, see Eq. (3.3)] the sum over the other pair satisfies an *a priori* estimate by which it is justified to consider only terms up to order c^2 in the expansion of the ground-state energy. In particular, one can take $N_i = 1$ in Eq. (2.17) and keep only terms up to order c^2 in the expansion of Ω_i in terms of the Ψ_i's. Thus we have

$$\begin{aligned} E_d(i,0) = V_{iioo} - \sum_n \left\{ c_n - \tfrac{3}{4} \sum_l c_{n-l} c_l \right\} \mathrm{Re}[V_{iion} + V_{ii+noo}] + \tfrac{1}{4} \sum_{ln} c_n c_1 \mathrm{Re}[V_{iiln} + V_{i+li+noo}] \\ + \tfrac{1}{2} \sum_{ln} c_n c_1 \mathrm{Re}[V_{ii+lon} + V_{ii+lno}] \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} E_s(i,0) = V_{iooi} - \sum_n \left\{ c_n - \tfrac{3}{4} \sum_l c_{n-l} c_l \right\} \mathrm{Re}[V_{ioni} + V_{i+nooi}] + \tfrac{1}{4} \sum_{ln} c_n c_l \mathrm{Re}[V_{inli} + V_{i+nooi+l}] \\ + \tfrac{1}{2} \sum_{ln} c_n c_l \mathrm{Re}[V_{inoi+l} + V_{ioni+l}] \, . \end{aligned} \tag{3.7}$$

Since c_n is of order higher than c^2, if the pair of indices $n = (n_1, n_2)$ satisfies $\mathbf{R}_n^2 > a^2$ we may consider only c_n with $\mathbf{R}_n^2 = a^2$ in the above sums.

Furthermore, in order to perform the sum over i, it is convenient to expand V_{ijkl} in δ and in $\nu^{1/2}$. Actually we may take the exact expression of V_{ijkl} when the matrix elements involve interactions between nearest neighbors or next to nearest, and it is justified to expand in δ and in $\nu^{1/2}$ otherwise. To this purpose, we note that by using the asymptotic expansions of the modified Bessel function I_o,[12]

$$I_o(z) \approx \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 + \frac{1}{8z} \right\} \quad \mathrm{Re}[z] \to \infty \tag{3.8a}$$

and

$$I_o(z) \approx \frac{e^{-z}}{\sqrt{-2\pi z}} \left\{ 1 - \frac{1}{8z} \right\} \quad \mathrm{Re}[z] \to -\infty \, , \tag{3.8b}$$

we get

$$\begin{aligned} V_{ijkl} \approx (-1)^{(i_1 i_2 + j_1 j_2 + k_1 k_2 + l_1 l_2)/\nu} \\ \times \Phi_1 e^{-(1/4)[2(A_1+A_2) - \mathbf{v}_c^2]} \frac{1}{\sqrt{\mathbf{v}_c^2}} \left\{ 1 + \frac{1}{\mathbf{v}_c^2} \right\} \\ \mathrm{Re}[v_c^2] \to \infty \end{aligned}$$

and

$$\begin{aligned} V_{ijkl} \approx (-1)^{(i_1 i_2 + j_1 j_2 + k_1 k_2 + l_1 l_2)/\nu} \\ \times \Phi_1 e^{-(1/2)(A_1+A_2)} \frac{1}{\sqrt{-\mathbf{v}_c^2}} \left\{ 1 - \frac{1}{\mathbf{v}_c^2} \right\} \\ \mathrm{Re}[v_c^2] \to -\infty \, . \end{aligned}$$

Note that estimate (3.5) guarantees that in the first case $2(A_1 + A_2)$ stays larger than $\mathrm{Re}[\mathbf{v}_c^2]$ when this becomes large.

