Fluctuation spectrum of the electron states in a two-dimensional system with a Gaussian potential and a transverse magnetic field

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(Received 7 December 1992; revised manuscript received 7 May 1993)

The results of the investigation of the fluctuation region of the electron spectrum in a two-dimensional system with a Gaussian white-noise potential and a perpendicular magnetic field are presented. We use the optimal fluctuation method and calculate both the density of states and the localization length in the whole fluctuation region below the lowest Landau level. We describe all possible structures of the quantum fluctuation region and establish limits on the applicability of the white-noise-potential model as well as on the approximation based upon the projection onto the lowest Landau level.

I. INTRODUCTION

The characteristics of the energy spectrum of a two-dimensional electron gas in a magnetic field and interacting with the static random potential is a problem of central interest in the realm of the physics of disordered systems. Like many other problems in this category, it is natural, first of all, to investigate it in some certain domains of the energy spectrum where it has more or less a simple and transparent structure. In the present work we concentrate on the so-called "fluctuation region," where the existence of the spectrum is due entirely to fluctuations of the random potential which occur with very small probability.[1-4] The density of states $\rho(E)$ is extremely small in this domain, forming the so-called "Lifshits tail." Roughly speaking, it is best represented by an exponential law

$$\rho(E) \propto \exp[-\Phi(E)] ,$$

where $\Phi(E) \gg 1$ is a relatively simple function of the energy and the space dimension d. For that reason it is the exponent function $\Phi(E)$ that is the main subject of interest in this portion of the spectrum.

To explain what region of the spectrum we have in mind, let us first consider the case in which there is no disordered potential. We denote the magnetic field by B, and the effective mass of the electron by m^*, and adopt units such that $\hbar = 2m^* = 1$ so that energy and magnetic field are given in units of inverse length square. The energy spectrum of the electron consists of the set of Landau levels $E_n = (2n+1)L^{-2}$, $n = 0, 1, \ldots$, where $L = \sqrt{\hbar c/(|e|B)}$ is the magnetic length. The effect of disorder is to broaden this spectrum from discrete levels to bands. If the magnetic field is much stronger than the disorder then the concept of Landau levels still makes sense (see, e.g., a recent paper[5]), and the fluctuation regions of the spectrum are located on both sides of each Landau level. On the other hand, we want to discuss also the situation where the magnetic field is not so strong. In that case, the fluctuation region is defined only below the lowest Landau level. This is the domain of the spectrum on which we will focus our attention, since it is the only part which contains a fluctuation region for any value of the magnetic field.

The study of the energy spectrum below the lowest band dates back to the middle 1970s.[6] In these works, the authors concentrated on the limit of strong magnetic field. They modeled the disorder by a Gaussian potential $V(\mathbf{R})$ with zero mean value and a correlation function $W(\mathbf{R}-\mathbf{R}') = \langle V(\mathbf{R})V(\mathbf{R}') \rangle$ with finite correlation radius R_c. In a later paper,[7] the limit of Gaussian white-noise potential

$$W(\mathbf{R}-\mathbf{R}') = W_0 \delta(\mathbf{R}-\mathbf{R}') ; \quad W_0 = \int W(\mathbf{R}) d\mathbf{R} \sim W(0) R_c^2 \tag{1}$$

was used to calculate the density of states within the fluctuation approach. Notice that this potential corresponds to the situation where the correlation radius is the smallest length parameter in the problem. The technique is based on a standard generalization of the variational procedure for white-noise potential to the case where a magnetic field is also present. Thereby, the authors obtained a modified nonlinear Schrödinger equation for the optimal wave function. In the far part of the fluctuation region, where $E_0 \ll |E|$, this equation was solved by treating the term containing the magnetic field as a small perturbation. The result for the exponential function of the density of states took the form

$$\Phi(E) = -11.6 \frac{E}{W_0} (1 + 0.64 E_0^2/E^2) . \tag{2}$$

In the nearest vicinity of the lowest Landau level a direct variational method was applied. The trial function was taken to be proportional to the s-wave function of the lowest Landau level, and the proportionality coefficient was treated as a variational parameter. The corresponding expression for the exponential function of the density of states for this part of a spectrum was obtained in the form[7]

$$\Phi(E) \approx 2\pi \frac{(E-E_0)^2}{E_0 W_0} . \tag{3}$$

Later on this result was confirmed in a paper[8] (see also Refs. 9 and 10) where the density of states of an electron in a magnetic field and the white-noise random potential was calculated exactly under the provision that only the states of free electrons in the lowest Landau level are taken into account. The exact solution of Ref. 8 agrees with the expression given by Eq. (3) in the limiting case, when $|E-E_0|^2 \gg E_0 W_0$. Thus, it would be an excellent approximation if the broadening of the Landau levels due to the random potential is small in comparison with the separation between adjacent Landau levels, and when we are considering the energy region close enough to the lowest Landau level.

