

Motion of a quantum particle in a random-flux field

Andrzej Łusakowski

Institute of Physics, Polish Academy of Science, Al. Lotników 32/46, 02-668 Warszawa, Poland

Łukasz A. Turski

Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/46, 02-668 Warszawa, Poland

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We consider a charged spinless quantum particle moving on a two-dimensional square lattice. Each plaquette of the lattice is penetrated by a random magnetic flux with values homogeneously distributed in the interval $(0, 2\pi)$ (in units of the elementary quantum flux h/e). The fluxes in different plaquettes are statistically independent. With the path integral method, within the saddle-point approximation, we evaluated the averaged density of states. Our results are compared with the recent numerical-simulation predictions of Pryor and Zee.

The problem of a quantum particle moving in the random potential field has been the subject of extensive experimental and theoretical investigation.¹ In contrast, very little is known about the behavior of the quantum particle in presence of a random magnetic field. Recently Pryor and Zee have analyzed the motion of a spinless quantum particle in the presence of a random-magnetic-flux arrangement using numerical methods.² In this paper we present an attempt to analyze this problem analytically.

The motion of a quantum particle in the field of random magnetic fluxes is just one example out of the variety of problems related to the motion of particles on manifolds with topological defects. Another problem of perhaps even greater applicability is the diffusion of a classical particle in the field of many randomly distributed dislocation lines. In a short paper³ we have set the framework for the general theory of such a process by proposing a Fokker-Planck equation for diffusion on the manifold with stochastic affine connection.

Imagine a two-dimensional square lattice (see Fig. 1)

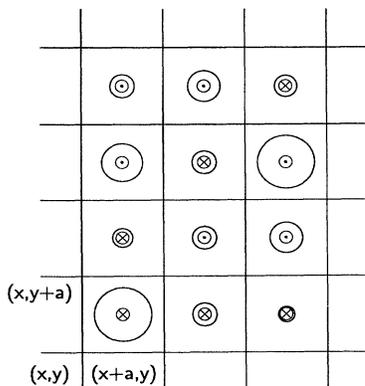


FIG. 1. Square lattice each cell of which contains a magnetic flux. Radii of the circles are meant to indicate the values of the fluxes which point in and out of the plane.

and assume that each plaquette is penetrated by a magnetic flux $\phi(x + a/2, y + a/2)$. The fluxes in different plaquettes are assumed to be independent and homogeneously distributed over the interval $(0, 2\pi)$. (The values of fluxes are measured in units of elementary flux h/e .) Furthermore, assume that a single, spinless, quantum particle moves over this lattice, and that its motion is governed by the tight binding Hamiltonian, which we write in the form

$$\begin{aligned}
 H = & -K_x (\delta_{x,x'+a} e^{iaA_x(x-a/2,y)} \\
 & + \delta_{x,x'-a} e^{-iaA_x(x+a/2,y)}) \delta_{y,y'} \\
 & -K_y (\delta_{y,y'+a} e^{iaA_y(x,y-a/2)} \\
 & + \delta_{y,y'-a} e^{-iaA_y(x,y+a/2)}) \delta_{x,x'} , \quad (1)
 \end{aligned}$$

where K_x and K_y are the coupling constants along the x and y directions of the lattice, respectively. A_x and A_y are the components of the magnetic field vector potential, which, as in the lattice gauge theory, are defined on the lattice bonds.

The main difference between the present problem and that of a particle moving in a random potential rests on later *locality*. Indeed, the particle needs to move just one lattice constant to experience changes induced by a random scalar potential. In our case of random magnetic fluxes the particle must traverse a closed loop around the region containing the flux to notice its presence at all. This nonlocality, known from the theory of the Aharonov-Bohm effect,⁶ results in technical difficulties in the analysis to follow.

If the magnetic fluxes were distributed periodically our model will be similar to that analyzed theoretically by Rammer and Schelankov⁴ and experimentally by Bending *et al.*⁵ We believe that experimental arrangements of fluxes analyzed in our paper can be achieved using one of the new high- T_c materials in which fluxes might form random arrangements in contrast to usual superconductors in which fluxes form periodic lattices.