We can now easily obtain an expansion of δE in $\delta = c^{1/4}$ and $\nu^{1/2}$.

$$\begin{aligned} \delta E = \sum_{i=(i_1,i_2)\neq 0} \Bigg\{ \frac{1}{2} \frac{\nu^{3/2}}{|\underline{\mathbf{R}}_i|^3} + \tfrac{3}{4} \sum_n c_n^2 \left[\frac{\nu^{1/2}}{|\underline{\mathbf{R}}_i|} + \frac{\nu^{3/2}}{|\underline{\mathbf{R}}_i|^3} \right] + \tfrac{1}{4} \sum_n^{(i\neq n)} c_n^2 \left[\frac{\nu^{1/2}}{|\underline{\mathbf{R}}_i - \underline{\mathbf{R}}_n|} + \frac{\nu^{3/2}}{|\underline{\mathbf{R}}_i - \underline{\mathbf{R}}_n|^3} \right] \\ - \sum_n^{(i\neq n)} c_n^2 \left[\frac{\nu^{1/2}}{|\underline{\mathbf{R}}_{in,c}|} + \frac{\nu^{3/2}}{|\underline{\mathbf{R}}_{in,c}|^3} \right] - \tfrac{1}{2} e^{-\underline{\mathbf{R}}_i^2/4} \left[\frac{\nu^{1/2}}{|\underline{\mathbf{R}}_i|} + \frac{\nu^{3/2}}{|\underline{\mathbf{R}}_i|^3} \right] - \tfrac{1}{4} c_i^2 \left[\frac{\nu^{1/2}}{|\underline{\mathbf{R}}_i|} + \frac{\nu^{3/2}}{|\underline{\mathbf{R}}_i|^3} \right] \Bigg\} , \end{aligned} \tag{3.9}$$

where

$$\underline{\mathbf{R}}_i^2 = a_o^2 (i_1^2 + i_2^2 + i_1 i_2) \equiv \underline{X}_i^2 + \underline{Y}_i^2$$

and

$$|\underline{\mathbf{R}}_{in,c}|^2 = [(\underline{\mathbf{R}}_i^2 - \underline{\mathbf{R}}_i \cdot \underline{\mathbf{R}}_n)^2 + (\underline{X}_i \underline{Y}_n - \underline{X}_n \underline{Y}_i)^2] \, .$$

It is worthwhile to mention that the infrared cancellations indeed have taken place order by order in our expansion; this is due both to the subtraction of $E_W(N)/N$ and to our careful bookkeeping of the order of the various terms in our expansion, e.g., cancellation occurs between

$$\tfrac{3}{4}\sum_n c_n^2 \frac{\nu^{1/2}}{|\underline{\mathbf{R}}_i|} + \tfrac{1}{4}\sum_n^{(i\neq n)} c_n^2 \frac{\nu^{1/2}}{|\underline{\mathbf{R}}_i - \underline{\mathbf{R}}_n|} - \sum_n^{(i\neq n)} c_n^2 \frac{\nu^{1/2}}{|\underline{\mathbf{R}}_{in,c}|} ,$$

namely between terms of different origin (in our expansion).

Finally, one may check that higher-order terms in the above expansion are indeed small. For example, we can estimate the contributions of order c^3 in expansions (3.6) and (3.7). The mechanism of infrared cancellation (after the subtraction of background contribution) implies that the Coulomb matrix elements can at most yield terms of order $|\underline{\mathbf{R}}_i|^{-3}$. The sum of such terms is of order $(6c)^3 A \Sigma_i |\underline{\mathbf{R}}_i|^{-3}$, with $A \sim 2$ for $\nu=\frac{1}{3}$; then such a contribution is 10^{-4} smaller than the one of order c^2.

IV. NUMERICAL RESULTS AND COMPARISON WITH THE PREVIOUS LITERATURE

In this section, we list the numerical results and, for better information, we split the direct and the exchange contribution, $\delta E = E_d - E_{\text{ex}}$. We obtain ($E_{\text{CDW}}$ denotes the result obtained by the CDW approach) the following.

