Symmetry breaking and stabilization of critical phase

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Anisotropic quantum XY spin chains in a quasiperiodic (QP) transverse field provide a very interesting and a rich class of one-dimensional models exhibiting extended (E), localized (L), and critical (C) phases in addition to a magnetic transition to long-range order. The interesting feature in this class of models is the fact that the intermediate C phase exists in a *finite* window of size determined by the magnitude of spin-space anisotropy. Furthermore, even a small amount of anisotropy in the spin space destroys the fractal phase boundary between the E and L phases of the isotropic model.

One-dimensional (1D) quasiperiodic (QP) systems have been the subject of many interesting studies in recent years.[1] These systems provide a link for understanding the crossover between random systems which cause localization in 1D and the periodic systems which lead to energy bands and extended states. In addition, in 1D QP systems, at the onset of transition between the E and L phases, the systems exhibit fractal spectra and wave functions known as the critical phase with power-law localization. Furthermore, the weak-coupling limit of this problem shares a common mathematical foundation with the small divisor perturbation theory of Kolmogorov, Arnold, and Moser (KAM).[2] Therefore, there is a close connection with the theory of nonintegrable Hamiltonian dynamics[3] relating KAM tori with the Bloch states and Cantori with the localized states.

In this paper, I study a QP spin model which, in addition to the E, L, and C phases, also exhibits a magnetic transition to long-range order (LRO) and, hence, provides a very interesting class of models which describe the interplay between magnetic and spectral properties. As shown below, the existence of a magnetic phase transition signaled by the appearance of a zero mode leads to a different behavior of the fattening of the critical phase. The model under investigation is the 1D anisotropic, quantum spin chain (anisotropic XY model) in a QP magnetic field,

$$H=-\sum_n [\sigma_n^x\sigma_{n+1}^x+(1+g)\sigma_n^y\sigma_{n+1}^y+h_n\sigma_n^z] \ . \qquad (1)$$

The magnetic field is chosen to be a sinusoidal function incommensurate with the periodicity of the lattice,

$$h_n=\lambda\cos(2\pi\sigma n) \ . \qquad (2)$$

Here σ is an irrational number. The irrationality of this parameter is the heart of this problem as it introduces two competing periodicities in the model. For convenience, I will choose the parameter σ to be the inverse golden mean $\sigma=[\sqrt{(5)}-1]/2$; however, none of the results of this paper depends on this choice. The parameter λ is the strength of the field and g is the spin-space anisotropy which breaks the $O(2)$ symmetry of the spin space in the model. For $g=0$, the model is an isotropic XY model, while for $g=-1$, the model reduces to the Ising model.[4] For any finite value of g, the model is Ising-like due to $O(1)$ symmetry with the easy axis (the axis of LRO in the absence of field) being the x axis for $-2<g<0$ and the y axis for other values of g. The axis perpendicular to the easy axis in the XY plane will be called the hard axis.

Using the methods described by Lieb *et al.*,[5] the spin models are mapped to fermion models, quadratic in fermion degrees of freedom. Therefore, the spectral properties of the spin model (1) can be determined by diagonalizing the following tight-binding model (TBM):

$$G(\psi_{n+2}+\psi_{n-2})+(Gh_{n-1}+h_n)\psi_{n-1} +(Gh_n+h_{n+1})\psi_{n+1}+(1+G^2+h_n^2)\psi_n=E^2\psi_n \ , \qquad (3)$$

where $1+g$ is set equal to G. In the Ising limit ($G=0$ or $g=-1$), the TBM involves only nearest-neighbor (NN) interaction among the fermions. In the pure XY limit ($G=1$), the above TBM reduces to the Harper equation squared. Hence, the self-dual Harper equation given by the following TBM,

$$\psi_{n-1}+\psi_{n+1}+h_n\psi_n=E\psi_n \ , \qquad (4)$$

with h_n given by (2), describes the physics of an isotropic XY model in a QP field. Using the self-duality of the model, it was shown[6] to exhibit a transition from the E to L phase at $\lambda=2$. In this transition, referred to as the breaking of analyticity, the E and L phases are separated from each other by a critical phase which exists only at the self-dual point $\lambda=2$. The anisotropy destroys the self-duality of the model and except for the Ising case, introduces next-nearest-neighbor (NNN) interaction among fermions. As shown below, the presence of the anisotropy introduces a new scenario for the breaking of analyticity by stabilizing the critical phase in a finite window in parameter space.