In this work we shall be interested in the averaged

density of states $\langle \varrho(\omega) \rangle$, where $\langle \dots \rangle$ denotes the average over the flux distribution. $\varrho(\omega)$ is given by the difference of the retarded and advanced propagators

$$\varrho(\omega) = i \sum_{\mathbf{r}} [G^R(\mathbf{r}, \mathbf{r}'; \omega) - G^A(\mathbf{r}, \mathbf{r}'; \omega)], \quad (2)$$

where

$$G^{R/A}(\mathbf{r}, \mathbf{r}'; \omega) = \sum_i \frac{\phi_i(\mathbf{r}) \phi_i^*(\mathbf{r}')}{\omega - \epsilon_i \pm i\eta}, \quad (3)$$

and $\phi_i(\mathbf{r})$ and ϵ_i are normalized eigenfunctions and eigenenergies of the one-particle Hamiltonian H .

In order to calculate the above quantity we adopted here a generating functional approach developed for the random potential problem by Bausch and Leschke.⁷ To

this end let us define two vectors:

$$\Phi(\mathbf{r}, \omega) = \begin{pmatrix} \psi(\mathbf{r}, \omega) \\ \hat{\psi}(\mathbf{r}, \omega) \end{pmatrix}, \quad (4)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \begin{pmatrix} \hat{\psi}^*(\mathbf{r}, \omega) \\ -\psi^*(\mathbf{r}, \omega) \end{pmatrix}, \quad (5)$$

where ψ and $\hat{\psi}$ are complex fields, and the star denotes complex conjugation. The generating functional $Z\{\hat{l}\}$ depends on the matrix source field \hat{l} and has the following form:

$$Z\{\hat{l}\} = \int D\Phi D\tilde{\Phi} \exp(J + S), \quad (6)$$

where

$$J = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_{\mathbf{r}} \left[\tilde{\Phi}(\mathbf{r}, \omega) \begin{pmatrix} \omega + i\eta & 0 \\ 0 & \omega - i\eta \end{pmatrix} \Phi(\mathbf{r}, \omega) + K_x e^{iaA_x(x+a/2, y)} \tilde{\Phi}(x, y; \omega) \cdot \Phi(x+a, y; \omega) \right. \\ \left. + K_x e^{-iaA_x(x+a/2, y)} \tilde{\Phi}(x+a, y; \omega) \cdot \Phi(x, y; \omega) + K_y e^{iaA_y(x, y+a/2)} \tilde{\Phi}(x, y; \omega) \cdot \Phi(x, y+a; \omega) \right. \\ \left. + K_y e^{-iaA_y(x, y+a/2)} \tilde{\Phi}(x, y+a; \omega) \cdot \Phi(x, y; \omega) \right]. \quad (7)$$

The functional S , the so-called source term, is given by

$$S = \frac{1}{2} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \sum_{\mathbf{r}} \sum_{\alpha, \beta} \tilde{\Phi}_{\alpha}(\mathbf{r}, \omega) l_{\alpha\beta}(\mathbf{r}; \omega, \omega') \Phi_{\beta}(\mathbf{r}, \omega'). \quad (8)$$

The virtue of the generating functional approach is that, by differentiating $Z\{\hat{l}\}$ with respect to source field $l_{\alpha\beta}$, we can obtain the relevant quantity. Indeed,

$$\left. \frac{\delta Z}{\delta l_{11}(\mathbf{r}; \omega, \omega')} \right|_{l_{\alpha\beta}=0} = -2\pi\delta(\omega - \omega') G^R(\mathbf{r}, \mathbf{r}; \omega). \quad (9)$$