For $\nu=\frac{1}{3}$,

$$E_d=0.0657,\quad E_{\text{ex}}=0.0030,\quad E_w=-0.4515 ,$$

$$E_o(\tfrac{1}{3})=-0.3888 ,$$

$$E_{\text{CDW}}(\tfrac{1}{3})=-0.3885 .$$

For $\nu=\frac{1}{5}$,

$$E_d=0.0276,\quad E_{\text{ex}}=0.0001,\quad E_w=-0.3497 ,$$

$$E_o(\tfrac{1}{5})=-0.3222 ,$$

$$E_{\text{CDW}}(\tfrac{1}{5})=-0.3219 .$$

For $\nu=\frac{1}{7}$,

$$E_d=0.0161,\quad E_{\text{ex}}=0,\quad E_w=-0.2956 ,$$

$$E_o(\tfrac{1}{7})=-0.2795 ,$$

$$E_{\text{CDW}}(\tfrac{1}{7})=-0.2794 .$$

For $\nu=\frac{1}{9}$,

$$E_d=0.0107,\quad E_{\text{ex}}=0,\quad E_w=-0.2607 ,$$

$$E_o(\tfrac{1}{9})=-0.2500 ,$$

$$E_{\text{CDW}}(\tfrac{1}{9})=-0.2499 .$$

The above results show that the exchange contribution is indeed very small already for $\nu=\frac{1}{5}$.

The numerical results obtained by the CDW approach[13] do not significantly differ from ours and are certainly not better.

Our careful control of the expansion in $\nu^{1/2}$ and in δ also proves that the contributions due to the nonvanishing overlap of the wave functions are small for $\nu<\frac{1}{3}$. This explains why our numerical results are indeed very close to those of Refs. 2, 3, 7, 13, and 14. We would like to emphasize again that since the comparison with Laughlin results involves differences of the order of a few percent, in our opinion a clear-cut control of the approximations and/or the expansion is not an academic issue, and it is actually one of the main motivations of our work.

As a function of ν, for small ν, we get

$$E_o(\nu)=-0.7821\sqrt{\nu}(1-0.3609\nu-0.1372\nu^2) .$$

(This formula, which we actually derived from the $\nu=1/m$, m = integer, could be used for arbitrary ν.)

One can show that our coefficients of the ν and ν^2 terms are the best for "localized" single-particle wave functions, i.e., whenever a multipole expansion of the Coulomb potential is allowed. The above result for $E_o(\nu)$ can be compared with the Laughlin ground-state energy (see Ref. 5)

$$E^L(\nu)=-0.7821\sqrt{\nu}(1-0.211\nu^{0.74}+0.012\nu^{1.7}) .$$

For convenience of comparison, we report the Laughlin ground-state energy for $\nu=\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, and $\frac{1}{9}$:

$$E^L(\tfrac{1}{3})=-0.410,\quad E^L(\tfrac{1}{5})=-0.328 ,$$

$$E^L(\tfrac{1}{7})=-0.2810,\quad E^L(\tfrac{1}{9})=-0.2500 .$$

Thus we can safely conclude that the (quantum) Wigner crystal picture yields a ground-state energy lower than that of the Laughlin state for $\nu<\frac{1}{9}$ (and the same ground-state energy for $\nu=\frac{1}{9}$), in agreement with the recent experimental results.[4]

We hope to be able to improve our approach to discuss the (possible) transition between the Wigner crystal and the Laughlin regime[4] claimed to occur at $\nu=\frac{1}{5}$.

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APPENDIX A

Here we prove that the set of states $\{\Psi_{n,m}\}$ defined in (2.8) is unitarily equivalent to the set $\{\Psi_{jk}^{\text{VN}}(z)\Psi_0(\Pi_x)\}$, where $\Psi_0(\Pi_x)=\exp(-\Pi_x^2/2)/\pi^{1/4}$ is the ground state of the Hamiltonian (2.1) and $\Psi_{jk}^{\text{VN}}(z)$ are the coherent states associated to the points of the Von Neumann lattice in the phase space defined by the canonical variables (Π_{cx},Π_{cy}) (see Ref. 9).