However, to the best of our knowledge, there is no full and complete picture of the electron density of states in the fluctuation region below the lowest Landau level. As we have just discussed, the existing results are valid only in two extreme asymptotic regions, and correspond only to some parts of the fluctuation region. Besides, the range of parameters for which they are valid is not clear enough. Moreover, the limits of applicability of the two models themselves (the white-noise potential and the projection on the lowest Landau level) are also not known. This is the motivation for our investigation.

In the present work we use the optimal fluctuation method, and study numerically the exponent function of the density of states pertaining to an electron in a magnetic field and the white noise random potential, in the whole fluctuation region below the lowest Landau level ($E<E_0$). The limit of applicability of the present (fluctuation) approach for different values of the energy is determined only by the condition $\Phi(E)\gg 1$. Within the fluctuation approach itself all the results which we obtain below are exact. There is, however, a second factor that may limit the validity of our results, namely the use of the white-noise model, which defines the so-called quantum part of the fluctuation region. The possibility of replacing a given random potential by a Gaussian white noise depends strongly on the magnetic field and the correlation radius. We obtain below the condition for the validity of such a replacement and analyze the general structure of the Gaussian fluctuation region. These results are illustrated in the form of a "phase diagram" in the (R_c, E_0) plane.

II. FORMULATION OF THE PROBLEM

Within the fluctuation approach the exponential function $\Phi(E)$ of the electron density of states for the potential (1) in the pertinent region is given by[4(a)]

$$\Phi(E)=\min_{\Psi}\frac{\left[\int \Psi(\mathbf{R})(E-T)\Psi(\mathbf{R})d\mathbf{R}\right]^2}{2W_0\int \Psi^4(\mathbf{R})d\mathbf{R}} .$$

Here $T=[\mathbf{p}-(e/c)\mathbf{A}]^2$ is the kinetic-energy operator expressed in terms of the momentum $\mathbf{p}$ and the vector potential $\mathbf{A}$. The minimum on the right-hand side of the expression for $\Phi(E)$ has to be taken on a set of monotonically decreasing radially symmetric functions $\Psi(R)$. It is useful at this point to introduce a dimensionless energy variable $\varepsilon=1-E/E_0$ which is positive within the pertinent region of the spectrum ($E<E_0$), and an energy-dependent dimensionless radial coordinate $r=R/L\sqrt{\gamma(\varepsilon)}$. The function $\gamma(\varepsilon)$ can be considered here as a parameter, and will be determined within the variational calculus. Performing the minimization procedure we may write the exponent as

$$\Phi(E)=\frac{E_0}{W_0}I(\varepsilon) , \tag{4}$$

$$I(\varepsilon)=\frac{4\pi}{\gamma(\varepsilon)}\int_0^\infty \Psi^4(r) r\,dr . \tag{5}$$

The "optimal" wave function $\Psi(r,\varepsilon)$ satisfies the nonlinear Schrödinger equation

$$-\frac{1}{r}\frac{d}{dr}\left[r\frac{d\Psi(r)}{dr}\right]-2\Psi^3(r)+\left[\frac{\gamma^2(\varepsilon)}{4}r^2-\gamma(\varepsilon)\right]\Psi(r)=-\varepsilon\gamma(\varepsilon)\Psi(r) , \tag{6}$$

with the initial conditions

$$\Psi(0)=1 , \quad \Psi'(0)=0 . \tag{7}$$

The unknown function $\gamma(\varepsilon)$ [beside the wave function $\Psi(r)$ itself] has to be found from the additional condition that $\Psi(r)$ should decrease monotonically to zero as r tends to infinity.

