Exactly soluble two-dimensional electron gas in a magnetic-field barrier

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The single-particle energy eigenstates of a two-dimensional electron gas confined to the x-y plane and in the presence of an external-magnetic-field barrier whose functional form is $\mathbf{B}(x,y)=B_0(1-\tanh^2 x/d)\hat{\mathbf{z}}$, with B_0 and d arbitrary, is solved exactly. It is found that the spectrum has bounded and unbounded states. The former are confined to the region where the magnetic field is appreciable. The lowest-lying eigenstates resemble the Landau levels of the constant-field case, but they also drift along the y axis with a speed proportional to the magnetic-field gradient. The unbounded states are extended either on one side of the barrier or on both sides, depending on their energy and asymptotic momenta. It is found that the discrete and continuum spectra overlap in an energy range. It is also argued that these results apply qualitatively to a general class of magnetic-field barriers.

Ever since the remarkable discovery of the quantum Hall effect[1] there has been enormous interest in the behavior of a two-dimensional electron system in the presence of an external homogeneous magnetic field. In addition to this field, other kinds of interactions have been amply studied, such as interelectron interaction, impurities, edge effects, etc. All these terms play an important role in the quantum Hall effect. The important point is that in the presence of the magnetic field these other interactions produce some unusual and remarkable effects.[2]

In the present work, we shall study a somewhat different problem which reveals some interesting physical phenomena and which has recently been under scrutiny.[3,4] This refers to the quantum-mechanical description of a two-dimensional electron gas subject to a magnetic-field barrier normal to the electron plane.

The dynamics of electrons in the presence of nonuniform magnetic fields has not bee studied much in the past, and most of the existing analyses have been based on the semiclassical approximation,[5] which requires that the field varies little in a region of the order of the local electron orbit. In addition, this approximation, though quite accurate for highly excited electron states, is not as good for the lowest-lying energy levels.

In this paper we will consider a special field configuration for which the electron states can be obtained analytically and in closed form. This result will allow us to check the accuracy of the semiclassical approximation, and will also provide us some results for which that method is clearly inadequate. Although we have chosen the functional form of the magnetic field in such a way that the resulting equation can be exactly solved, we can expect that many physical results will qualitatively apply as well to more general situations, where the analytic problem becomes intractable.

The magnetic field we consider is given by

$$\mathbf{B}(x)=\begin{bmatrix} 0 \\ 0 \\ B_0(1-\tanh^2 x/d) \end{bmatrix}, \tag{1}$$

where B_0 and d are arbitrary parameters. We will assume, for simplicity, that the electrons are confined to the x-y plane. The extension to the three dimensional case is straightforward. We note that since $\nabla\times\mathbf{B}\neq 0$, this field configuration cannot exist in free space. Nevertheless, we expect to extract physical consequences of the exact solution which transcend this particular case and apply to a more realistic field of a similar type. As a possible experimental array which would produce such a field, let us imagine that we have the electron plane being shielded by parallel superconducting plates except along a slot in the direction of the x axis, Fig. 1. If a constant magnetic field is applied normal to the plates, it will penetrate it through the slots and eventually produce a field configuration resembling that in Eq. (1).

The vector potential corresponding to this magnetic field is

$$\mathbf{A}(x)=\begin{bmatrix} 0 \\ Bd\tanh x/d \\ 0 \end{bmatrix}, \tag{2}$$

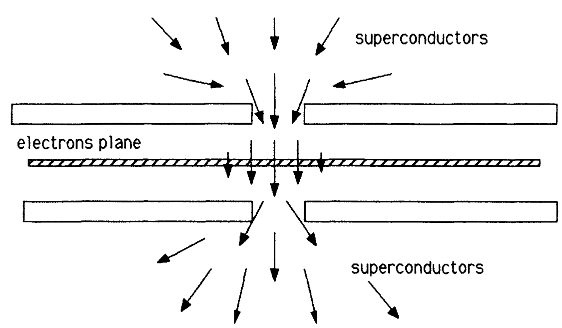

FIG. 1. Sectional view of a device for producing a magnetic wall. The top and bottom plates represent superconductors. The middle plane contains the two-dimensional electron system. The external magnetic field, indicated by arrows, penetrates the superconductors through the slots, creating a magnetic-field barrier on the electron plane.