The numerical study of the model (1) involves studying a sequence of periodic systems with periods corresponding to various rational approximants of the golden mean. This requires diagonalizing the TBM [Eq. (3)] for various Fibonacci orders. In my systematic study, for different values of g, the energy spectrum and wave functions were

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calculated. The spectral properties of the model were determined from the wave functions and the scaling of the bandwidths with the size of the system: the C and E phases are, respectively, distinguished by the algebraic and exponential decay of the total bandwidth (TBW) with the size of the system.[7] Using the methods of Lieb *et al.*,[5] the long-range correlations between the site n and $N+n$, $Cr(n,N)=\langle\sigma_n\sigma_{n+N}\rangle$, along the x, y, and z axes were computed.

In analogy with the self-dual isotropic model, the anisotropic models for all values of g were found to exhibit a pure spectrum, i.e., all the quantum states were either E, C, or L. Figure 1 shows the TBW, the scaling ratio R (which is the ratio of the lowest gap to the lowest bandwidth), and the magnetic correlation along the easy axis as a function of λ for various values of g. Except in the special cases of XY and Ising limits, the system is found to exhibit three phases E, C, and L and makes transitions between the E-C and C-L phases as the strength of the magnetic field is varied. As the anisotropy g increases, the zero measure critical phase of the isotropic model is found to exist for a *finite* window in the λ space, where the width of the window is always equal to twice the anisotropy. The transition between the C-L phase is always accompanied by the appearance of a zero mode [see Fig. 2(a)] in the energy spectrum and the vanishing of long-range correlations along the easy axis (Fig. 1). The transition between the E-C phase has no bearing on the magnetic properties of the system. The numerical results show that the onset of the E-C transition is determined by the strength of the spin-spin interaction along the hard axis while the onset of the C-L transition is determined by the strength of the spin-spin interaction along the easy axis. In the isotropic limit, where the easy and hard axes degenerate, the E-C and C-L transitions also degenerate to a single transition. As g increases from zero to a positive value, the onset of the E-C transition remains locked at $\lambda=2$, twice the strength of the spin interaction along the hard axis, for all positive values of g, while the onset of the C-L transition is found to vary with g and is given by $\lambda=2+2g$, which is equal to twice the strength of the spin interaction along the easy axis. As g varies between -2 and 0, the x axis now becomes the easy axis and the onset of the C-L transition remains locked at $\lambda=2$, while the onset of the E-C transition varies, and is now given by $\lambda=2+2g$. This explains why the transition from the E-C phase does not affect the long-range correlations in the system. In the Ising case,

