Migdal-Kadanoff study of the random-field Ising model

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The random-field Ising model is studied within a Migdal-Kadanoff renormalization-group scheme. For dimension $d=3$ the recursion relations and the zero-temperature fixed point are studied numerically. There is a continuous phase transition with magnetization exponent $\beta=0.02$. The magnetization as a function of temperature displays an abrupt crossover from pure Ising to random-field behavior. Analytic results are obtained within a $d=2+\varepsilon$ expansion.

I. INTRODUCTION

Considerable progress has been made in understanding the random-field Ising model (RFIM).[1,2] It is now known that the low-temperature phase in three dimensions is ferromagnetic[3,4] and that the critical properties are controlled by a zero-temperature fixed point with modified scaling behavior.[5–9] However, the critical behavior is still not fully understood. Since equilibration is very slow[7,8] and samples are macroscopically inhomogeneous,[10] experimental results[10–12] are difficult to interpret. Monte Carlo simulations also suffer from equilibration problems. However, zero-temperature properties have been studied numerically using an exact polynomial time algorithm for finding ground states.[13]

The Migdal-Kadanoff renormalization group (MKRG) has been applied to a variety of disordered systems.[14–23] Though mathematically simple, the method has proved to be quite successful in predicting qualitative features of critical phenomena. In the present paper we apply the MKRG to the RFIM using both numerical and analytic techniques. In applying the MKRG to disordered systems, it is necessary to follow the distribution of couplings and fields under renormalization. In our numerical work a Monte Carlo sample is used to represent this distribution.[20,21] A similar study was recently carried out by McKay and Berker.[22] They found a "hybrid-order" transition with no latent heat but a discontinuous magnetization. Boechat and Continentino [23] also obtain a hybrid-order transition using a transmissivity approximation. By contrast, we find a continuous transition with a very small value of β, in agreement with Ref. 13. We believe the difference between our work and Refs. 22 and 23 lies in a more accurate representation of the distribution.

The numerical work is supplemented by an analysis of the MKRG near the lower critical dimension. We show that the transition is necessarily continuous above two dimensions and obtain a $2+\varepsilon$ expansion for the zero-temperature fixed point and associated exponents. The presence of exponentially rare islands of frozen-in spins plays a central role in this analysis. Within the MKRG approach these islands are represented by a finite weight at zero coupling in the distribution of couplings.

Recent experiments on the liquid-vapor transition in aerogel[24] reveal a coexistence curve similar to the pure Ising model but slow dynamics suggestive of random-field effects. Because the aerogel creates a random network which attracts the fluid, the RFIM should be an appropriate model[25] for this system. Here we determine the coexistence curve for the RFIM and find an abrupt crossover from pure Ising behavior to RFIM behavior essentially at T_c. This abrupt crossover may explain why the RFIM value of β was not observed in the experiments. An alternative explanation was given by Maritan *et al.*,[26] who proposed that the RFIM with an asymmetric distribution of random fields would lie in a different universality class from the symmetric RFIM. Our calculations do not support this idea; we find that asymmetric distributions flow to the same zero-temperature fixed point as symmetric distributions.

II. THE MODEL AND RECURSION RELATION

The RFIM is defined by the Hamiltonian

$$-\beta H=\sum_{\langle ij\rangle} K_{ij}S_iS_j+\sum_i H_iS_i \ , \tag{2.1}$$

where $\langle ij\rangle$ indicates a sum over nearest-neighbor pairs and H_i is chosen from a probability distribution

$$P(H)=p\delta(H-h_a)+(1-p)\delta(H+h_b) \ , \tag{2.2}$$

where $p=\frac{1}{2}$, $h_a=h_b=h/T$, and $K_{ij}=k/T$ unless otherwise stated.