and the resulting Schrödinger equation is

$$\frac{1}{2m}[\mathbf{P}+(e/c)\mathbf{A}]^2\Psi=E\Psi \ , \tag{3}$$

where $-e$ is the electron charge. The Hamiltonian is independent of the y coordinate and, consequently, p_y is conserved. The wave functions can be written as $\Psi(x,y)=\phi(x)\exp(ip_y y/\hbar)$, with $\phi(x)$ satisfying

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2m}\left[p_y+\frac{eB_0 d}{c}\tanh(x/d)\right]^2\right]\phi(x)=E\phi(x) \ . \tag{4}$$

It is convenient to convert this equation to dimensionless form by dividing it by $\hbar^2/2md^2$ and introducing the dimensionless variable $x/d=z$. Then Eq. (4) becomes

$$\left[-\frac{d^2}{dx^2}+\alpha^2[\tanh z+(p_y/p_0)]^2\right]\phi(z)=\epsilon\phi(z) \ , \tag{5}$$

with $p_0=eB_0 d/c$, $\alpha=p_0 d/\hbar$, and $\epsilon=2md^2E/\hbar^2$. This eigenvalue equation arises in other contexts and its solution is well known.[6] The details of this solution are discussed in Ref. 6 and, therefore, we will only reproduce the main ideas here. Equation (5) can be converted into a hypergeometric differential equation by introducing, in the first place, a new dependent variable $F(z)$ defined by

$$\phi(z)=e^{-az}\text{sech}^b z F(z) \ , \tag{6}$$

and then changing the independent variable z as follows:

$$u=(1-\tanh z)/2 \ . \tag{7}$$

The resulting function $\tilde{F}(u)=F(z)$ satisfies

$$u(1-u)\frac{d^2\tilde{F}}{du^2}+[a+b+1-2(b+1)u]\frac{d\tilde{F}}{du}+[\alpha^2-b(b+1)]\tilde{F}=0 \ ,$$

with

$$(b\pm a)^2=\alpha^2(1\pm p_y/p_0)^2-\epsilon \ . \tag{8}$$

It is evident from Eq. (5) that a necessary condition for bound states is that $|p_y/p_0|<1$. We will first consider bound states and denote $p_y/p_0=\tanh z_0<1$. The parameter $-z_0$ corresponds to the minimum of the "potential" term in Eq. (5). The requirement that the solution of Eq. (5) be finite in the limit $z\to+\infty$ implies that the solution of Eq. (8) is the hypergeometric function

$$F\{b+\tfrac{1}{2}-[\alpha^2+\tfrac{1}{4}]^{1/2}, b+\tfrac{1}{2}+[\alpha^2+\tfrac{1}{4}]^{1/2} \,|\, a+b+1 \,|\, e^{-z}/(e^z+e^{-z})\} \ , \tag{9}$$

where a and b are given by Eq. (8) with $a+b>0$. In the limit $z\to-\infty$, the corresponding solution of Eq. (5) has the following asymptotic behavior:[6]

$$\frac{\Gamma(b-a)e^{(a-b)z}}{\Gamma[b+\frac{1}{2}+(\alpha^2+\frac{1}{4})^{1/2}]\Gamma[b+\frac{1}{2}-(\alpha^2+\frac{1}{4})^{1/2}]}+\frac{\Gamma(a-b)e^{(b-a)z}}{\Gamma[a+\frac{1}{2}-(\alpha^2+\frac{1}{4})^{1/2}]\Gamma[a+\frac{1}{2}+(\alpha^2+\frac{1}{4})^{1/2}]} \ . \tag{10}$$

For bounded solutions we must have $\epsilon<[\alpha(1\pm\tanh z_0)]^2$ and, consequently, a and b are real. With no loss of generality, let us assume $b>a$ and $z_0>0$.

The condition for finiteness of the solution at $z_0\to-\infty$ requires

$$b+\tfrac{1}{2}-[\alpha^2+\tfrac{1}{4}]^{1/2}=-n \ , \tag{11}$$

where $n=0,1,2,\ldots$, so that the divergent term in Eq. (10) is absent. This condition implies that for each n, the parameters a and b are discretized:

$$b=b_n=[\alpha^2+\tfrac{1}{4}]^{1/2}-(n+\tfrac{1}{2}) \ , \quad a=a_n=(\alpha^2\tanh z_0)/b_n \ . \tag{12}$$