Now, since $Z\{\hat{l} = 0\} = 1$ we may perform the averaging over the (quenched) disorder *before* attempting to calculate either of the propagators. In the process of evaluation of the mean value of $Z\{\hat{l}\}$ one encounters the problem of averaging expressions like

$$\exp \left(\sum_{\mathbf{r}} [U(\mathbf{r})\chi_1(\mathbf{r}) + U^*(\mathbf{r})\chi_2(\mathbf{r}) + V(\mathbf{r})\chi_3(\mathbf{r}) + V^*(\mathbf{r})\chi_4(\mathbf{r})] \right), \quad (10)$$

with $U(\mathbf{r}) = \exp\{iaA_y(x, y + a/2)\}$ and $V(\mathbf{r}) =$

$\exp\{iaA_x(x + a/2, y)\}$, and $\chi_i(\mathbf{r})$, $i = 1, \dots, 4$ being arbitrary functions of \mathbf{r} .

For an arbitrary type of magnetic disorder, the calculation of those averages becomes a formidable mathematical task. In the case of the magnetic disorder we have chosen, the average is conveniently done in the Landau gauge:

$$A_x = 0, \quad (11)$$

$$A_y(x, y + a/2) = \sum_{x_1=-\infty}^{x-a} \phi(x_1 + a/2, y + a/2),$$

what implies that U factors, for different values of y coordinate, are statistically independent. In the following analysis outlined in the Appendix we obtain

$$\left\langle \exp \left(\sum_{\mathbf{r}} [U(\mathbf{r})\chi_1(\mathbf{r}) + U^*(\mathbf{r})\chi_2(\mathbf{r})] \right) \right\rangle = \prod_{\mathbf{r}} I_0 \left(2\sqrt{\chi_1(\mathbf{r})\chi_2(\mathbf{r})} \right), \quad (12)$$

where I_0 is the modified Bessel function.

The above result permits us to express the effective functional $J_1 = \ln\langle \exp J \rangle$ as

$$J_1 = \frac{1}{2} \int \frac{d\omega}{2\pi} \sum_{\mathbf{r}} \left[\tilde{\Phi}(\mathbf{r}, \omega) \begin{pmatrix} \omega + i\eta & 0 \\ 0 & \omega - i\eta \end{pmatrix} \Phi(\mathbf{r}, \omega) + K_x \left(\tilde{\Phi}(x, y; \omega) \Phi(x+a, y; \omega) + \tilde{\Phi}(x+a, y; \omega) \Phi(x, y; \omega) \right) \right. \\ \left. + \sum_{\mathbf{r}} \ln I_0 \left(2K_y \sqrt{\text{Tr} R(x, y) R(x, y+a)} \right) \right], \quad (13)$$

where we have introduced the matrix \hat{R}

$$R_{\alpha\beta}(\mathbf{r}; \omega; \omega') = \frac{1}{2} \Phi_{\alpha}(\mathbf{r}; \omega) \tilde{\Phi}_{\beta}(\mathbf{r}; \omega'). \quad (14)$$

Here and in what follows Tr denotes usual matrix trace and the integration over ω 's.

Notice that the term with $\ln I_0$ produces an infinite number of vertices. This should be compared with the random potential case, where after averaging we get only one vertex of the fourth order in the fields ψ and $\hat{\psi}$. This is the direct manifestation of the nonlocality of the present problem.

The averaged value of the Z is then

$$\langle Z\{\hat{l}\} \rangle = \int D\Phi D\tilde{\Phi} \exp\{J_1 + S\}, \quad (15)$$

where the source term is written as $S = \sum_{\mathbf{r}} \text{Tr} \hat{l}(\mathbf{r}) \hat{R}(\mathbf{r})$.