The MKRG approximation is equivalent to an exact solution on a hierarchical structure. In the present paper we use the necklace hierarchical structure.[27] Each bond (ij) carries a coupling constant K_{ij} and two fields (H_i,H_j) at its ends. For a scale factor b and dimension d, the renormalization-group transformation consists of adding b^{d-1} couplings and b^{d-1} fields at each of the two ends of the parallel bonds and then decimating the resulting b bonds and $b-1$ fields in series by eliminating the internal degrees of freedom. The procedure is illustrated in Fig. 1 for $b=2$, $d=3$. For $b=2$, the fields and couplings enjoy the recursion relations

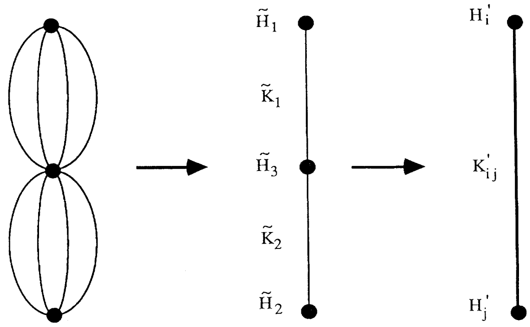

FIG. 1. Iterative construction of the necklace hierarchical structure. For the structure on the left, the top and bottom spins are subject to four fields, while the middle spin is subject to eight fields.

$$K'_{ij}=\frac{a+b-c-d}{4}\ ,\quad H'_i=\frac{a-b+c-d}{4}\ ,\quad H'_j=\frac{a-b-c+d}{4}\ , \tag{2.3a}$$

with

$$\begin{aligned} a&=\ln\{\exp(\tilde H_1+\tilde H_2+\tilde H_3+\tilde K_1+\tilde K_2)+\exp(\tilde H_1+\tilde H_2-\tilde H_3-\tilde K_1-\tilde K_2)\}\ ,\\ b&=\ln\{\exp(-\tilde H_1-\tilde H_2+\tilde H_3-\tilde K_1-\tilde K_2)+\exp(-\tilde H_1-\tilde H_2-\tilde H_3+\tilde K_1+\tilde K_2)\}\ ,\\ c&=\ln\{\exp(\tilde H_1-\tilde H_2+\tilde H_3+\tilde K_1-\tilde K_2)+\exp(\tilde H_1-\tilde H_2-\tilde H_3-\tilde K_1+\tilde K_2)\}\ ,\\ d&=\ln\{\exp(-\tilde H_1+\tilde H_2+\tilde H_3-\tilde K_1+\tilde K_2)+\exp(-\tilde H_1+\tilde H_2+\tilde H_3-\tilde K_1-\tilde K_2)\}\ . \end{aligned} \tag{2.3b}$$

$\tilde K_1$ and $\tilde K_2$ are the two resulting couplings obtained by adding 2^{d-1} couplings on parallel bonds. $\tilde H_1$, $\tilde H_2$, and $\tilde H_3$ are the resulting fields acting on each of the three spins in the unit cell. $\tilde H_1,\tilde H_2$ are each the sum of 2^{d-1} fields, while $\tilde H_3$ is the sum of 2^d fields.

If the variance of H becomes large, the sums of exponentials in Eq. (2.3b) may be replaced by the exponential with the largest argument. The resulting zero-temperature recursion relations are obtained by making the following substitutions in Eq. (2.3a):

$$\begin{aligned} a&\to\max\{\tilde H_1+\tilde H_2+\tilde H_3+\tilde K_1+\tilde K_2\ ,\ \tilde H_1+\tilde H_2-\tilde H_3-\tilde K_1-\tilde K_2\}\ ,\\ b&\to\max\{-\tilde H_1-\tilde H_2+\tilde H_3-\tilde K_1-\tilde K_2\ ,\ -\tilde H_1-\tilde H_2-\tilde H_3+\tilde K_1+\tilde K_2\}\ ,\\ c&\to\max\{\tilde H_1-\tilde H_2+\tilde H_3+\tilde K_1-\tilde K_2\ ,\ \tilde H_1-\tilde H_2-\tilde H_3-\tilde K_1+\tilde K_2\}\ ,\\ d&\to\max\{-\tilde H_1+\tilde H_2+\tilde H_3-\tilde K_1+\tilde K_2\ ,\ -\tilde H_1+\tilde H_2-\tilde H_3+\tilde K_1-\tilde K_2\}\ . \end{aligned} \tag{2.3c}$$

III. NUMERICAL RESULTS FOR $d=3$

The distribution of couplings and fields is represented by a Monte Carlo sample of N triplets $\{(K_{ij},H_i,H_j)\}$, initially chosen according to Eq. (2.2). The distribution is renormalized by mapping $\{(K_{ij},H_i,H_j)\}$ into N renormalized triplets $\{(K'_{ij},H'_i,H'_j)\}$ where each renormalized triplet is obtained, using Eq. (2.3), from eight triplets chosen randomly from $\{(K_{ij},H_i,H_j)\}$. We used $N=80\,000$ except in the calculation of the magnetization curve where $N=20\,000$. Note that under renormalization correlations develop among the couplings and fields.