In order to simplify the computation, we use square lattices, both in configuration and in phase space. In configuration space the eigenstates of H_o in (2.1) have the form

$$\Psi_{n_1n_2}(\mathbf{r})=\frac{(-1)^{n_1n_2}}{(2\pi)^{-1/2}}e^{i(Y_nx-X_ny)/2}e^{-(\mathbf{r}^2-\mathbf{R}_n)/4} , \tag{A1}$$

with

$$\mathbf{R}_n=(X_{n_1n_2},Y_{n_1n_2})=-a_o(n_1,n_2) .$$

For the square lattice, the rationality condition (2.7) with $M=1$ implies $a_o=\sqrt{2\pi}$.

To compare the wave functions (A1) with the wave functions $\{\Psi_{jk}^{\mathrm{VN}}(z)\Psi_0(\Pi_x)\}$, we note that, in terms of the two couples of canonical variables introduced in (2.3) and (2.4), the Hamiltonian takes the form

$$H_o=\tfrac{1}{2}(\Pi_x^2+\Pi_y^2)$$

and the magnetic translations are given by

$$T(\mathbf{a}_1)=\exp\{i\Pi_{c_x}a_o\},\quad T(\mathbf{a}_2)=\exp\{i\Pi_{c_y}a_o\}\ .$$

The first Landau level corresponds to the ground-state wave function of H_o, as far as the variables Π_x and Π_y are concerned, e.g.,

$$\Psi_o(\Pi_x)=\frac{1}{\pi^{1/4}}e^{-\Pi_x^2/2}\ ,$$

and its degeneracy is described by a set of wave functions for the canonical system (Π_{c_x},Π_{c_y}).

The comparison with Von Neumann coherent states suggests to introduce the operators

$$A=\frac{\Pi_{c_x}+i\Pi_{c_y}}{\sqrt{2}},\quad A^+=\frac{\Pi_{c_x}-i\Pi_{c_y}}{\sqrt{2}}$$

which satisfy $[A,A^+]=1$, and to describe the degeneracy of the first Landau level in terms of eigenstates of A centered on the Von Neumann lattice sites:

$$\begin{aligned}&\Psi_o(\Pi_x,z)\equiv\Psi_{jk}^{\mathrm{VN}}(z)\Psi_o(\Pi_x)\ ,\\ &A\Psi_{jk}^{\mathrm{VN}}(z)=z\Psi_{jk}^{\mathrm{VN}}(z),\quad z=\sqrt{\pi}(j+ik)\ ,\end{aligned}\tag{A2}$$

k and j being integers. The factor $\sqrt{\pi}$ is crucial in order to have a complete set of vectors (which still remain complete if only one vector is removed) (see Ref. 10).

In order to show the equivalence between the states in (A1) and the ones in (A2) we represent the Von Neumann state $\Psi_{jk}^{\mathrm{VN}}(z)$ as a wave function on the spectrum of Π_{c_y}:

$$\Psi_{jk}^{\mathrm{VN}}(\Pi_{c_y})=e^{-\Pi_{c_y}^2/2}e^{-i\sqrt{2}z\Pi_{c_y}}e^{-(|z|^2-z^2)/2}\ .$$

Finally, $\Psi_{jk}^{\mathrm{VN}}(z)\Psi_o(\Pi_x)$ can be rewritten in the configuration space using the formula discussed in Refs. 6 and 15:

$$\Phi_{jk}(x,y)=\frac{1}{2\pi}\int d\Pi_{c_y}d\Pi_x e^{i(xy/2+x\Pi_x+y\Pi_{c_y}+\Pi_{c_y}\Pi_x)}\times\Psi_{jk}^{\mathrm{VN}}(\Pi_{c_y})\Psi_o(\Pi_x)\ .$$

Once this integration is performed, one gets exactly the wave functions (A1): $\Phi_{jk}(x,y)=\Psi_{-kj}(x,y)$.