The permitted energy eigenvalues are then

$$\epsilon_n=\alpha^2(1+\tanh z_0)^2-(a_n+b_n)^2 \tag{13}$$

for all integers n such that $b_n>a_n$. This last condition imposes a constraint on α, z_0, and n. Using Eq. (12) one readily derives that

$$[\alpha^2+1/4]^{1/2}-\tfrac{1}{2}-\alpha\tanh^{1/2}z_0>n\geq 0 \ . \tag{14}$$

The solution for the case $z_0<0$ yields the same results as above, indicating that the energy eigenstates are even functions of $\tanh z_0$. We will be interested in the $\alpha\gg 1$ limit. In fact, we can rewrite α as follows: $\alpha=eB_0d^2/\hbar c=\pi(e/hc)(2B_0d^2)=\pi\Phi/\Phi_0$, where $\Phi_0=hc/e$ is the quantum of flux and Φ is the total flux on a length d of the magnetic barrier along the y axis. (If we chose $d=0.1$ cm and $B_0=10$ G, then $\alpha\simeq 10^5$). Under this condition we can derive asymptotic expressions for the energy levels. In the first place, the condition for the maximal eigenvalue, consistent with Eq. (14), becomes

$$\alpha(1-\tanh^{1/2}z_0)>n_{\max} \ . \tag{15}$$

Thus we conclude that for $z_0=0$ the maximum number of states is $\simeq\alpha$. For large z_0 only one state will be permitted with $n=0$, and provided

$$1-\tanh z_0>1/\alpha-1/2\alpha^2 \ . \tag{16}$$

The expression for energy eigenvalues is

$$\epsilon_n=[2\alpha(n+1/2)+2(n+\tfrac{1}{2})^2](1-\tanh^2 z_0)-4(n+1/2)^2+\cdots \ , \tag{17}$$

where, in addition, we assumed $n\ll\alpha$, and deleted terms are of order α^{-1}. In terms of the original variables this is

$$E_n=\hbar\omega_c(z_0)(n+\tfrac{1}{2})-(\hbar^2/md^2)(n+\tfrac{1}{2})^2\times(1+\tanh^2 z_0) \ . \tag{18}$$

The first term in Eq. (18) is just the Landau formula adapted to the local value of the magnetic field $B(z_0)$, where the orbit is centered. This result was derived using the adiabatic approximation in Ref. 4. We note that corrections are of order α^{-1}. For large values of $n \simeq \alpha$ the correction terms become important. For example, in the case $z_0=0$, $n \gg 1$, we find

$$E_h \simeq \hbar\omega_c(0)(n+\tfrac{1}{2})-(\hbar^2/2md^2)(n+\tfrac{1}{2})^2 \ . \tag{19}$$

A classical charged particle moving in a inhomogeneous magnetic field is known to drift[7] in a direction perpendicular to the local values of the magnetic field and its gradient, according to the following expression:

$$v_{\text{drift}}=cE|\nabla B|/eB^2 \ , \tag{20}$$

where E is the particle energy. Let us derive a similar quantum-mechanical expression. According to the classical result we expect to find a drift velocity along the y axis. The corresponding expression is

$$v_y=\frac{\partial E_n}{\partial p_y}=(1/p_0)\frac{\partial E_n}{\partial(\tanh z_0)}$$
$$=(\hbar^2 c/2meB_0d^3)\frac{\partial \epsilon_n}{\partial(\tanh z_0)} \ . \tag{21}$$

Using Eq. (18) we obtain

$$v_y=[cE_n(z_0)/eB^2(z_0)]\frac{\partial B(z_0)}{\partial x}$$
$$+[2\hbar^2 c(n+\tfrac{1}{2})^2/2med^2B^2(z_0)]$$
$$\times\frac{\partial B(z_0)}{\partial x}(1-\tanh^2 z_0)+\cdots \ , \tag{22}$$

where deleted terms are of order α^{-1}. Thus we find that the leading term coincides with the classical expression and the second one is a quantum correction.