We would like to emphasize that in the fluctuation region the energy itself is asymptotically a good quantum number,[3] and the optimal wave function is asymptotically a true eigenfunction. Therefore we can obtain the localization length $\xi(E)$ from the solution of Eq. (6) in the quantum fluctuation region,

$$\xi(E)=Lg(\varepsilon) , \quad g(\varepsilon)=\sqrt{\gamma(\varepsilon)}f(\varepsilon) ,$$
$$f^2(\varepsilon)=\int_0^\infty \Psi^2(r)r^3dr \Big/ \int_0^\infty \Psi^2(r)r\,dr . \tag{8}$$

In the next section we solve Eq. (6) numerically without any additional assumptions or approximations, and obtain the exponent $\Phi(E)$ as well as the localization length $\xi(E)$ in the whole quantum fluctuation region.

III. RESULTS AND DISCUSSION

The choice of an appropriate normalization (7) for the wave function $\Psi(r)$ enables us to reduce the problem to the solution of an ordinary differential equation (6) with initial conditions as in Eq. (7). However, the procedure of the numerical solution of Eq. (6) has some peculiarities. To understand its nature we first consider the simpler case where the magnetic field is absent. It corresponds to the limit $\gamma(\varepsilon)\to 0, \varepsilon\to\infty, \varepsilon\gamma(\varepsilon)\to\varepsilon_0=\text{const}$. In the present two-dimensional case, there is an infinite set of solutions $\{\Psi_i(r),\varepsilon_i<0\}$ corresponding to exponentially localized functions $\Psi_i(r)$, but only one of them $[\Psi_0(r),\varepsilon_0]$ corresponds to the positive and monotonically decreasing solution which we are looking for.[11] Thus, in the absence of the magnetic field, only one "nonlinear eigenvalue" ε_0 has to be found (according to Ref. 3, $\varepsilon_0=0.41$). The situation becomes much more difficult in the presence of a

magnetic field. It is now necessary to obtain the function $\gamma(\varepsilon)$ for as many values of ε as possible.

However, at least in the deepest tail of the density of states, where $|E| \gg E_0$ or $\varepsilon \gg 1$, the magnetic field is effectively small and the electron almost does not feel it. Therefore to first order in E_0/E the exponent function of the density of states depends very weakly on E_0, and hence the first term on the right-hand side of Eq. (2) should be an excellent approximation. The only way to achieve it in a self-consistent manner is to require that, asymptotically as ε tends to infinity, the function $I(\varepsilon)$ defined in Eq. (5) will be linear in ε with an appropriate coefficient

$$I(\varepsilon) \rightarrow 11.6\varepsilon \ .$$

In the opposite limiting case (in the closest vicinity of the lowest Landau level), $E_0 - E \ll E_0$ or $\varepsilon \rightarrow 0$, the electron does not feel the higher Landau levels and $\Phi(E)$ has to coincide with its asymptotic expression (3). The only way by which it can happen is that the asymptotic form of the integral (4) for $I(\varepsilon)$ (as $\varepsilon \rightarrow 0$) will be

$$I(\varepsilon) \rightarrow 2\pi\varepsilon^2 \ .$$

Thus we can use both of the limiting cases (namely $\varepsilon \rightarrow 0$ and $\varepsilon \rightarrow \infty$) as checks for the numerical procedure.

The results of our calculations are presented in Figs. 1–3.

The functions $I(\varepsilon)/\varepsilon$ for $10^{-1} < \varepsilon < 10^7$ and $I(\varepsilon)/\varepsilon^2$ for $10^{-2} < \varepsilon < 10^{-1}$ are plotted in Figs. 1 and 2. The function $I(\varepsilon)$ may be replaced for rough estimates by two simple functions

$$I(\varepsilon) = \begin{cases} 10\varepsilon^2 & \varepsilon < 1 \quad \text{(near subregion)} \\ 10\varepsilon & \varepsilon > 1 \quad \text{(far subregion)} \ . \end{cases} \tag{9}$$

In both limiting cases $\varepsilon \rightarrow 0$ and $\varepsilon \rightarrow \infty$ the results for the exponent of the density of states coincide perfectly with the ones obtained earlier, except that we found $C \approx 11.7$ in Eq. (2) instead of $C \approx 11.8$ (Ref. 3) and $C \approx 11.6$ (Ref. 8). Nevertheless we note that our numerical result for exponent $\Phi(E)$ in the vicinity of the lowest