The initial distribution with $p=\frac{1}{2}$ and $h_a=h_b$ is an even function of H, and this symmetry is preserved under the exact renormalization-group transformation. However, since the distribution is represented by a finite sample, fluctuations cause $|\langle H\rangle|$ to grow and ultimately run away to large values. The breaking of inversion symmetry is avoided without modifying the correlations between K and H by reversing the signs of both fields in 50% of the renormalized triplets. This procedure is only used when the initial distribution is symmetric.

At the critical point both the average coupling $\langle K\rangle$ and the standard deviation of the field, $\sigma_H=\langle H^2\rangle^{1/2}$, flow to infinity, confirming the notion that the critical behavior of the RFIM is controlled by a zero-temperature fixed point.[28] In the high-temperature phase $\sigma_H/\langle K\rangle\to\infty$, whereas in the low-temperature phase $\sigma_H/\langle K\rangle\to 0$. The phase boundary is obtained by searching for values of h and k such that under renormalization the ratio approaches a plateau. The fixed-point ratio is found to be $\sigma_H/\langle K\rangle=1.005$. In practice, this value is held for three or four iterations before finite-size fluctuations cause this ratio to run away toward zero or infinity. To study the zero-temperature fixed distribution which controls the critical point, we use the recursion relations starting from the initial phase boundary. After ten iterations both $\langle K\rangle$ and σ_H reach a value of roughly 20, at which point we switch from the exact recursion relations to the zero-temperature recursion relations.

The zero-temperature recursion relations are invariant under an arbitrary scale factor. We take advantage of this and rescale K and H by a factor λ after each renormalization, where λ is chosen to keep σ_H fixed at unity. At the zero-temperature fixed point, the distribution rescaled by λ is invariant under renormalization.

We used the following procedure to obtain the fixed distribution. Starting from the phase boundary, we observed that the average coupling approaches the value 0.995 under the rescaling described above. If the mean value of the couplings deviates from 0.995 under renormalization, then a constant times the deviation is subtracted from each coupling of the ensemble; i.e., the distribution of the couplings is shifted in the opposite direction. The constant we choose here is 0.8. The shifts at each state are of order 10^{-3} with roughly half in each direction. The fixed distributions of couplings and fields are shown in Figs. 2 and 3, respectively. Figure 4 displays the joint distribution of fields and couplings. Each point represents a value of $(H_i+H_j)/2$ and K_{ij} in a Monte Carlo sample representing the fixed distribution.

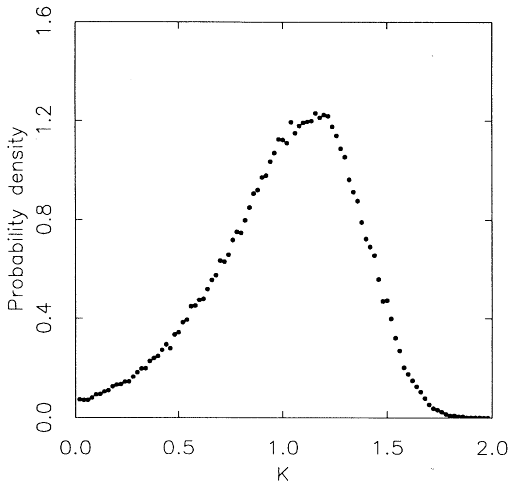

FIG. 2. Fixed distribution for the couplings. The component of the distribution at $K=0$ with weight $p_0=0.015$ is not shown.