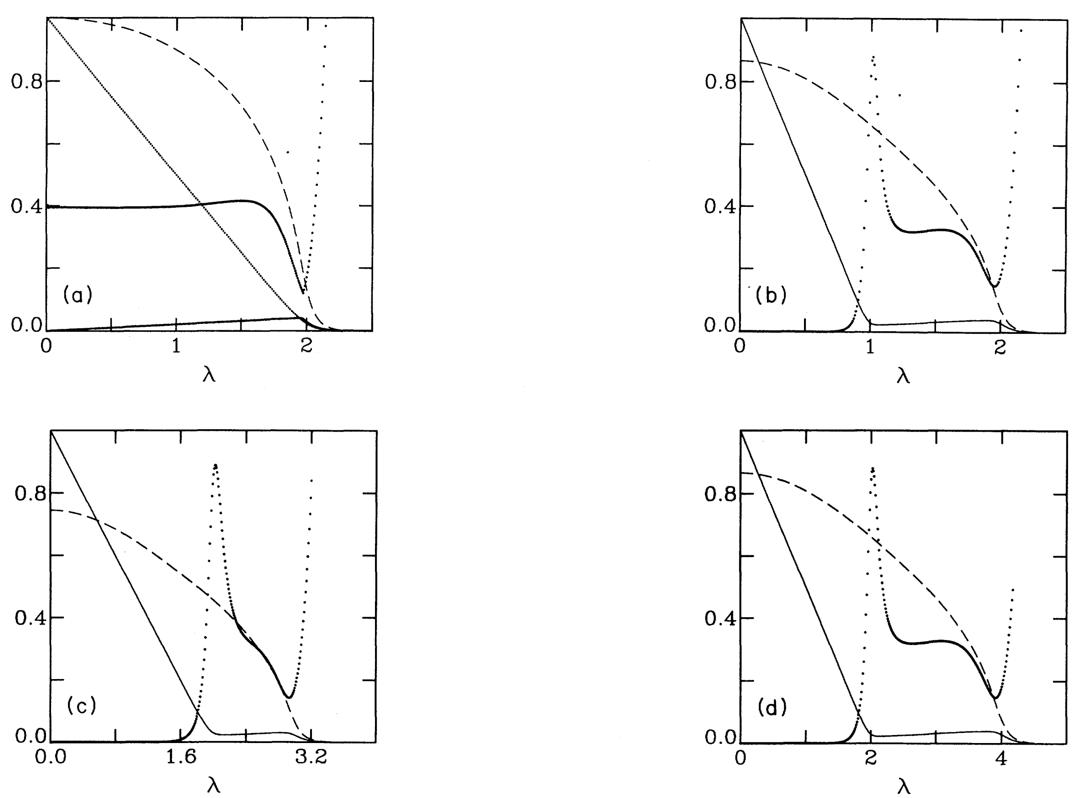

FIG. 1. (a) TBW (solid line), R (dotted line), and Cr (dashed line) as a function of λ for Ising model ($g=-1$). Cr is the correlation between the spin at site 1 and the spin at the midpoint of the periodic chain (in general, it varies from site to site) along the easy axis. Note that all the quantities are normalized to unity. In the C phase, TBW increases from zero to a maximum value as λ approaches 2, the onset of C-L transition. The dark squares show the TBW for the XY model, showing the E and L phases. (b)–(d) Same as (a) for $g=-0.5, g=0.5$, and $g=1$. These figures show E, C, and L phases. As λ is increased, TBW monotonically decreases in the E phase, monotonically increases in the C phase and then drops exponentially to zero in the localized phase.

there is no hard axis and, hence, no E phase and E-C transition but there is a C-L transition at $\lambda=2$, accompanied by a magnetic transition as reported previously.[4]

It is interesting to note that the E-C and C-L transitions were respectively signaled by maxima and minima in the ratio R (Fig. 1). In analogy with the previously known results,[8] the minimum value of R signaling the onset of the C-L transition is always found to converge to a universal value R_1 or R_2, exhibiting period 3 as a sequence of Fibonacci periods are used. On the contrary, the maxima in R signaling the transition from the E-C phase show no convergence to any definite value.

I next examine the question of the smoothness of boundaries between the various phases of the system as the single harmonic modulating field is replaced by a more generic periodic function containing multiple harmonics. In a recent study,[9,10] the Harper or XY model with the magnetic field containing two harmonics, was studied, i.e.,

$$h_n=\frac{\lambda}{\sqrt{(1+\alpha^2)}}[\cos(2\pi\sigma n)+\alpha+\cos(4\pi\sigma n)] \ . \qquad (5)$$

The system was found to exhibit cascades of E-L transitions in the lowest quantum states and the boundary between the existence and nonexistence of the E and L phases was a fractal in the λ-α plane. Associated with the fractal boundary were the cascades of band crossings sandwiched between E-L phases which were found to obey additive rules and a self-similar pattern of scaling ratios R. My numerical studies confirmed that the spectral states near zero energy (the lowest state for the XY model) also exhibited a devil-fork phase diagram in the two-parameter space. Underlying these cascades of transitions are the cascades of band crossings between the two lowest-energy bands of the XY model. In view of the exotic behavior of the isotropic XY model, I investigate the effect of spin-space anisotropy on the fractal boundary and the band crossings. One could, in principle, speculate on the existence of a fractal boundary between the E-C phase as well as between the C-L phase. However, the absence of any convergence in R at the E-C transition suggests that the C and L phases are more likely candidates for the existence of a fractal boundary between them. The single harmonic anisotropic model in analogy with the isotropic model exhibits no band crossings [see Fig. 2(a)]. Figures 2(b)–(d) show how the band crossings of the two harmonic isotropic models begin to disappear as an anisotropy is introduced in the system. Extensive numerical studies showed that the anisotropy destroys the additive rules for the band crossings and hence the