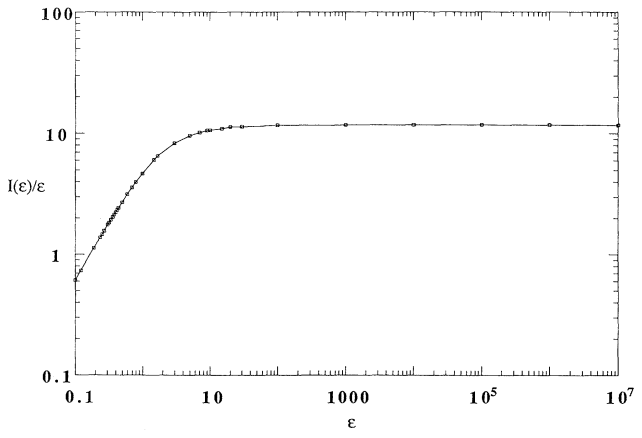

FIG. 1. The function $I(\varepsilon)/\varepsilon = \Phi(E)W_0/(E_0 - E)$ [see Eq. (5)] in the intermediate and far parts of the fluctuation region $(10^{-1} < \varepsilon < 10^7)$.

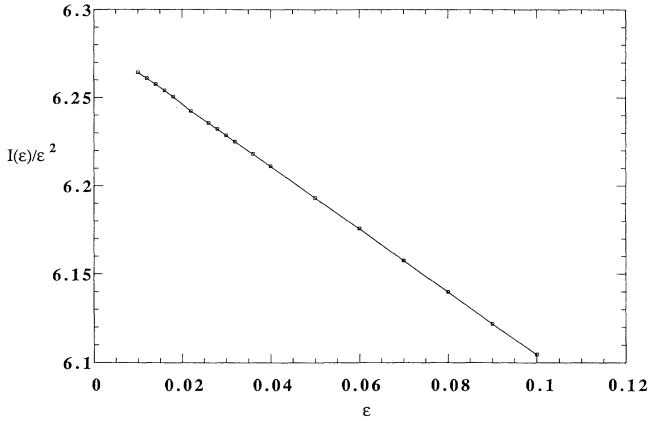

FIG. 2. The function $I(\varepsilon)/\varepsilon^2 = \Phi(E)W_0E_0/(E_0 - E)^2$ [see Eq. (5)] in the nearest part of the fluctuation region $10^{-2} < \varepsilon < 10^{-1}$.

Landau level is always somewhat less than one given in Eq. (3) and predicted in papers.[7,8] This difference is not surprising, and it is due to the inclusion of higher Landau levels. In the special case where the disorder is very weak ($W_0 \ll E_0$), this difference may be crucial.

The dimensionless localization length $g(\varepsilon)$ defined in Eq. (8) is depicted in Fig. 3. In the interval $1 < \varepsilon < 10$,[7] the localization length (up to a factor of order unity) can easily be approximated by the relation

$$\xi(E) \propto \frac{L}{\left[1 - \dfrac{E}{E_0}\right]^{1/2}} \cong |E|^{-1/2} \ .$$

For small ε (say $0.01 < \varepsilon < 1.7$, namely close to the lowest Landau level from below) we have $g(\varepsilon) \sim 1$, hence the localization length coincides with the magnetic length, namely

$$\xi(E) \sim L \ .$$

This result shows that for the Gaussian white-noise potential the localization length is always finite in fluctuation region. Therefore the divergence of localization length may occur only in the nearest vicinity of the Landau level.[12]

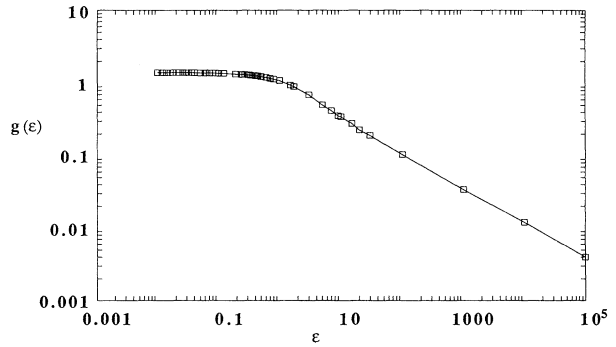

FIG. 3. The dimensionless localization length $g(\varepsilon) = \xi(E)/L$ [see Eq. (8)] in the whole fluctuation region.