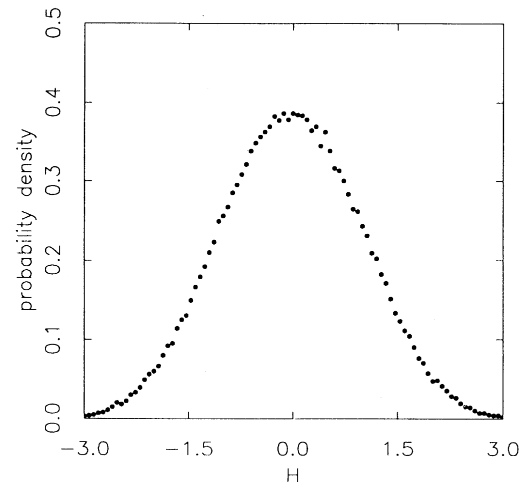

FIG. 3. Fixed distribution for random fields.

Note that small couplings are correlated with large fields and vice versa, reflecting the competition between fields and couplings. Figure 4 also shows that the fixed distribution of the couplings has a nonvanishing component at $K=0$. This component has a weight $p_0=0.015$. The $K=0$ component of the distribution plays a central role in the analysis of Sec. IV.

Having the zero-temperature fixed distribution in hand, we are ready to extract the critical exponents. At a zero-temperature fixed point, there are three independent exponents. Following Bray and Moore,[6] we refer to these as x, y, and ν. The exponent y describes how the width of the distribution grows under renormalization. Using the method described above to hold the distribution near the fixed point, we evaluate the rescaling factor λ many times and then take the mean $\bar{\lambda}$:

$$y=\frac{\ln\bar{\lambda}}{\ln(b)} . \tag{3.1}$$

The value of y is 1.49 ± 0.01. Note $y \leq d/2$, which is the upper bound obtained by Berker and McKay.[29]

The exponent x describes the rescaling of an infinitesimal symmetry-breaking field under renormalization and can be calculated from the definition

$$x=\frac{\ln\langle dH'/dH\rangle}{\ln(b)} , \tag{3.2}$$

where H' is the mean value of the renormalized fields at the two ends of the bond. The average is evaluated at the zero-temperature fixed point, and the derivative is taken with respect to a shift in all fields on the right-hand side of Eq. (2.3c). The value of x is 2.990 ± 0.001, which is less than d ensuring that the magnetization is continuous at the critical temperature.

To compute the correlation-length exponent ν, we use a method similar to that of Ref. 21. After obtaining an ensemble which represents the fixed distribution, we make a second copy of the ensemble and shift each coupling by a small amount $\delta_0=10^{-5}$. The two copies of the distribution are then transformed simultaneously according to the procedure described above so that the original distribution of the couplings and fields is held near the fixed point while the shifted copy flows away. The difference t_n between the values of σ_H/K obtained from the two copies is recorded at each successive iteration n. The correlation-length exponent is defined from the flow away from the fixed distribution:

$$1/\nu=\frac{\ln(t_{n+1}/t_n)}{\ln(b)} . \tag{3.3}$$

The value for $1/\nu$ is 0.45 ± 0.4.

The values of x, y, and ν depend on an accurate representation of the fixed distribution. To check the accuracy of the results, we used the procedure described above to find the fixed distribution except that $\sigma_H/\langle K\rangle$ was held at 0.95 rather than the best value 1.005. Now $\sigma_H/\langle K\rangle$

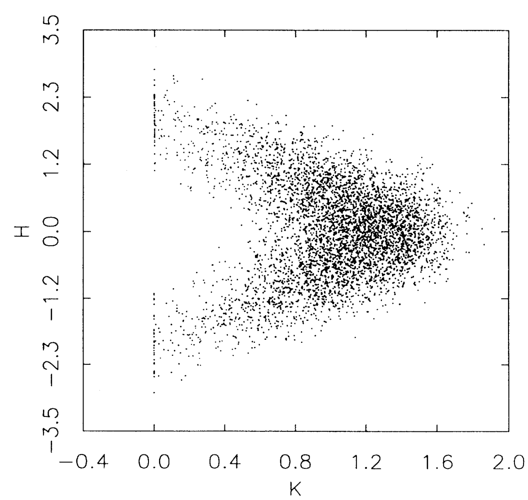

FIG. 4. Fixed distribution of K and $H=(H_i+H_j)/2$ displayed as an ensemble of 5000 points.

always deviates toward smaller values under renormalization and the value of x is shifted to 2.993. When the value of $\sigma_H/\langle K\rangle$ was held at 1.05, the deviations were always toward larger values and the value of x becomes 2.985, and so the correct value of x lies in this range. The exponent y is insensitive to changes of this size.