We can easily check that for each state centered at $-z_0$, there will be another state of the same energy, centered at z_0, having a drift velocity of the same strength but opposite direction. Therefore, the total electric current along the y direction of a full Landau level will vanish. It is interesting, though, to evaluate the net current flowing in the range $x<0$. If we assume that the system is a strip of width L in the y direction, and that the wave functions satisfy periodic boundary conditions, then $p_yL=2\hbar N$, with $N=0,\pm1,\pm2,\ldots$. The number of states in the range dp_y is then

$$(2\pi\hbar)^{-1}dp_y=(2\pi\hbar)^{-1}p_0 d(\tanh z_0) \ . \tag{23}$$

Thus the current becomes

$$I_n=-e\int_0^1 v_y^n(\tanh z_0)p_0(2\pi\hbar)^{-1}d(\tanh z_0)=-ep_0(2\pi h)^{-1}\int_0^1 p_0^{-1}\frac{\partial E_m}{\partial(\tanh z_0)}d(\tanh z_0)$$
$$=-e(2\pi h)^{-1}E_n(\tanh z_0)|_0^1$$
$$=(e^2B_0/4\pi mc)[2(n+\tfrac{1}{2})-(1/\alpha)][(n+\tfrac{1}{2})^2+\tfrac{1}{4}]+O(1/\alpha^2) \ . \tag{24}$$

Therefore if we ignore terms of order α^{-1}, and sum the contribution of a set of Landau levels up to some $n_{\max}$, we obtain the following expression:

$$\frac{1}{B_0}\sum_{n=0}^{n_{\max}} I_n=I/B_0=(e^2/4\pi mc)(n_{\max}+1)^2 \ . \tag{25}$$

This relationship is similar to the integer quantum Hall resistance.[8] It relates the ratio of the total current to the difference of the asymptotic values of the magnetic field to universal constants times a function of the filling factors. Presumably, in a real system such as a heterostructure like GaAs-$Al_xGa_{1-x}As$, a plateau type of behavior could be displayed as a function of the electron density.[4]

Let us next consider unbounded states. For $|z|\gg1$ the particles will be asymptotically free and the eigenstates can be properly characterized by the asymptotic values of the kinetic momenta π_x and $\pi_y=p_y+p_0\tanh z|_{z=\pm\infty}=p_y\pm p_0$. Let us denote by $+/-$ the momenta at the far left/right of the magnetic barrier. Clearly $\pi_x^{-2}+\pi_y^{-2}=\pi_x^{+2}+\pi_y^{+2}=2mE=(mv)^2$. If we start with a particle at the far left approaching the barrier, classically two things could happen. Either the particle crosses the barrier or bounces back. We easily establish the conditions for these events. At the far left, we have $\pi_x^-(\text{in})$, $\pi_y^-(\text{in})$. For $z\to\infty$ we must have

$$\pi_y^+(\text{out})=\pi_y^-(\text{in})+2p_0 \ ,$$
$$\pi_x^+(\text{out})=[(mv)^2-\pi_y^+(\text{out})]^{1/2} \ . \tag{26}$$

The kinematics is best seen graphically, see Fig. 2. Thus we immediately conclude that the condition for a bounce is $(mv)^2<[\pi_y^-+2p_0]^2$. Moreover, a simple relationship between the angle of incidence θ_{in} and scattering θ_{out} can be established. The resulting expression is

$$\sin\theta_{\text{out}}^+-\sin\theta_{\text{in}}^-=2p_0/mv. \tag{27}$$

It is important to note that because of the lack of time-reversal invariance, these relationships do not apply for a particle incident from the far right. Nevertheless the corresponding equations are easily derived observing that in this case

$$\pi_y^-(\text{out})=\pi_y^+(\text{in})-2p_0 \ ,$$
$$\pi_x^-(\text{out})=[(mv)^2-\pi_y^-(\text{out})^2]^{1/2} \ . \tag{28}$$

It is interesting to note that if $d=0.1$ cm then $p_0=B_0$ (G) 10^{-20}. On the other hand the typical Fermi momentum is 10^{-18} gr cm/sec. Thus if $B_0\simeq10^4$ (G), $p_0\gg p_f$, but for $B_0\simeq10$ (G), $p_0\ll p_f$. So changing the field within this range produces a current switch.

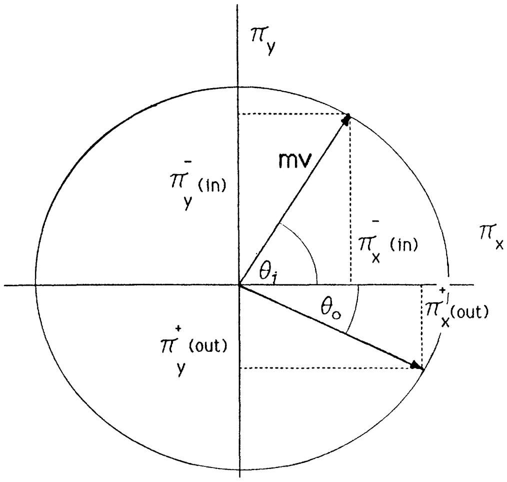

FIG. 2. Geometrical construction of the kinematics of a particle crossing the magnetic barrier. The radius of the circle is m times the speed of the particle, which is assumed to be incident from the far left. The axes are the x and components of the kinetic momenta and θ_i, θ_0 represent the asymptotic incoming and outgoing angles. The transformation laws for $\pi_{x,y}^{\pm}$ are given by Eq. (26).