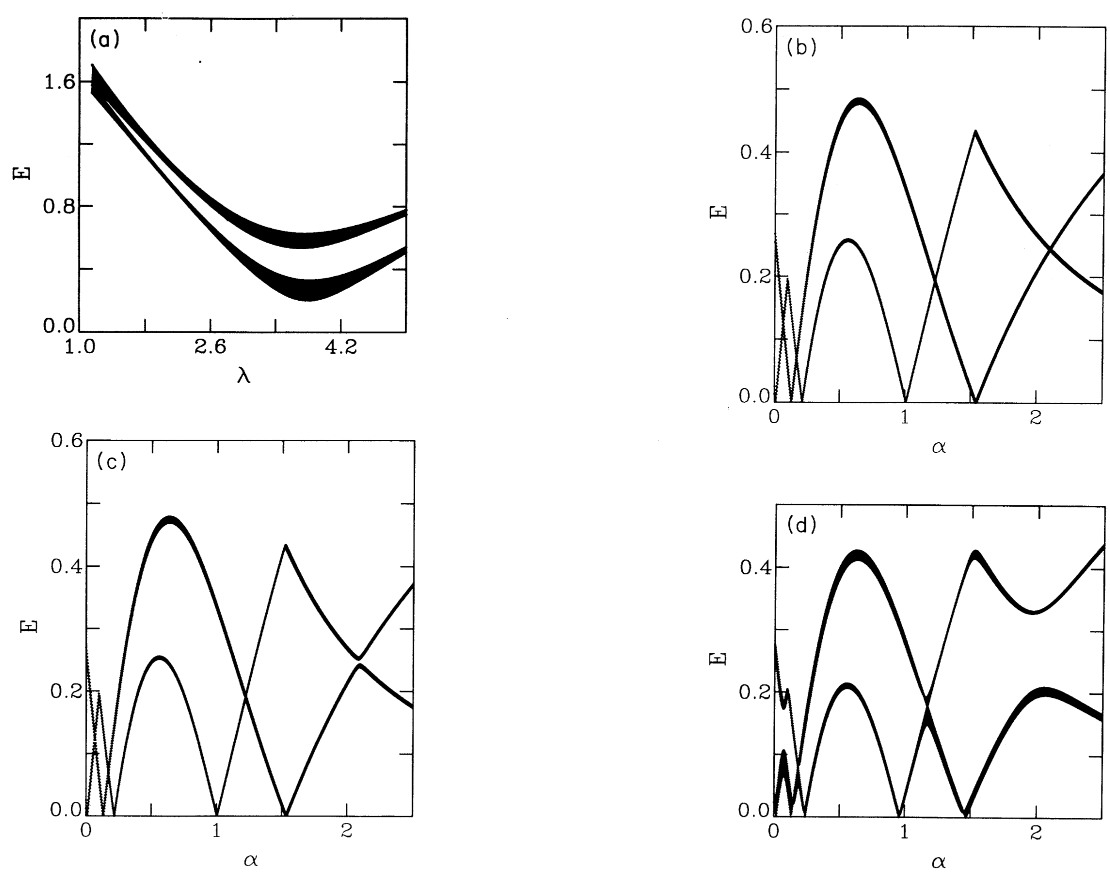

FIG. 2. (a) Two lowest-energy bands vs λ, for $g=1$, illustrating no band crossings in the single harmonic case. (b)–(d) Two lowest bands in the two harmonic models for fixed $\lambda=2$, for $g=0$ (showing four band crossings), $g=0.01$ (showing three crossings), and $g=0.1$ (showing two band crossings), showing how the band crossings are wiped out as g increases.

fractal boundary. This was also found to be the case when in the g-h parameter space, where both g and h are modulating functions of lattice sites with the same period, which is incommensurate with the periodicity of the lattice.

In conclusion, the breaking of $O(2)$ symmetry of the spin space, which is responsible for the long-range correlations among spins, destroys the self-duality of the isotropic model and stabilizes the critical phase (sandwiched between the E and L phases) in a *finite* window in parameter space of size equals to twice the magnitude of anisotropy. Whereas the easy axis determines the transition between the C-L phase (accompanied by infinite correlation length and LRO), the hard axis is responsible for the transition between the E-C phase. This explains why the transition between the E-C phase does not lead to any magnetic transition in the system. In the fermion representation, the anisotropy introduces NNN interactions among fermions. The fact that in the Ising model, the NNN interaction term disappears suggests that the origin of the NNN term in the fermion picture may be linked to the presence of both easy and hard axes.

In addition to fattening the critical phase, it is interesting that the long-range correlation preserves the smoothness of the boundary between the E,C, and L phases in contrast with the isotropic systems, where such perturbations lead to a fractal boundary between the E and L phases.

In view of the similarity between the TBMs and area-preserving maps of the dynamical system, it is interesting to conjecture what analogies these results may have with the Hamiltonian maps. One could speculate a new scenario for the breakdown of KAM tori in the Hamiltonian systems in which, contrary to the existing scenario where a given KAM torus breaks at a critical value of the nonlinear parameter, a torus will remain critical for a finite measure of nonlinear parameter space before breaking. If the NNN interaction among fermions is a key for the three-phase diagram, with each phase existing for a finite measure of parameter values, one may need to search for the fat critical regime in the four-dimensional Hamiltonian maps. If long-range correlations are the key requirements for this behavior, the fat critical phase may be found in the area-preserving maps obtained by studying the ground-state configurations of classical spin models exhibiting LRO. Furthermore, it will be interesting to see this stable critical phase in quantum maps such as the Harper map[11] which exhibits both E and L phases.

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