The critical indices α and β are obtained from the exponent relations[6,7]

$$\alpha=2-(d-y)\nu\ ,\quad \beta=(d-x)\nu\ . \tag{3.4}$$

The results for α, β, and ν are listed in Table I. The MKRG value of the specific-heat exponent is much less than the values obtained by other methods.[6,8,13] MKRG for the pure Ising model also yields a very negative value $\alpha=-1.19$, in contrast to the accepted value $\alpha=0.11$ for Euclidean lattices.[30] Nevertheless, our results suggest that the value of α for the RFIM is less than that for the corresponding pure system. The very small value of β is consistent with the numerical study of Ref. 13 and theoretical expectations.[2]

We computed the magnetization versus temperature for various values of the random field. If the free energy is differentiated with respect to the external field, a relation is obtained between the original magnetization and the renormalized magnetization. This leads to an infinite-product expression for the spontaneous magnetization:[16,22]

$$m=\prod_n b^{-d}\left\langle\frac{dH'}{dH}\right\rangle^{(n)}\ , \tag{3.5}$$

where the superscript n indicates that the average is carried out with respect to the distribution of fields and couplings at the nth iteration of the RG. The mean value of the couplings flows to infinity under renormalization in the ferromagnetic phase, and so we switch to the zero-temperature recursion relations when the fields and couplings become large.

Figure 5 shows the magnetization for several values of h compared with the pure Ising model. The magnetization curves of the RFIM are quite close to that of the pure system except very near the critical temperature. Roughly speaking, the RFIM magnetization curve follows the pure Ising curve until T_c, at which point it falls precipitously to zero. In practice, it would be difficult to obtain the exponent β by fitting the magnetization curve and the evidence for RFIM behavior lies in the truncation of the pure Ising curve. As the strength of the random field becomes small, the size of the "discontinuity" becomes small. We suspect that this effect explains why the power-law fits to the coexistence curves in Ref. 24 yield the pure Ising value of β. It should be noted in this context that the random field may be very weak in aerogel and that the majority of the suppression of T_c in this system may be due to large-scale structure in the field, whose effects will be discussed elsewhere.

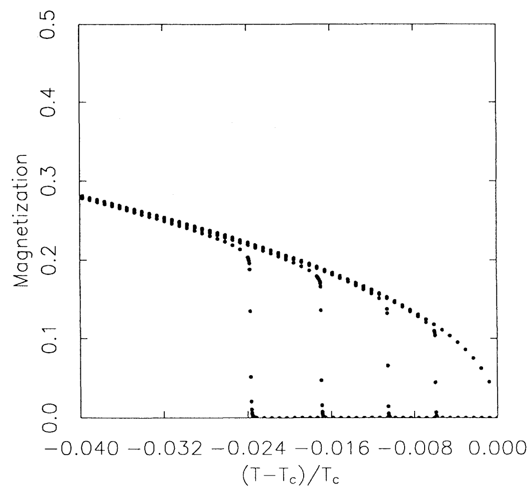

FIG. 5. Magnetization curve for four values of h (0.000 24, 0.000 48, 0.000 84, and 0.0013) compared with the pure Ising curve. T_c is the critical temperature of the pure Ising model.

TABLE I. Critical exponents obtained here for the random-field Ising model in $d=3$ and $d=2+\varepsilon$ in the Migdal-Kadanoff approximation compared to values obtained in other studies.

	ν	α	β
$d=3$	2.25	-1.37	0.02
$d=3^{a}$	1.0	≤ 0.5	0.05
$d=2+\varepsilon$	$2/\varepsilon$	$1-2/\varepsilon$	0
$d=2+\varepsilon^{b}$	$1/\varepsilon$	$\frac{3}{2}-1/\varepsilon$	0

[a]Reference 13.
[b]Reference 6.