Let us next obtain the wave function of a bouncing solution incident from the left with asymptotic momenta π_x^- and π_y^-. Using Eq. (8) we derive

$$a+b=[2md^2/\hbar^2]^{1/2}(\pi_y^{\pm 2}/2m-E)^{1/2}\ ,$$
$$ab=(\pi_y^-+p_0)p_0d^2/\hbar^2\ . \tag{29}$$

Thus we see that if $\pi_y^{-2}/2m<E<\pi_y^{+2}/2m$, $a+b$ is real and $b-a$ is imaginary. Let us write $b-a=i\pi_x^- d/\hbar$ and $b+a=\kappa d/\hbar$. Let $\gamma=[\alpha^2+\frac{1}{4}]^{1/2}$, then the wave function becomes

$$\phi(x)=\{e^{-(x/2\hbar)(\kappa-i\pi_x^-)}/(e^{x/d}+e^{-x/d})^{d(\kappa+i\pi_x^-)}\}$$
$$\times F[b+\tfrac{1}{2}-\gamma,b+\tfrac{1}{2}+\gamma\ |d/h+1|$$
$$e^{-x/d}/(e^{x/d}+e^{-x/d})]\ . \tag{30}$$

For large x/d, this function decays as $\exp(-\kappa x/d)$. For $x\to-\infty$ one obtains

$$\phi\simeq\left[\frac{\Gamma(i\pi_x^- d/\hbar)e^{i\pi_x^- x/\hbar}}{\Gamma(b+\frac{1}{2}-\gamma)\Gamma(b+\frac{1}{2}+\gamma)}+\text{c.c.}\right]\ . \tag{31}$$

So we conclude that these states represents a plane wave traveling towards the magnetic wall plus a reflected wave with the same squared amplitude. Since π_y is unchanged, we see that the particles are reflected specularly by the field.

There is one aspect of the spectrum of our solutions which is worth noting. We saw that the energy of the bouncing states forms a continuum starting at $E=0$. On the other hand, we found that in the energy range $0<E<p_0^2/2m$, there is a collection of bounded states with a discrete energy spectrum given by Eq. (13), which overlaps with the continuum. Hence, the density of states as a function of E will have an overlapping contribution from bounded and extended states. See Fig. 3. Although this phenomena is known to exist for Schrödinger equations with rather pathological potentials,[9] an exact analytic expression of this effect has, to our knowledge, not been reported in the literature before.

Another unusual aspect of this spectrum is the fact that there will be a continuum of bouncing states on either side of the magnetic wall in the range $E<p_0^2/2m$, which do not mix due to tunneling, in spite of the fact that the barrier has finite 'width and height." This property is a consequence of the asymptotic behavior of the vector potential that cannot be gauged away in the limits $x\to\pm\infty$ simultaneously.

We will next consider extended states of particles crossing the magnetic barrier. Let us consider a state which is incident from the far left, having asymptotic momenta $\pi_x^-(\text{in})$, $\pi_y^-(\text{in})$. The condition for crossing the barrier is $\pi_x^-(\text{in})^2>4p_0[\pi_y^-(\text{in})+p_0]$. We shall see below that the incident wave is partially transmitted and partially reflected. In this case Eq. (29) yields a and b purely imaginary:

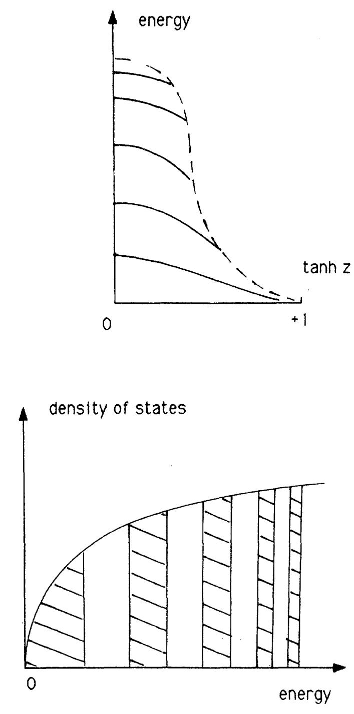

FIG. 3. Energy spectrum of the bounded states as a function of the parameter z_0 is shown schematically on the top graph. We have assumed that a maximum of five levels is permitted; see Eq. (15). The bottom graph represents the total density of states as a function of the energy. For the continuum spectrum it increases as $E^{1/2}$, as is usual for a free particle. The hatched columns represent the overlapping with the bounded states.