In order to carry out the ψ fields integration in Eq. (15) we use the following functional identity:⁸

$$1 = \int \{D\hat{R}D\hat{s}\} \exp\left(\sum_{\mathbf{r}, \alpha, \beta} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} s_{\alpha\beta}(\mathbf{r}, \omega, \omega') \left(R_{\beta\alpha}(\mathbf{r}; \omega'; \omega) - \frac{1}{2} \tilde{\Phi}_{\alpha}(\mathbf{r}; \omega) \Phi_{\beta}(\mathbf{r}; \omega')\right)\right). \quad (16)$$

In the above equation the functional integration $\{D\hat{R}D\hat{s}\}$ is understood as the integration over independent elements of complex matrices satisfying the conditions

$$R_{\beta\alpha}^*(\mathbf{r}; \omega', \omega) = -\sigma_{\alpha\gamma}^y R_{\gamma\delta}(\mathbf{r}; \omega, \omega') \sigma_{\delta\beta}^y, \quad (17)$$

$$s_{\beta\alpha}^*(\mathbf{r}; \omega', \omega) = \sigma_{\alpha\gamma}^y s_{\gamma\delta}(\mathbf{r}; \omega, \omega') \sigma_{\delta\beta}^y, \quad (18)$$

where $\sigma_{\alpha\beta}^i$ ($i = x, y, z$) are Pauli's matrices.

Performing the integration over the ψ and $\hat{\psi}$ fields we obtain

$$\langle Z\{\hat{l}\} \rangle = \int \{D\hat{R}D\hat{s}\} \exp\{J_{\text{eff}} + S\}, \quad (19)$$

where

$$J_{\text{eff}} = -\ln \det G^{-1} + \sum_{\mathbf{r}} \text{Tr}(\hat{s}\hat{R}) + \sum_{\mathbf{r}} \ln I_0 \left(2K_y \sqrt{\text{Tr}R(x, y) R(x, y + a)}\right) \quad (20)$$

and

$$G_{\alpha\beta}^{-1}(\mathbf{r}, \omega; \mathbf{r}', \omega') = \{[(\omega\delta_{\alpha\beta} + i\eta\sigma_{\alpha\beta}^z)\delta_{xx'} + K_x(\delta_{x+ax'} + \delta_{x-ax'})]2\pi\delta(\omega - \omega') - s_{\alpha\beta}(\mathbf{r}; \omega, \omega')\delta_{xx'}\}\delta_{yy'}. \quad (21)$$

Now we calculate the averaged density of states using the saddle-point approximation. The saddle-point equations for the functional J_{eff} read

$$R_{\alpha\beta}(\mathbf{r}, \omega, \omega') = -G_{\alpha\beta}(\mathbf{r}, \omega; \mathbf{r}, \omega'), \quad (22)$$

$$s_{\alpha\beta}(\mathbf{r}, \omega, \omega') = - \sum_{\mu=\pm 1} F_{\alpha\beta}(K_y; x, y + \mu a). \quad (23)$$

The quantity $F_{\alpha\beta}(K_y; x, y + \mu a)$ is defined as

$$F_{\alpha\beta}(K_y; x, y + \mu a) = K_y \frac{I_0'(2K_y Q) R_{\alpha\beta}(x, y + \mu a, \omega, \omega')}{I_0(2K_y Q) Q}$$

with

$$Q = \sqrt{\text{Tr}R(x, y) R(x, y + \mu a)}.$$

The prime here denotes the differentiation with respect to the function's argument.

Since our physical system is spatially homogeneous, we are looking for solutions of Eqs. (22) and (23) having the form

$$R_{\alpha\beta}^0(\mathbf{r}, \omega, \omega') = 2\pi\delta(\omega - \omega') \delta_{\alpha\beta} R_{\alpha}(\omega),$$

$$s_{\alpha\beta}^0(\mathbf{r}, \omega, \omega') = 2\pi\delta(\omega - \omega') \delta_{\alpha\beta} s_{\alpha}(\omega).$$

The poles of the retarded (advanced) propagator lie in the lower (upper) half of the complex plane, and since Eq. (22) says that R_{α} is just the propagator, we expect that the integral $\int d\omega [R_{\alpha}(\omega)]^2$ and consequently $\text{Tr}\hat{R}^2$ in Eq. (23) should vanish. The vectors $R_{\alpha}(\omega)$ and $s_{\alpha}(\omega)$ obey then the following equations:

$$s_{\alpha}(\omega) = -2K_y^2 R_{\alpha}(\omega), \quad (24)$$

$$R_{\alpha}(\omega) = -a \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} \frac{1}{\omega + i\eta_{\alpha} + 2K_x \cos(ka) - s_{\alpha}(\omega)}, \quad (25)$$

where $\eta_1 = \eta$ and $\eta_2 = -\eta$.