In a recent paper, Maritan *et al.*[26] studied an asymmetric RFIM within mean-field theory. Their results suggest that the asymmetric RFIM may be in a different universality class from the symmetric RFIM. We examined the asymmetric case by letting $p\neq 0.5$ and $h_a\neq h_b$ in Eq. (2.2). For all values of p and h_a which we tested, we found that the RG flow at the critical point is always toward the same symmetric, zero-temperature fixed point. We believe that the distinction between the symmetric and asymmetric RFIM is an artifact of mean-field theory.

IV. ANALYTIC ARGUMENTS AND EXACT RESULTS NEAR THE LOWER CRITICAL DIMENSION

The zero-temperature recursion relations [Eqs. (2.3a) and (2.3c)] can be written in the following simpler form:

$$K'=0 \quad \text{if } |\tilde{H}_3|>\tilde{K}_1+\tilde{K}_2\ , \tag{4.1a}$$

$$K'=\tfrac{1}{2}(\tilde{K}_1+\tilde{K}_2-|\tilde{H}_3|) \quad \text{if } \tilde{K}_1+\tilde{K}_2>|\tilde{H}_3|>|\tilde{K}_1-\tilde{K}_2|\ , \tag{4.1b}$$

$$K'=\min\{\tilde{K}_1,\tilde{K}_2\} \quad \text{if } |\tilde{K}_1-\tilde{K}_2|>|\tilde{H}_3|\ , \tag{4.1c}$$

and

$$H'=\tfrac{1}{2}\{\tilde{H}_1+\tilde{H}_2+(\tilde{K}_1+\tilde{K}_2)\operatorname{sgn}(\tilde{H}_3)\} \quad \text{if } |\tilde{H}_3|>\tilde{K}_1+\tilde{K}_2\ , \tag{4.1d}$$

$$H'=\tfrac{1}{2}(\tilde{H}_1+\tilde{H}_2+\tilde{H}_3) \quad \text{if } \tilde{K}_1+\tilde{K}_2>|\tilde{H}_3|\ , \tag{4.1e}$$

where $H'=\frac{1}{2}(H_i'+H_j')$. From these recursion relations and several plausible assumptions, we obtain analytic results for the RFIM on necklace hierarchical structures in $d=2+\varepsilon$ dimensions. The most important assumption is that p_0, the fraction of bonds with $K=0$ at the fixed point, approaches zero faster than any power of ε as $d\to 2$. We will give some justification for this later.

Let us suppose that $p_0=0$; then, Eq. (4.1a) shows that the inequality $\tilde{K}_1+\tilde{K}_2>|\tilde{H}_3|$ always holds and Eq. (4.1e) governs the flow of H. The exponent y is obtained from the rescaling of the fixed distribution under renormalization. Since then H' is the sum of b^d random fields, we have $b^y=\sigma'_H/\sigma_H=d^{d/2}$ or

$$y=d/2=1+\varepsilon/2 \ . \tag{4.2}$$

The exponents x and ν are obtained from the two leading eigenvalues of the RG transformation. If $p_0=0$, then, with probability 1, $dH'/dH=b^d$, $dK'/dK=b^{d-1}$, and $dH'/dK=0$. Thus shifts in the H and K directions are eigenfunctions of the recursions relations with eigenvalues b^d and b^{d-1}, respectively. The flow in the H direction yields x:

$$x=d=2+\varepsilon \ . \tag{4.3}$$

The correlation-length exponent ν is defined in terms of the flow in the K direction due to the recursion relations composed with the rescaling by b^y. Thus

$$1/\nu+y=d-1 \tag{4.4a}$$

or, from Eq. (4.2),

$$\nu=2/\varepsilon \ . \tag{4.4b}$$

These results are exact for the MKRG if $p_0=0$. We have seen that p_0 is very small in $d=3$. We also examined p_0 for the necklace with three parallel bonds ($d=2.58$) and found $p_0\approx 10^{-4}$. This observation lends weight to the hypothesis that $p_0\to 0$ decays faster than any power of ε. Here we give a self-consistent argument that p_0 goes to zero like e^{-C/ε^2} as $d\to 2$. Suppose that p_0 is small and that σ_K/σ_H approaches a finite constant as $d\to 2$. Under these assumptions, $|\tilde{H}_3|>\tilde{K}_1+\tilde{K}_2$ is a rare event and approximate recursion relations for $\langle K\rangle$ and σ_H can be obtained from Eqs. (4.1b), (4.1c), and (4.1e):