APPENDIX B

We have to prove that the sequence

$$\Omega_n^N(\mathbf{r})=\sum_i^N f_i\Psi_{n+i}(\mathbf{r})\ ,\tag{B1}$$

where

$$f_{nm}=\left\{\frac{a^2}{2\pi}\right\}\int_0^{2\pi/a}dk\int_0^{2\pi/a}dq\, e^{-ikna-iqma}\frac{1}{\sqrt{C(k,q)}}\tag{B2}$$

and

$$C(k,q)\equiv\sum_{n\equiv(n_1,n_2)}d_n e^{ikn_1a+iqn_2a}=C(-k,-q)\ ,\tag{B3}$$

is a Cauchy sequence in $\mathbf{L}^2(\mathbb{R}^2)$ and therefore it defines a vector $\Omega_n(\mathbf{r})$.

It is not difficult to show that series (B3) is uniformly convergent together with all its derivatives, since the coefficients d_n are exponentially decreasing in n [by Eqs. (2.18) and (2.9), $|d_n|=\exp(-\mathbf{R}_n^2/4)\leq\exp\{(-\pi/2\nu\sqrt{3})(n_1^2+n_2^2)\}$].

Furthermore, by exploiting the fact that the c_n's are small for $\nu\leq\frac{1}{3}$, one can estimate that $C(k,q)$ is always different from zero. Hence, $C(k,q)$ and $[C(k,q)]^{-1/2}$ are C^∞ functions, and, by a standard theorem,[16] the Fourier coefficients f_{nm} decrease faster than any inverse power of n and m. Therefore, for M and N large enough,

$$\begin{aligned}|I_{NM}|&\equiv\left|\int_V|\Omega_n^N(\mathbf{r})-\Omega_n^M(\mathbf{r})|^2dx\,dy\right|\\&=\left|\sum_{ij=M}^N f_if_j^*d_{i-j}\right|\leq\left[\sum_{i=M}^N|f_i|\right]^2<\varepsilon\ .\end{aligned}$$

It is straightforward to prove that $\Omega_n(\mathbf{r})$ satisfies the orthogonality condition (2.16).

The expression of the Ω_n's in terms of the Ψ_n's, Eq. (2.15), can be inverted for $\nu<1$, and one gets

$$\Psi_n(\mathbf{r})=\sum_i h_i\Omega_{n+i}(\mathbf{r})\ ,\tag{B4}$$

with

$$h_{nm}=\left\{\frac{a^2}{2\pi}\right\}\int_0^{2\pi/a}dk\int_0^{2\pi/a}dq\, e^{-ikna-iqma}\sqrt{C(k,q)}\ .$$

The convergence of the expression (B4) follows from the properties of $C(k,q)$ by an argument similar to the one used above.

APPENDIX C

In this appendix, we will prove that in the thermodynamical limit the difference

$$\begin{aligned}&\frac{1}{N}(\langle H_B^{(N)}\rangle_{\Lambda_o}-H_B^{(N)_{\mathrm{cl}}})\\&\qquad\equiv\frac{1}{N}\left[\frac{e}{2}\sum_i^N\rho_B\int_V\frac{|\Psi_i(\mathbf{x})|^2}{|\mathbf{x}-\mathbf{y}|}d^2\mathbf{x}\,d^2\mathbf{y}-\frac{e}{2}\sum_i^N\rho_B\int_V\frac{1}{|\mathbf{R}_i-\mathbf{y}|}d^2\mathbf{y}\right]\end{aligned}\tag{C1}$$

tends to zero. Here, the wave function Ψ_i is the one defined in (2.8).

We note that, since both integrals are divergent in the limit of $V\to\infty$, a finite result requires a suitable cancellation between the two divergences.

We start introducing the Fourier transform of the Coulomb potential in the first term, and we observe that the integration in **x** can safely be extended to the whole space due to the Gaussian behavior of the wave function $\Psi_i(\mathbf{x})$. This is not the case for the integration in **y**, because of its divergent limit for $V \to \infty$.