We are now in a position to discuss the general structure of the Gaussian fluctuation spectrum. Consider a random potential as in Eq. (1) with fixed values of $W(0)$ and R_c. Let us first recall that the region for which the fluctuation approach is applicable is determined by the inequality.

$$\Phi(E) \gg 1 , \tag{10}$$

where, in the quantum fluctuation region, the function $\Phi(E)$ is defined by Eq. (4). As was shown, it is possible to use the white-noise approximation (1) if the correlation radius is the smallest length scale in the problem, so that the quantum fluctuation region is bounded from below due to the inequality[4]

$$\xi(E) \gg R_c . \tag{11}$$

If we now combine the conditions formulated in Eqs. (10) and (11) and our numerical results displayed in Figs. 1–3, we can analyze the general structure of the Gaussian fluctuation region, and summarize it in the form of the "phase diagram" in the (R_c, E_0) plane (see Fig. 4). In this figure, the various domains of applicability of the two models are drawn. To elaborate on the results presented in this figure, we keep the value of $W(0)$ fixed, and this quantity defines a characteristic energy scale $[W(0)]^{1/2}$.

First, let us assume that there are long correlations in this system, so that the inequality $R_c^{-2} \ll [W(0)]^{1/2}$ is satisfied (see region **I** in Fig. 4). Then the white-noise approximation is not valid, the quantum fluctuation domain is empty, and the exponent of the density of states has to be described by the classical asymptotics.[6]

Second, if there are short correlations, then the inequality $R_c^{-2} \ll [W(0)]^{1/2}$ holds (regions **II**–**IV** of Fig. 4). Then a new characteristic energy appears in the problem:

(i) Weak magnetic field $E_0 \ll W_0$ (region **II** of Fig. 4). Here the characteristic energy interval between the fluctuation region and the lowest Landau level is large compared with the cyclotron energy. It means that all the Landau levels have to be taken into account. Therefore, the projection approximation[8] is not valid. On the other hand, up to energies of order R_c^{-2} the correlation radius is much smaller than the localization length and, therefore, the white-noise model is applicable. As for the density of states, only the far asymptotic form for $\Phi(E)$ [Eq. (2)] manifests itself. It is valid in energy domain $W_0 \ll |E| \ll R_c^{-2}$.

(ii) Intermediate strength of the magnetic field, namely

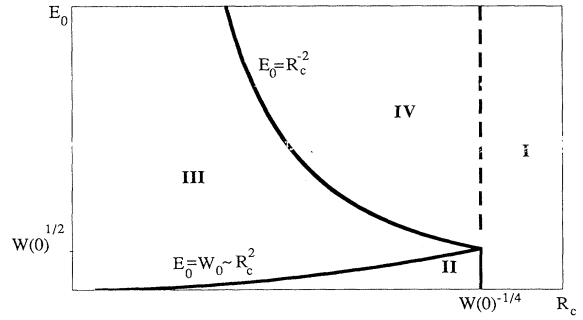

FIG. 4. "Phase diagram" in the (R_c, E_0) plane. In regions **I** and **IV** the white-noise approximation is not valid. In region **II** the projection approach is not valid and only the far asymptotic expression for $\Phi(E)$ [Eq. (2)] manifests itself. In region **III** the quantum fluctuation region consists of two subregions, where both asymptotic expressions (3) and (2) manifest themselves.

$W_0 \ll E_0 \ll R_c^{-2}$ (region **III** of Fig. 4). Here the disorder is comparatively weak, and the characteristic energy interval between the fluctuation region and the lowest Landau level is small compared with the cyclotron energy. Therefore, near the upper bound of the fluctuation region defined by $E_0 - E \sim \sqrt{E_0 W_0} \ll E_0$, one can neglect the contribution of higher Landau levels. In other words, the projection method[8] is applicable in this subregion. That leads to the asymptotic expression for $\Phi(E)$ given by Eq. (3). In the deeper region, where $|E| \geq E_0$, one cannot neglect the contribution from higher Landau levels. Nevertheless, the white-noise model is applicable up to energies of order R_c^{-2}. As for the density of states, the far asymptotic form for $\Phi(E)$ [Eq. (2)] manifests itself. It is valid in an energy domain $E_0 \ll |E| \ll R_c^{-2}$.

(iii) The limit of the strong magnetic field, $R_c^{-2} \ll E_0$ (region **IV** in Fig. 4). Here the magnetic length is extremely small compared with the correlation radius. Therefore, an approximation of the random potential by a δ-correlated one is not valid, and the fluctuation region is identical to the classical one.

ACKNOWLEDGEMENTS

We would like to thank Professor A. G. Aronov and Professor A. L. Efros for helpful discussions, and Professor L. A. Pastur for useful literary references. This work was partially supported by a grant from the Israeli Ministry of Science and Development.

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