$$\langle K\rangle'=b^{d-1}(\langle K\rangle-A\sigma_H) \tag{4.5a}$$

and

$$\sigma'_H=b^{d/2}\sigma_H \ , \tag{4.5b}$$

where A is a constant which depends on the ratio of σ_K/σ_H. Both $\langle K\rangle$ and σ_H diverge, but the zero-temperature fixed point can be located from the recursion relations for the ratio $v=\langle K\rangle/\sigma_H$:

$$v'=b^{\varepsilon/2}(v-A) \ . \tag{4.6}$$

The fixed point is thus at $v^*=2A/\varepsilon$. This result differs from Ref. 6 where $v^*\sim 1/\sqrt{\varepsilon}$. Note that the result $\nu=2/\varepsilon$ follows in the usual way from Eq. (4.6).

If p_0 is small, the fixed distribution for H is nearly a Gaussian of width σ_H since the recursion relations reduce to sums of independent random variables. The distribution for K is more complicated since it is controlled by both Eqs. (4.1b) and (4.1c). Since values of K far from the mean can only be generated by the recursion relations from values of H equally far from the mean, we surmise that the tail of the K distribution decays at least as fast as a Gaussian of width σ_K. The dominant mechanism for generating weight in the distribution at $K=0$ requires that the sum of b^d fields is less than the sum of b^d couplings. The probability for this is controlled by the tails of the distribution, so that

$$p_0\sim e^{-C'v^2}\sim e^{-C/\varepsilon^2} \ , \tag{4.7}$$

with C and C' constants of order unity. Thus we expect that the corrections to Eqs. (4.2), (4.3), and (4.4b) are of order e^{-C/ε^2}. These arguments help to explain why p_0 and β are so small, even for $d=3$. It would be useful, however, to have a stronger argument that p_0 is exponentially small near $d=2$.

Although p_0 is small, it is nonvanishing for $d>2$ as long as the distribution for $|H|$ has no sharp upper cutoff since there is a nonvanishing probability that $|\tilde{H}_3|>\tilde{K}_1+\tilde{K}_2$. The fact that $p_0>0$ implies that the leading behavior of x can be cast as a strict inequality. From the definition Eq. (3.2) and the recursion relations Eqs. (4.1d) and (4.1e), we have

$$b^x=(1-p_0)b^d-p_0b^{d-1} \ , \tag{4.8a}$$

and so

$$x<d \ , \tag{4.8b}$$

which ensures that the magnetization is continuous at the RFIM transition. Previous approximate studies of the RFIM on hierarchical structures[22,23] have yielded results which violate inequality (4.8b).

Renormalized bonds at length scale L with $K=0$ represent islands of spins of size L that are decoupled from the macroscopic phase and controlled by the local random field. At the critical point, such islands appear with a density p_0 at each length scale. The complement of the set of islands or the macroscopic phase is thus a random fractal with a Hausdorf dimension $d_f=d+\ln(1-p_0)/\ln b$.

The RFIM on hierarchical structures near the lower critical dimension behaves in most ways like the RFIM on Euclidean lattices. Table I compares our $2+\varepsilon$ results to the estimates obtained by Bray and Moore.[6] The $2+\varepsilon$ expansions of x and y are the same as that found in Refs. 6, 8, and 9. There is disagreement in the literature concerning the correct ε expansion for ν with Ref. 6 giving $\nu=1/\varepsilon$ and Refs. 8 and 9 giving $\nu=3/2\varepsilon$. The distinction between these answers can be traced to different assumptions about the energy of a domain wall in two dimensions. On the hierarchical structures studied here, the surface energy of a domain wall scales trivially as L^{d-1}, implying the relation $1/\nu+y=d-1$, while Refs. 6, 8, and 9 assume nontrivial, though differing, scaling for the surface energy. The behavior of the domain-wall energy is probably the most important difference between the RFIM on hierarchical structures and on Euclidean lattices and leads to estimates for ν and α within the Migdal-Kadanoff approximation which are probably not very good. In other regards, however, the Migdal-Kadanoff approach yields a surprisingly accurate and illuminating picture of the random-field Ising model.

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