If ΔI denotes the difference between the two integrals in (C1), we get

$$\Delta I = \frac{1}{2\pi}\int_V d^2\mathbf{y}\int \frac{d^2\mathbf{k}}{|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{y}} e^{-i\mathbf{k}\cdot\mathbf{R}_i} e^{-\mathbf{k}^2/2} - \int_V \frac{1}{|\mathbf{R}_i-\mathbf{y}|} d^2\mathbf{y} \, .$$

The integration in **k**, performed in polar coordinates, yields the modified Bessel function I_o (see Ref. 12). Therefore, after the change of variables $\mathbf{p}=\mathbf{R}_i-\mathbf{y}$, we obtain

$$\Delta I = 2\pi\int_0^L \left[\left[\frac{\pi}{2}\right]^{1/2} p e^{-p^2/4} I_o(-p^2/4) - 1\right] dp \, .$$

The first integration can be performed, and it essentially defines the degenerate hypergeometric function $\Phi(\alpha;\gamma;z)$ (see Ref. 12). Therefore

$$\Delta I = 2\pi\left\{\frac{L^2}{2}\left[\frac{\pi}{2}\right]^{1/2}\Phi(\tfrac{1}{2};2;-L^2/2) - L\right\} .$$

Since we are interested in the value of ΔI for large values of L, we can use the form of the asymptotic behavior of the function $\Phi(\alpha;\gamma;z)$ (see Ref. 17):

$$\Phi(\alpha;\gamma;z) \approx \frac{(-z)^{-\alpha}}{\Gamma(\gamma-\alpha)}\Gamma(\gamma)[1+O(z^{-1})] \qquad \mathrm{Re}[z] \to -\infty \, ,$$

and therefore we get

$$\lim_{L\to\infty}\Delta I = \lim_{L\to\infty} O(1/L) = 0 \, .$$

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[1] L. Bonsall and A. A. Maradudin, Phys. Rev. B **15**, 1959 (1977).

[2] K. Maki and X. Zotos, Phys. Rev. B **28**, 4349 (1983).

[3] D. Yoshioka and H. Fukuyama, J. Phys. Soc. Jpn. **47**, 394 (1979); D. Yoshioka and P. A. Lee, Phys. Rev. B **27**, 4986 (1983).

[4] A. Khurana, Phys. Today **43** (12), 17 (1990); V. J. Goldman, M. Santos, M. Shayegau, and J. E. Cunningham, Phys. Rev. Lett. **65**, 2189 (1990).

[5] D. Levesque, J. J. Weis, and A. H. MacDonald, Phys. Rev. B **30**, 1056 (1984).

[6] I. Dana and J. Zak, Phys. Rev. B **28**, 811 (1983).

[7] R. Ferrari, Phys. Rev. B **42**, 4598 (1990); and (unpublished).

[8] P. Wigner, Phys. Rev. **46**, 1002 (1934).

[9] R. Kubo *et al.*, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic, New York, 1965), Vol. 17, p. 269.

[10] H. Bacry, A. Grossmann, and J. Zak, Phys. Rev. B **12**, 1118 (1975); V. Bargmann, P. Butera, L. Girardello, and J. R. Klauder, Rep. Math. Phys. **2**, 221 (1971); A. M. Perelomov, Teor. Mat. Fiz. **6**, 213 (1971).

[11] E. Tosatti and M. Parrinello, Lett. Nuovo Cimento **36**, 289 (1983); E. Tosatti, M. Parrinello, and C. D. Chen (unpublished).

[12] I. S. Gradshteyn and I. M. Ryzhik, in *Table of Integrals, Series and Products*, edited by A. Jeffrey (Academic, New York, 1980).

[13] P. K. Lam and S. M. Girvin, Phys. Rev. B **30**, 473 (1984).

[14] G. Fano and F. Ortolani, Phys. Rev. B **37**, 8179 (1988).

[15] M. Moshinsky and C. Quesne, J. Math. Phys. **12**, 1772 (1971).

[16] R. Courant and D. Hilbert, *Method of Mathematical Physics* (Interscience, New York, 1953).

[17] F. G. Tricomi, *Funzioni Ipergeometriche Confluenti* (Cremonese, Roma, 1954).