From Eqs. (24) and (25) we obtain the equation for $R_{\alpha}(\omega)$:

$$R_{\alpha}(\omega) = - \frac{1}{\sqrt{[\omega + 2K_x + 2K_y^2 R_{\alpha}(\omega)][\omega - 2K_x + 2K_y^2 R_{\alpha}(\omega)]}} \quad (26)$$

from which we may obtain the averaged density of states. It is given by the imaginary part of $R_\alpha(\omega)$. $\text{Im}R_\alpha(\omega)$ is nonzero for $|\omega| < \omega_0$ where ω_0 is given by the following formula:

$$\omega_0/K_x = 4(Q_+ + Q_- - \frac{1}{3}\alpha^{4/3})^{-3/2} + \alpha^{4/3}(Q_+ + Q_- - \frac{1}{3}\alpha^{4/3})^{1/2}, \quad (27)$$

$Q_\pm = (1 \pm \sqrt{1 - \alpha^4/27})^{2/3}$ and $\alpha = K_y/K_x$. In principle, because Eq. (26) reduces to the polynomial equation of fourth order, it is possible to obtain the analytical expression for the averaged density of states. However, this formula is not particularly transparent and we decided to present the solution in the graphical form. In Fig. 2 we display $\rho(\omega)$ for three different values of α .

Now, the main result is the narrowing of the band. In the case of the free particle the allowed energies lie between $-2K_x - 2K_y$ and $2K_x + 2K_y$. In the presence of magnetic disorder the band is shrinker ($\omega_0 < 2K_x + 2K_y$). This is in contrast with the potential fluctuation case where we observe the tails of the averaged density of states in the energy range forbidden for the free particle.

For the symmetric lattice, $K_y = K_x$, we may compare our result with those of Pryor and Zee.² In Ref. 2 the motion of a quantum particles was analyzed on a finite lattice in the presence of the magnetic flux disorder analogous to that discussed in our paper. The calculated quantity was $p(E)$, defined as the probability density of finding the state with its energy between E and $E + dE$. The main results in Ref. 2 were the narrowing of the band. States with eigenenergies $3.4K_y < |E| < 4K_y$ were found to be extremely improbable. This result coincides with our band shrinking (lowest curve in Fig. 2) obtained within the mean field approximation. Furthermore, Pryor and Zee² considered a different type of magnetic disorder than those discussed so far. Namely, they assumed that each plaquette of the lattice may be penetrated by the magnetic flux which takes two values, 0 or π only, and with equal probability. Our analysis can be directly applied also to that case by averaging the generating functional $Z\{\hat{l}\}$ over such magnetic disorder. Instead of the term $\ln I_0 \left(2K_y \sqrt{\text{Tr}R(x, y) R(x, y + a)} \right)$ in Eq. (13), we obtain

$$\ln \cosh \frac{1}{2}K_y (\tilde{\Phi}(x, y; \omega) \cdot \Phi(x, y + a; \omega) + \tilde{\Phi}(x, y + a; \omega) \cdot \Phi(x, y; \omega)). \quad (28)$$

In the saddle-point method, which must be slightly modified in the present case, we obtain the same density of states as previously. It is tempting to interpret results such as a mean field indicator of some sort of statistical universality.