$$a = i[\pi_x^+(\text{out}) - \pi_x^-(\text{in})]d/2\hbar ,$$
$$b = i[\pi_x^+(\text{out}) + \pi_x^-(\text{in})]d/2\hbar . \quad (32)$$

Then the resulting wave function becomes

$$\phi = \frac{e^{-i[\pi_x^+(\text{out})-\pi_x^-(\text{in})]x/2\hbar}}{(e^{x/d}+e^{-x/d})^{i[\pi_x^+(\text{out})+\pi_x^-(\text{in})]d/2\hbar}} \times F[b+\tfrac{1}{2}-\gamma, b+\tfrac{1}{2}+\gamma \, | 1+i\pi_x^+(\text{out})d/\hbar | \, e^{-x/d}/(e^{x/d}+e^{-x/d})] . \quad (33)$$

The asymptotic form of this function as $x \to +\infty$ is $\exp[ix\pi_x^+(\text{out})/\hbar]$. For $x \to -\infty$ we obtain

$$\phi \simeq \left[\frac{\Gamma[id\pi_x^-(\text{in})/\hbar] e^{i\pi_x^-(\text{in})x/\hbar}}{\Gamma(b+\frac{1}{2}-\gamma)\Gamma(b+\frac{1}{2}+\gamma)} + \frac{\Gamma[-id\pi_x^-(\text{in})/\hbar] e^{-i\pi_x^- x/\hbar}}{\Gamma(a+\frac{1}{2}-\gamma)\Gamma(a+\frac{1}{2}+\gamma)} \right] . \quad (34)$$

It is easy then to derive the expressions for the reflection and transmission coefficients. From Eq. (34) we obtain

$$R = |\Gamma(b+\tfrac{1}{2}+\gamma)\Gamma(b+\tfrac{1}{2}-\gamma)/\Gamma(a+\tfrac{1}{2}+\gamma) \times \Gamma(a+\tfrac{1}{2}-\gamma)|^2 . \quad (35)$$

Using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, we obtain after some straightforward algebra the following expression:

$$R = \frac{\cosh\left[\frac{\pi d}{\hbar}\right][\pi_x^+(\text{out}) - \pi_x^-(\text{in})] + \cos 2\pi\gamma}{\cosh\left[\frac{\pi d}{\hbar}\right][\pi_x^+(\text{out}) + \pi_x^-(\text{in})] + \cos 2\pi\gamma} . \quad (36)$$

The transmission coefficient is obtained from the relationship $T = 1 - R$ and yields

$$T = \frac{2\sinh\left[\frac{\pi d}{\hbar}\right]\pi_x^+(\text{out})\sinh\left[\frac{\pi d}{\hbar}\right]\pi_x^-(\text{in})}{\cosh\left[\frac{\pi d}{\hbar}\right][\pi_x^+(\text{out}) + \pi_x^-(\text{in})] + \cos 2\pi\gamma} . \quad (37)$$

Let us consider the special case $\pi_y^- = 0$, π_x^- fixed, and $\alpha \gg 1$. If $\pi_x^-(\text{in})d/\hbar \gg 1$ and $\pi_x^+(\text{out})d/\hbar \gg 1$, the transmission coefficient can be approximated by

$$T = 1 - (\exp\{-[\pi_x^+(\text{out}) + \pi_x^-(\text{in})]\pi d/\hbar\})\cos 2\pi\alpha . \quad (38)$$

Thus we see that the transmission coefficient has a small oscillatory behavior as a function of α.

Let us finally note that the results we obtained above can be expected to hold qualitatively in more general cases where the energy spectrum cannot be solved in closed form. We expect that for any one-dimensional magnetic-field barrier converging rapidly to 0 in the perpendicular direction, and having a characteristic width d and field strength B_0, will exhibit properties similar to our solution. In addition, it is easy to see that the electron spin can be included in the exact solution.

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