In conclusion, we have analyzed the hopping of a quantum particle over the lattice penetrated by the randomly distributed magnetic fluxes. Using a properly tailored

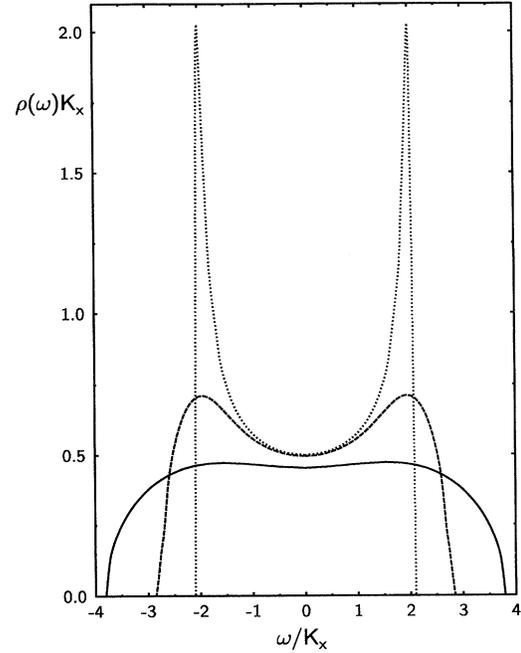


FIG. 2. The averaged density of states plotted for various values of the parameter α . Dotted, broken, and solid lines correspond to $\alpha = 0.1, 0.5$, and 1, respectively.

Bausch and Leschke formulation,⁷ we have derived the mean field expression for the density of states which exhibits properties analogous to those suggested in computer simulations of Pryor and Zee.² In order to appreciate the physical significance of the predicted band narrowing we should analyze two-point Green functions for our particle. Only then we would be able to compare the case of random fluxes to that discussed in Ref. 4. Whether the departure form $|\mathbf{B}|$ (periodic fluxes) or the usual \mathbf{B}^2 (homogeneous field) (Ref. 9) behavior of the magnetoresistivity will be observed also in that case remains an open question. Some work along this line is now in progress.

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APPENDIX

Following Eq. (10) we have introduced the quantities U which, in the Landau gauge and for different values of the y coordinate, are statistically independent. It is sufficient, therefore, to consider the average of $\exp\{\sum_x (U_x \chi_{1x} + U_x^* \chi_{2x})\}$ (recall that $x \in Z$)

$$\exp\left(\sum_x (U_x \chi_{1x} + U_x^* \chi_{2x})\right) = \prod_x \sum_{n_x=0}^{\infty} \sum_{m_x=0}^{\infty} \frac{1}{n_x!} \frac{1}{m_x!} (\chi_{1x}^{n_x} \chi_{2x}^{m_x}) (U_x^{n_x} U_x^{*m_x}). \quad (A1)$$

For a given set $\{n_x, m_x\}$ the average of

$$\prod_x U_x^{n_x} U_x^{*m_x} \quad (\text{A2})$$

is equal to 1 provided $n_x = m_x$, and vanishes otherwise. Indeed, let us choose the greatest x_0 for which

$$n_{x_0} \neq 0 \text{ or } m_{x_0} \neq 0. \quad (\text{A3})$$

If $n_{x_0} \neq m_{x_0}$ then the product (A2) contains the factor

$$\exp\{i(n_{x_0} - m_{x_0})\phi(x_0 - a/2, y)\},$$

the average of which is zero. Thus n_{x_0} must be equal to m_{x_0} , otherwise the average of (A2) vanishes. Next we

choose the greatest $x < x_0$ for which condition (A3) is satisfied and in the same way we prove that $n_x = m_x$. Repeating this procedure we find

$$\left\langle \prod_x U_x^{n_x} U_x^{*m_x} \right\rangle = \prod_x \delta_{n_x m_x}, \quad (\text{A4})$$

and, therefore,

$$\left\langle \exp \left(\sum_x (U_x \chi_{1x} + U_x^* \chi_{2x}) \right) \right\rangle = \prod_x I_0(2\sqrt{\chi_{1x} \chi_{2x}}), \quad (\text{A5})$$

where I_0 is the modified Bessel function.

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